THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 7

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Name:

1. Use the differential of an appropriate function to obtain an approximate value and then compare it with the actual one. You may use a calculator.

$$\sin 29^\circ \times \tan 46^\circ \ .$$

Solution. Let $f(\theta, \phi) = \sin\theta \tan\phi$; $(\theta, \phi) = (30^\circ, 45^\circ) = (\pi/6, \pi/4)$; $(d\theta, d\phi) = (-1^\circ, 1^\circ) = (-\pi/180, \pi/180)$. Therefore,

$$\begin{aligned} df &= \cos\theta \tan\phi d\theta + \sin\theta \sec^2\phi d\phi \\ &= \frac{\sqrt{3}}{2} \times 1 \times (-\frac{\pi}{180}) + \frac{1}{2} \times \frac{1}{(\frac{\sqrt{2}}{2})^2} \times \frac{\pi}{180} \\ &= (1 - \sqrt{3}/2)\frac{\pi}{180} \ . \end{aligned}$$

and hence the approximate value is given by

$$f(\pi/6, \pi/4) + (1-\sqrt{3}/2)\frac{\pi}{180} = \frac{1}{2} + (1-\sqrt{3}/2)\frac{\pi}{180} \ .$$

2. The height and the radius of the base of a cylinder are measured with error up to 0.1 and 0.2 respectively. Find the approximate and exact maximum error of its volume.

Solution. Let the volume be V, the radius be r and the height be h. They are related by

$$V = \pi r^2 h$$

For small changes in r and h, we use the approximate change

$$dV = 2\pi rh dr + \pi r^2 dh$$

To calculate the approximate maximum error,we take $dr = 0.2r$ and $dh = 0,1h$,

$$dV = 2\pi rh(0.2r) + \pi r^2(0.1h) = 0.5V$$

Therefore, the approximate maximum error is given by $0.5V/V = 0.5$.

On the other hand, the exact maximum error is

$$\frac{\Delta V}{V} = \frac{\pi(r+0.2r)^2(h+0.1h) - \pi r^2 h}{V} = \frac{0.584V}{V} = 0.584 \ .$$

3. Find the tangent plane and the normal line of each of the surfaces at the given point:

(a)

$$xy^2 - yz^2 + 6xyz = 6 \ , \quad P(1,1,1) \ .$$

(b)

$$x^2yz - e^{xy+1} = 0 \ , \quad P(-1,1,1) \ .$$

You should verify that they are differential surface near the given point first.

Solution.

(a) Let $f(x,y,z) = xy^2 - yz^2 + 6xyz$. Then $\nabla f = (y^2+6yz, 2xy - z^2 + 6xz, -2yz + 6xy)$ and $\nabla f(1,1,1) = (7,7,4) \neq (0,0,0)$. By Theorem 6.2 $f = 6$ defines a surface near $(1,1,1)$. The tangent plane at $(1,1,1)$ is given by

$$(7,7,4)\cdot((x,y,z)-(1,1,1)) = 0,$$

that is, $7x+7y+4z = 18$. The normal line at $(1,1,1)$ is given by

$$(1,1,1) + t(7,7,4) \ , \quad t \in \mathbb{R} \ .$$

(b) Write $g(x,y,z) = x^2yz - e^{xy+1}$. Then

$$\nabla g = (2xyz - ye^{xy+1}, x^2z - xe^{xy+1}, x^2y) \ .$$

We have $\nabla g(-1,1,1) = (-3,2,1) \neq (0,0,0)$. Hence $g = 0$ defines a surface near $(-1,1,1)$. The tangent plane at $(-1,1,1)$ is given by

$$(-3,2,1)\cdot((x,y,z)-(-1,1,1)) = 0 \ ,$$

or $-3x+2y+z = 6$. The normal line at $(-1,1,1)$ is given by

$$(-1,1,1) + t(-3,2,1) \ , \quad t \in \mathbb{R} \ .$$

4. Use implicit differentiation to find the first and $\frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial x^2}$ of $z = z(x,y)$:

(a)

$$x + y + z = e^z \ ,$$

(b)

$$z^3 - 3xyz = -1 \ ,$$

(c)

$$\sin(x+y) - 6\cos(y+z) = x \ .$$

Solution.

(a) First we get $1 + z_x = z_x e^z$, or

$$z_x = \frac{1}{e^z - 1}.$$

Then $z_{xx} = z_{xx}e^z + (z_x)^2 e^z$ which gives

$$z_{xx} = \frac{(z_x)^2 e^z}{1 - e^z} \ .$$

Similarly we get

$$z_y = \frac{1}{e^z - 1},$$

and

$$z_{yy} = \frac{(z_y)^2 e^z}{1 - e^z} \ .$$

Finally, differentiate both sides with respect to y to $1 + z_x = z_x e^z$ yields

$$z_{xy} = \frac{z_x z_y e^z}{1 - e^z} \ .$$

(b)

$$z_x = \frac{yz}{z^2 - xy} \ .$$

$$z_{xx} = \frac{-2z(z_x)^2 + 2yz_x}{z^2 - xy} \ .$$

$$z_y = \frac{xz}{z^2 - xy} \ .$$

$$z_{yy} = \frac{-2z(z_y)^2 + 2xz_y}{z^2 - xy} \ .$$

$$z_{xy} = \frac{-2zz_xz_y + 3z + 3yz_y + 3xz_x}{z^2 - xy} \ .$$

(c)

$$z_x = \frac{1 - \cos(x+y)}{6\sin(y+z)} \ .$$

$$z_{xx} = \frac{\sin(x+y) - 6\cos(y+z)z_x^2}{6\sin(y+z)} \ .$$

$$z_y = -\frac{\cos(x+y) + 6\sin(y+z)}{6\sin(y+z)} \ .$$

$$z_{yy} = \frac{\sin(x+y) - 6\cos(y+z)(1+z_y)^2}{6\sin(y+z)} \ .$$

$$z_{xy} = \frac{\sin(x+y) - 6\cos(y+z)z_x(1+z_y)}{6\sin(y+z)} \ .$$