MATH 2010F Advanced Calculus I, 2016-17

Solution of Test 1

1. (20 points) Consider the points $(1, 3, -2), (2, 4, 5)$ and $(-3, -2, 2)$.

   (a) Find the magnitude and direction of $(1, 3, -2) \times (2, 4, 5)$.

   (b) Find the area of the triangle with vertices at these points.

   (c) Find the volume of the parallelepiped with vertices at these points together with $(0, 0, 0)$.

   Solution.

   (a) By definition,
   $$(1, 3, -2) \times (2, 4, 5) = (23, -9, -2)$$
   Hence the magnitude is
   $$\sqrt{23^2 + 9^2 + 2^2} = \sqrt{614}$$
   and the direction is
   $$\frac{1}{\sqrt{614}}(23, -9, -2)$$

   (b) Let
   $$\mathbf{u} = (2, 4, 5) - (1, 3, -2) = (1, 1, 7) \quad \text{and} \quad \mathbf{v} = (-3, -2, 2) - (1, 3, -2) = (-4, -5, 4)$$
   Then the area is
   $$\frac{1}{2}|\mathbf{u} \times \mathbf{v}| = \frac{1}{2}|(39, -32, -1)| = \frac{1}{2}\sqrt{2546}$$

   (c) The volume is
   $$|(-1, -3, 2) \cdot (\mathbf{u} \times \mathbf{v})| = 55$$

2. (15 points) The line
   $$l: \ x = 3 + 2t, y = 2t, z = t$$
   intersects the plane
   $$\mathcal{P}: \ x + 3y - z = -4$$
   at a point A.

   (a) Find the coordinates of A;

   (b) Further find the equation of the line which is contained in the plane $\mathcal{P}$, passes through A, and is perpendicular to the line l.

   Solution.

   (a) To find the coordinates of A, it suffice to solve the following equation:
   $$(3 + 2t) + 3(2t) - t = -4$$
   which gives $t = -1$. Thus, $A = (1, -2, -1)$.

(b) Observe that the direction of l is $\mathbf{u} := (2,2,1)$ and the normal direction of the plan $\mathcal{P}$ is $\mathbf{v} := (1,3,-1)$. Thus, the direction of the line we want is

$$\mathbf{u} \times \mathbf{v} = (-5,3,4)$$

Thus, the line which is contained in the plane $\mathcal{P}$, passes through A, and is perpendicular to the line l is:

$$l' : x = 1 - 5t, y = -2 + 3t, z = -1 + 4t$$

3. (15 points) Let l_1 be the line passing $A(2,4,0)$ and $B(3,1,1)$, and l_2 be the line passing $C(1,1,3)$ and $D(0,5,1)$. Let $E \in l_1$ and $F \in l_2$, find points E and F so that the line passing E and F is perpendicular to l_1 and l_2. **Solution.**
Note that

$$l_1 : x = 2 + t, y = 4 - 3t, z = t$$

and

$$l_2 : x = 1 - t, y = 1 + 4t, z = 3 - 2t$$

Thus, we can denote E and F by

$$E = (2 + e, 4 - 3e, e), \quad F = (1 - f, 1 + 4f, 3 - 2f)$$

Let $\mathbf{u}$ and $\mathbf{v}$ be direction of l_1 and l_2 respectively. Then

$$(E - F) \cdot \mathbf{u} = 0$$
$$(E - F) \cdot \mathbf{v} = 0$$

Consequently, we obtain that

$$\begin{cases} e = -4 \\ f = \frac{11}{3} \end{cases}$$

Thus,$E = (-2, 16, -4)$ and $F = (\frac{-8}{3}, \frac{47}{3}, \frac{-13}{3})$.

4. (15 points) Find the "standard forms" of the quadratic equation

$$x^2 + 2xy + y^2 + 2y = -6,$$

and describe its solution set.

Solution.
The matrix associated to the quadratic form is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} .$$

Its determinant vanishes, so the curve is a parabola. To remove the mixed term, we first note that its eigenvalues are given by $\lambda_1 = 0$ and $\lambda_2 = 2$.

For each of the above eigenvalues, we solve for an associated eigenvector:

For λ_1, an eigenvector is given by $(1, -1)$. For λ_2, an eigenvector is given by $(1, 1)$. Therefore, by the change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} ,$$

we have

$$\begin{aligned} x^2 + 2xy + y^2 + 2y + 6 &= (x' + y')^2 + 2(x' + y')(-x' + y') + (-x' + y')^2 + 2(-x' + y') + 6 \\ &= (x'^2 + y'^2 + 2x'y') + (-2x'^2 + 2y'^2) + (x'^2 + y'^2 - 2x'y') \\ &\quad +2(-x' + y') + 6 \\ &= 4y'^2 - 2x' + 2y' + 6 . \end{aligned}$$

Finally, let $\tilde{x} = -2x'$ and $\tilde{y} = 2y'$, we have

$$\begin{aligned} x^2 + 2xy + y^2 + 2y &= 4y'^2 - 2x' + 2y' + 6 \\ &= \tilde{y}^2 + \tilde{x} + \tilde{y} + 6 \\ &= \left(\tilde{y} + \frac{1}{2}\right)^2 + \tilde{x} + \left(6 - \frac{1}{4}\right) \\ &= u + v^2 + c , \end{aligned}$$

where $u = \tilde{x}$, $v = \tilde{y} + 1/2$ and $c = 23/4$.

5. (15 points) The folium of Descartes in parametric form is given by

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}, \quad a > 0 .$$

(a) Show that it defines a regular curve on $(-\infty, -1)$ and $(-1, \infty)$;

(b) Find its velocity and tangent line at $t = -2$;

(c) Verify that it is the solution set to

$$x^3 + y^3 = 3axy .$$

Solution.

(a) We differentiate x, y to get

$$x'(t) = \frac{3a - 6at^3}{(1+t^3)^2} , \quad y'(t) = \frac{6at - 3at^4}{(1+t^3)^2} .$$

One finds that $(x'(t), y'(t)) \neq (0, 0)$ for all $t \neq 1$. It defines a regular curve.

(b) Its velocity at $t = -2$ is

$$(x'(-2), y'(-2)) = (\frac{51a}{49}, \frac{-60a}{49})$$

and the tangent line at that point is

$$l : x = \frac{6a}{7} + \frac{51a}{49}t, y = \frac{-12a}{7} - \frac{60a}{49}t$$

(c) When $x = \dfrac{3at}{1+t^3}, y = \dfrac{3at^2}{1+t^3}$, we have

$$\begin{aligned} x^3 + y^3 &= \frac{27a^3t^3(1+t^3)}{(1+t^3)^3} \\ &= \frac{27a^3t^3}{(1+t^3)^2} \\ &= 3axy \ . \end{aligned}$$

6. (10 points) Given a parametric curve $\boldsymbol{\gamma}(t) = (\cos^3 t, \sin^3 t)$, $\quad t \in [0, \pi]$, find its length over $t \in [0, \pi]$.

Solution.
The length function of γ is

$$\begin{aligned} \Lambda(t) &= \int_0^t |\gamma'(z)| dz \\ &= \int_0^t \sqrt{(3\cos^2 z \sin z)^2 + (3\sin^2 z \cos z)^2} dz \\ &= \frac{3}{2}\int_0^t |\sin 2z| dz \\ &= \frac{3}{4}\int_0^{2t} |\sin z| dz \\ &= \begin{cases} \frac{3}{4}(1 - \cos 2t), & t \in [0, \frac{\pi}{2}) \\ \frac{3}{4}(3 + \cos 2t), & t \in [\frac{\pi}{2}, \pi] \end{cases} \end{aligned}$$

so

$$L = \Lambda(\pi) = 3$$

7. (10 points) A projectile is fired with angle of inclination $\pi/6$ from a 1 km-cliff to reach 3 km from the base of the cliff. What should be the initial velocity ?

Solution.
Denote g the gravitational constant. Now we let $\mathbf{a}(t) = (0, -g), \mathbf{r}(0) = (0, 1), \mathbf{v}(0) = c(\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$, where $c \in \mathbb{R}$ is the initial velocity.
Integrating $\mathbf{a}(t)$ to get $\mathbf{v}(t)$:

$$\begin{aligned} \mathbf{v}(t) &= \int_0^t \mathbf{a}(\tau) d\tau + \mathbf{v}(0) \\ &= (0, -gt) + c(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) \\ &= (\frac{\sqrt{3}c}{2}, \frac{c}{2} - gt). \end{aligned}$$

Again, integrating $v(t)$ to get $x(t)$:

$$\begin{aligned}\mathbf{x}(t) &= \int_0^t v(\tau)d\tau + x(0)\\ &= (\frac{\sqrt{3}c}{2}t, \frac{c}{2}t - \frac{1}{2}gt^2) + (0,1)\\ &= (\frac{\sqrt{3}c}{2}t, \frac{c}{2}t - \frac{1}{2}gt^2 + 1).\end{aligned}$$

From $\mathbf{x}(t_0) = (3,0)$, we find

$$\frac{\sqrt{3}c}{2}t_0 = 3 \quad \text{and} \quad t_0 = \sqrt{\frac{2\sqrt{3}+2}{g}}.$$

Thus

$$c = \frac{2\sqrt{3}}{t_0} = \frac{2\sqrt{3}\sqrt{g}}{\sqrt{2\sqrt{3}+2}}.$$

and the initial velocity is

$$v(0) = \left(\frac{3\sqrt{g}}{\sqrt{2\sqrt{3}+2}}, \frac{\sqrt{3}\sqrt{g}}{\sqrt{2\sqrt{3}+2}}\right).$$