Advanced Calculus I

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MATH2010E (Summer Term, 2017), Classroom: LSB C5

Course Web Page : http://www.math.cuhk.edu.hk/course/math2010f/

Outline

Introduction

Advanced Calculus I is all about differentiation of functions of multiple variables This course consists of two parts:

- 1). The background which includes Euclidean space; straight lines, plane and quadric surfaces; parametric curves.
- 2). Differentiation theory which contains partial differentiation, differentiation; curves, surfaces, and hypersurfaces; and extremal problems.

References

(1) [Thomas] MD Weir and J Hass, *Thomas' Calculus*, 12th edition, Pearson.

(2) [Au] T Au, *Differential Multivariable Calculus*, McGraw Hill.

(3) [Fitz] P Fitzpatrick, *Advanced Calculus: A Course in Mathematical Analysis*, 2nd edition, PWS.

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Meeting Schedule

- May 17, 19, 22, 24, 29, 31,
- June 5, 7, 12 14, 19, 21, 23, 26.

Grade

- 10% Assignments (in- and off-class)
- 50% Two tests(no make-up):
  Test 1, May 31, Wed 1:30-3:00;
  Test 2, June 19, Mon, 1:30-3:00;
- 40% Final Examination (TBA).

Assignment Policy

Problem sets will be assigned and deadline for submission will be posted. You must attend the tutorial classes where you work out and hand in additional exercises given by the tutors in class.

Chapter 1:
The Euclidean Space

Section 1.1: The Dot Product

• One dimensional space $\mathbb{R}$: real line. Every point on the real line is a real number x.

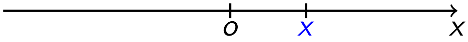


Figure: Real line

• Two dimensional space $\mathbb{R}^2$: plane. Every point on the plane is denoted by the coordinate (x, y);

• Set notation: $\mathbb{R}^2 = \{(x, y) \ : \ x, y \in \mathbb{R}\}$.

Cartesian coordinate system:

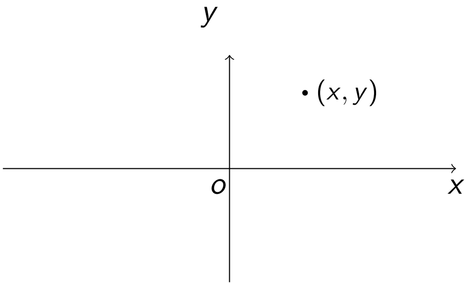
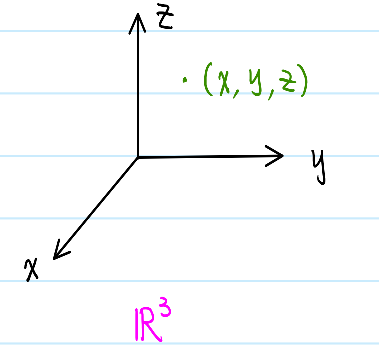


Figure: Plane

- Three dimensional space $\mathbb{R}^3 = \{(x, y, z) \ : \ x, y, z \in \mathbb{R}\}$.

Cartesian coordinate system:



More generally, we have:

Definition 1.1.1 (Euclidean space $\mathbb{R}^n$)

We define the n-dimensional Euclid space to be

$$\mathbb{R}^n =: \{(x_1, \cdots, x_n) \ : \ x_1, \cdots, x_n \in \mathbb{R}\},$$

$$x_i : \text{ith coordinate.}$$

- The zero n-tuple, is denoted as $\mathbf{0} = (0, 0, \cdots, 0)$ from time to time.

- Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, there are two algebraic operations defined on $\mathbb{R}^n$:
  - The addition:

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \cdots, x_n + y_n) \in \mathbb{R}^n ,$$

  - The scalar multiplication:

$$\alpha\mathbf{x} = (\alpha x_1, \cdots, \alpha x_n) , \quad \alpha \in \mathbb{R} .$$

- $\mathbb{R}^n$ becomes a n-dimensional **vector space** over the field of real number.

The **canonical basis** of $\mathbb{R}^n$ is given by

$$\begin{cases} \mathbf{e}_1 = (1, 0, 0, \cdots, 0), \\ \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \\ \cdots \\ \mathbf{e}_n = (0, 0, 0, \cdots, 1). \end{cases}$$

Using this basis every n-tuple can be written as the linear combination of the basis elements in a very simple way,

$$\mathbf{x} = (x_1, x_2, \cdots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots x_n\mathbf{e}_n \ .$$

Definition 1.1.2 (**Dot product**)

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the **dot product** between $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \sum_{j=1}^{n} x_j y_j \\ &= x_1 y_1 + x_2 y_2 + \cdots x_n y_n \ . \end{aligned}$$

We recall the definition inner product on a vector space V over reals:

Definition 1.1.3 (Inner product)

Let $u, v, w \in V$, and $\alpha, \beta \in \mathbb{R}$. We call $\langle \cdot, \cdot \rangle$ to be an inner product if the following three axioms hold:

(1) $\langle u, u \rangle \geq 0$ and equals to 0 iff $u = 0$,

(2) $\langle u, v \rangle = \langle v, u \rangle$,

(3) $\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$.

Noting that (2) and (3), we have that

$$\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle \ .$$

One has no difficulty in verifying the dot product is indeed an inner product operation over $\mathbb{R}^n$. Alternatively one may use $\langle \mathbf{x}, \mathbf{y} \rangle$ to denote $\mathbf{x} \cdot \mathbf{y}$.

With the help of dot product, we can define

Definition 1.1.4 (Euclidean norm)

Let $\mathbf{x} \in \mathbb{R}^n$, then we define Euclidean norm to be

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{j=1}^{n} x_j^2 \right)^{1/2} = \sqrt{x_1^2 + \cdots + x_n^2}\ .$$

Theorem 1.1.5 (Cauchy-Schwarz Inequality)

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then we have

$$\left| \sum_{j=1}^{n} x_j y_j \right| \leq \sqrt{\sum_{j=1}^{n} x_j^2} \sqrt{\sum_{j=1}^{n} y_j^2},$$

$$\left(\textit{equivalently}, \ |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| \cdot |\mathbf{y}| \right).$$

Furthermore, equality sign holds if and only if either one of $\mathbf{x}, \mathbf{y}$ is zero n-tuple or there is some $\alpha \neq 0$ such that $\mathbf{y} = \alpha \mathbf{x}$.

Proof of Theorem 1.1.5:

First assume not all x_j's are zero in $\mathbf{x}$. Consider the expression

$$\sum_{j=1}^{n} (x_j t - y_j)^2 \ ,$$

which is a sum of squares and so must be non-negative for all $t \in \mathbb{R}$. We can express it as a quadratic polynomial in t as

$$p(t) \equiv at^2 - 2bt + c \ ,$$

where

$$a = \sum_{j=1}^{n} x_j^2 \ , \quad b = \sum_{j=1}^{n} x_j y_j, \quad c = \sum_{j=1}^{n} y_j^2 \ .$$

Since $a > 0$, $p(t)$ tends to ∞ as $t \to \pm\infty$. Therefore, it is non-negative if and only if its discriminant is non-positive, that is, $4b^2 - 4ac \leq 0$, which yields $|b| \leq \sqrt{ac}$ after taking square root. Our inequality follows. Moreover, the equality sign holds if and only if $4b^2 - 4ac = 0$. In this case the quadratic equation $at^2 - 2bt + c = 0$ has a (double) root, say, t_1. Going back to the original expression, we have

$$\sum_{j=1}^{n} (x_j t_1 - y_j)^2 = 0 \ ,$$

which forces $t_1 x_j = y_j$ for all $j = 1, \cdots, n$. So we can take $\alpha = t_1$ in case $c = \sum_j y_j^2 > 0$.

When all x_j's vanish but not all y_j's, we exchange $\mathbf{x}$ and $\mathbf{y}$ to get the same conclusion. $\square$

Definition 1.1.6 (Euclidean distance)

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the Euclidean distance is defined as

$$|\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

In mathematics, a distance is a rule to assign a non-negative number to any pair of elements in a set under consideration. The rule consists of three "axioms": For a, b, c in this set,

(1) $d(a, b) \geq 0$, and equal to 0 iff $a = b$,

(2) $d(a, b) = d(b, a)$, and

(3) $d(a, b) \leq d(a, c) + d(c, b)$.

Example 1.1.7

Taking $d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|$ for $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$, please verify that $d(\cdot,\cdot)$ is a distance, i.e.

(1) $|\mathbf{x}-\mathbf{y}| \geq 0$ and equal to 0 if and only if $\mathbf{x}=\mathbf{y}$,

(2) $|\mathbf{x}-\mathbf{y}| = |\mathbf{y}-\mathbf{x}|$,

(3) $|\mathbf{x}-\mathbf{y}| \leq |\mathbf{x}-\mathbf{z}| + |\mathbf{z}-\mathbf{y}|$.

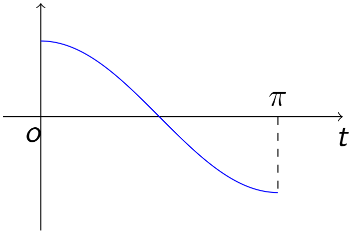
Sol. $\cdots$

Analytical angle

- By Cauchy-Schwarz Inequality, we know that

$$\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|} \in [-1, 1].$$

- $\cos t$ is strictly decreasing from 1 to -1 as t goes from 0 to π, i.e.



Therefore, by what we just said, there exists a unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \mathbf{x} \cdot \mathbf{y} / |\mathbf{x}||\mathbf{y}| \Leftrightarrow \mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta.$$

Definition 1.1.8 (Angle between $\mathbf{x}$ and $\mathbf{y}$)

Let non-zero $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we define the angle θ between $\mathbf{x}$ and $\mathbf{y}$ to be

$$\theta = \arccos\left(\frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}\right) \in [0, \pi].$$

• Two n-tuples $\mathbf{x}$ and $\mathbf{y}$ are **perpendicular** or **orthogonal** to each other if $\mathbf{x} \cdot \mathbf{y} = 0$.

• In terms of the angle, they are perpendicular if and only if their angle is $\pi/2$. The zero n-tuple is perpendicular to all n-tuples.

• Let $\mathbf{x} = c\mathbf{y}$ with $c \neq 0$, then we have

- $\theta = 0$, if $c > 0$;
- $\theta = \pi$, if $c < 0$.

Example 1.1.9

Find all n-tuples $\mathbf{x}$ that are perpendicular to $(1, -1, 2)$ and $(-1, 0, 3)$.

Sol: These points satisfy

$$(1, -1, 2) \cdot \mathbf{x} = 0 \ , \quad (-1, 0, 3) \cdot \mathbf{x} = 0 \ ,$$

that is, the linear system

$$\begin{cases} x - y + 2z & = 0 \ , \\ -x \quad\ + 3z & = 0 \ . \end{cases}$$

We solve this system (see Comments at the end of this chapter) to get $\mathbf{x} = (x, y, z) = a(3, 5, 1), a \in \mathbb{R}$. By varying a, we obtain infinitely many solutions. $\square$

Section1.2: Vector Representation of $\mathbb{R}^n$

• Visualizing n-tuples for $n = 2, 3$ as vectors has been used widely in physics. For instance:

- Physical quantities with magnitude & direction: velocity, force, displacement.

• Geometrically, vectors represented by a directed segment $\overrightarrow{AB}$ with initial point A and terminal point B.

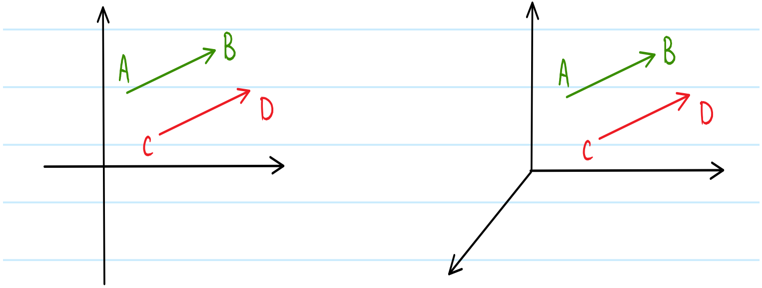


Figure: Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

• Two vectors are equal if there have same length and direction, so in the above two figures, $\overrightarrow{AB} = \overrightarrow{CD}$.

• Given the coordinate of $A(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, it is convenient to denote the vector by $\overrightarrow{OA}$, where O is the origin point. More generally,

Definition 1.2.1

Given two $A(x_1, \cdots, x_n), B(y_1, \cdots, y_n) \in \mathbb{R}^n$, the vector $\overrightarrow{AB}$ algebraically represented by

$$\overrightarrow{AB} = (y_1 - x_1, \cdots, y_n - x_n).$$

This is also called component form of vector.

Definition 1.2.2 (Magnitude or length)

Given vector $\vec{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$, the magnitude(length) of vector $\vec{u}$ defined to be

$$|\vec{u}| = \sqrt{u_1^2 + \cdots + u_n^2}\ .$$

So the magnitude of $\overrightarrow{AB}$ is indeed the distance between A and B.

- Any vector with unit length is called a **direction.**

Now we interpret the algebraic operations of $\mathbb{R}^n$:

(1) Indeed, by drawing pictures, it is easy to know that the "addition operation "of $\mathbb{R}^n$ is accomplished by the "parallelogram law".

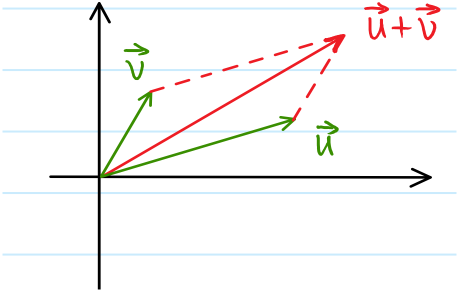


Figure: Parallelogram law: addition

(2) The "scalar multiplication" means changing the vector by a scale of α.

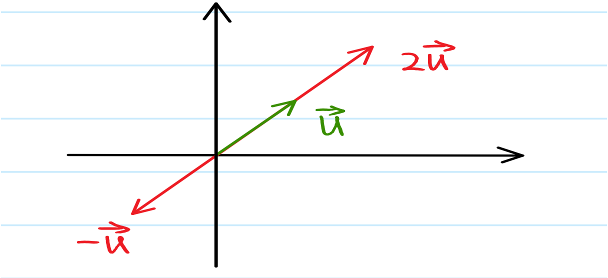


Figure: Scalar multiplication

• How about the substraction ?

Let $\vec{u} = (u_1, \cdots, u_n), \vec{v} = (v_1, \cdots, v_n)$ be two vectors in $\mathbb{R}^n$, then $(1) + (2)$ implies that

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v} = (u_1 - v_1, \cdots, u_n - v_n).$$

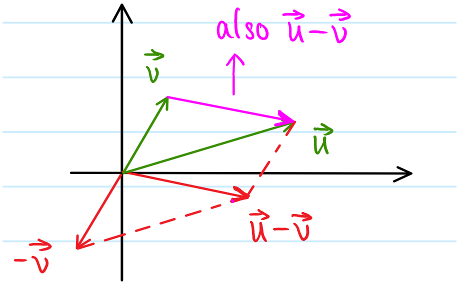


Figure: Parallelogram law: subtraction

- For $\mathbf{u}, \mathbf{v}$, its midpoint is given by $(\mathbf{u}+\mathbf{v})/2$.

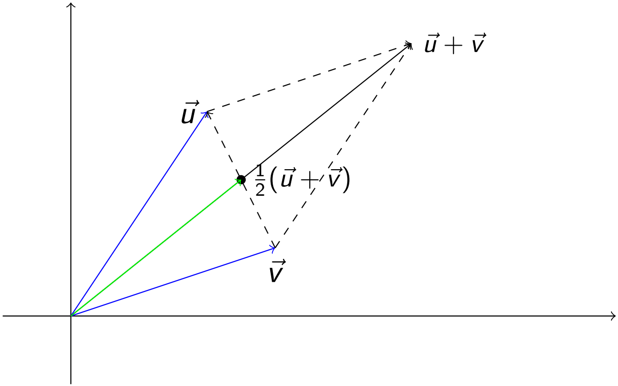


Figure: Midpoint

• Now we consider the **geometric meaning of analytical angle between $\mathbf{x}, \mathbf{y}$**. Indeed, such angle is the same as the "geometric angle"

To see it, let $\mathbf{x} = (a, b)$ and $\mathbf{y} = (c, d)$ be two non-zero vectors in the plane. By the Law of Cosines in trigonometry (see Comments at the end of this chapter),

$$(c-a)^2+(d-b)^2 = (a^2+b^2)+(c^2+d^2)-2\sqrt{c^2+d^2}\sqrt{a^2+b^2}\cos\phi\,,$$

where $\phi \in [0, \pi]$ is the "geometric angle" between $\mathbf{x}$ and $\mathbf{y}$. Simplifying, we have

$$-2(ac+db) = -2\sqrt{c^2+d^2}\sqrt{a^2+b^2}\cos\phi\,,$$

which is equal to

$$\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos\phi\,.$$

Comparing with the definition of θ, we have $\cos\phi = \cos\theta$ so that $\phi = \theta$. In other words, *the geometric angle coincides with the analytical angle.*

• A vector is uniquely determined by its magnitude and direction. Each non-zero vector $\mathbf{x}$ can be written as

$$\mathbf{x} = |\mathbf{x}|\ \boldsymbol{\xi},$$

where $|\mathbf{x}|$ is its magnitude and $\boldsymbol{\xi} = \dfrac{\mathbf{x}}{|\mathbf{x}|}$ its direction.

• Every direction $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ can further be expressed as

$$\boldsymbol{\xi} = \left(\cos\alpha_1, \cos\alpha_2, \cdots, \cos\alpha_n\right)\ ,$$

where $\alpha_k \in [0, \pi]$ are called the **direction angles** of $\boldsymbol{\xi}$. From $\boldsymbol{\xi} \cdot \mathbf{e}_k = \cos\alpha_k$ we see that α_k is the angle between $\boldsymbol{\xi}$ and the e_k-axis. These $\cos\alpha_k$'s are called the **direction cosines** of $\mathbf{x}$.

Example 1.2.3

Find the magnitude and direction of $(1, 2, -7)$ and determine the vector $(2, a, 6)$ that is perpendicular to $(1, 2, -7)$.

Sol: The magnitude of $(1, 2, -7)$ is

$$|(1, 2, -7)| = \sqrt{1^2 + 2^2 + (-7)^2} = \sqrt{54} \ ,$$

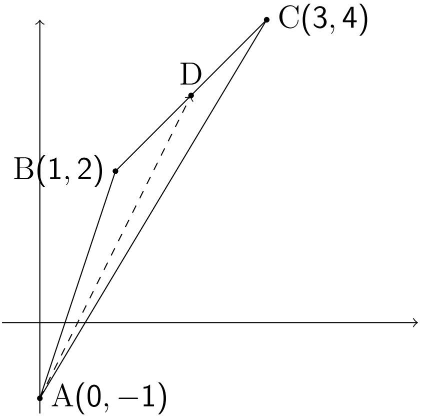
and its direction is $(1, 2, -7)/\sqrt{54}$. By orthogonality,

$$0 = (1, 2, -7) \cdot (2, a, 6) = 2 + 2a - 42 = 0 \ ,$$

which implies $a = 20$. The vector $(2, 20, 6)$ is perpendicular to $(1, 2, -7)$. $\square$

Example 1.2.4

Consider the triangle with vertices at $A(1,2)$, $B(3,4)$, $C(0,-1)$. Find the direction of the vector pointing at the midpoint of the side connecting $(1,2)$ and $(3,4)$ from $(0,-1)$.



Sol: Firstly, we translate $(0, -1)$ to the origin so that the triangle is congruent to the one whose vertices are $((1, 2) - (0, -1), (3, 4) - (0, -1), (0, -1) - (0, -1)$, that is, $(1, 3), (3, 5), (0, 0)$. The midpoint of the side from $(1, 3)$ and $(3, 5)$ is given by

$$\frac{1}{2}\left((1, 3) + (3, 5)\right) = (2, 4) \ ,$$

and its direction is given by

$$\frac{(2, 4)}{\sqrt{2^2 + 4^2}} = \frac{(2, 4)}{\sqrt{20}} = \frac{(1, 2)}{\sqrt{5}} \ .$$

(No need to simplify further.) □

Example 1.2.5

(a). Find the magnitude and direction of the vector from $(1, -1)$ to $(-2, 5)$.

(b). Find all directions that are perpendicular to the vector in (a).

Sol: (a). The magnitude and direction of the vector from $(1, -1)$ to $(-2, 5)$ are the same as those of the position vector $(-2, 5) - (1, -1) = (-3, 6)$. Its magnitude is given by

$$|(-3, 6)| = \sqrt{(-3)^2 + 6^2} = 3\sqrt{5}\ ,$$

and the direction is given by

$$\frac{(-3, 6)}{3\sqrt{5}} = \frac{(-1, 2)}{\sqrt{5}}\ .$$

(No need to simplify further.)

(b). A vector (a, b) perpendicular to $(-3, 6)$ satisfies

$$(-3, 6) \cdot (a, b) = -3a + 6b = 0 \ .$$

For instance, we may take $a = 2, b = 1$ so $(2, 1)$ is one choice. Then the direction vectors can be taken to be :

$$\frac{(2,1)}{\sqrt{5}} \ , \quad -\frac{(2,1)}{\sqrt{5}} \ .$$

(Again no need to simplify.) $\square$

Section 1.3: Euclidean Motions

Definition 1.3.1 (Euclidean motion)

A **Euclidean motion** is a map from $\mathbb{R}^n$ to itself of the form

$$T\mathbf{x} = A\mathbf{x} + \mathbf{b} \ ,$$

where $\mathbf{b} \in \mathbb{R}^n$ and A is an $n \times n$-matrix, that preserves the distance between two points, that is, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|T\mathbf{x} - T\mathbf{y}| = |\mathbf{x} - \mathbf{y}| \ .$$

Here in $A\mathbf{x}$ the vector $\mathbf{x}$ should be understood as a column vector.

• Recall that a square matrix R is called an orthogonal matrix if $R'R = RR' = I$, where R' is the transpose of R and I is the identity matrix.

Proposition 1.3.2

A map $T\mathbf{x} = A\mathbf{x} + \mathbf{b}$ is a Euclidean motion if and only if A is an orthogonal matrix.

Proof. In the following we use $\langle \mathbf{x}, \mathbf{y} \rangle$ instead $\mathbf{x} \cdot \mathbf{y}$ to denote the dot product. First of all, let T be a Euclidean motion. Then it follows from the definition that

$$|\mathbf{x} - \mathbf{y}| = |T\mathbf{x} - T\mathbf{y}| = |A\mathbf{x} - A\mathbf{y}| = |A(\mathbf{x} - \mathbf{y})|,$$

which yields immediately that

$$\begin{aligned}|A(\mathbf{x}+\mathbf{y})|^2 &= \langle A(\mathbf{x}+\mathbf{y}), A(\mathbf{x}+\mathbf{y})\rangle \\ &= |\mathbf{x}+\mathbf{y}|^2 \\ &= |\mathbf{x}|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + |\mathbf{y}|^2 \ .\end{aligned}$$

On the other hand, a direct calculation shows that

$$\begin{aligned}\langle A(\mathbf{x}+\mathbf{y}), A(\mathbf{x}+\mathbf{y})\rangle &= \langle A\mathbf{x}+A\mathbf{y}, A\mathbf{x}+A\mathbf{y}\rangle \\ &= |A\mathbf{x}|^2 + 2\langle A\mathbf{x}, A\mathbf{y}\rangle + |A\mathbf{y}|^2.\end{aligned}$$

By comparing, we see that for all $\mathbf{x}, \mathbf{y}$,

$$\langle A'A\mathbf{x}, \mathbf{y}\rangle = \langle A\mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle \ ,$$

which implies that $A'A = I$. Thus A is a orthogonal matrix.

Finally, this relation also shows that T is a Euclidean motion whenever A is orthogonal. $\square$

• Here are some examples of Euclidean motions.
(1). Take A to be the identity and $\mathbf{b}$ a nonzero vector. Then $T\mathbf{x} = \mathbf{x} + \mathbf{b}$ is a translation. The origin is moved to $\mathbf{b}$ after the motion.
(2). When $n = 2$, the Euclidean motion

$$T\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is the reflection with respect to the x-axis and

$$T\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is the reflection with respect to the y-axis. (In matrix form the vector $\mathbf{x}$ is understood as a column vector.) On the other hand, given any plane in $\mathbb{R}^3$, one may consider the reflection with respect to this plane. For instance,

$$T\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is the reflection with respect to the xy-plane in $\mathbb{R}^3$. The reflection with respect to any straight line in $\mathbb{R}^2$ or with respect to any plane in $\mathbb{R}^3$ can be defined similarly.

(3). The (counterclockwise) rotation of angle θ in $\mathbb{R}^2$ is given by the Euclidean motion

$$T\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} , \quad \theta \in (0, 2\pi) .$$

In $\mathbb{R}^3$, one can perform a rotation around a fixed axis. For instance, the rotation

$$T\mathbf{x} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

takes the z-axis as its axis of rotation.

• The composition of two Euclidean motions is still Euclidean motion.

Indeed, let $T\mathbf{x} = A\mathbf{x} + \mathbf{b}$ and $S\mathbf{x} = B\mathbf{x} + \mathbf{c}$ be two Euclidean motions. Its composition is given by

$$ST\mathbf{x} = B(A\mathbf{x} + \mathbf{b}) + \mathbf{c} = C\mathbf{x} + \mathbf{d}\ , \quad C \equiv BA, \quad \mathbf{d} = B\mathbf{b} + \mathbf{c}\ .$$

As

$$\begin{aligned} C'C &= (BA)'BA \\ &= A'B'BA \\ &= A'IA \\ &= I, \end{aligned}$$

we conclude that ST is again a Euclidean motion.

• **Each Euclidean motion admits an inverse.** Indeed, letting $U\mathbf{x} = A^{-1}\mathbf{x} - A^{-1}\mathbf{b}$ which is obviously an Euclidean motion, we have

$$UT\mathbf{x} = A^{-1}(A\mathbf{x} + \mathbf{b}) - A^{-1}\mathbf{b} = \mathbf{x} \; .$$

• In the following we study the structure of Euclidean motions for $n = 2, 3$. Apparently it suffices to look at the orthogonal matrix A.

Theorem 1.3.3

In $\mathbb{R}^2$, every orthogonal matrix can be written as

(1)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

or

(2)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \theta \in [0, 2\pi).$$

Remark: Case (1) is a genuine rotation for $\theta \in (0, 2\pi)$ and reduces to the identity at $\theta = 0$. Case (2) is the reflection with respect to the x-axis and then followed by a rotation.

Proof. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

By orthogonality $A'A = I$ we have

$$a^2 + c^2 = 1 \,, \quad b^2 + d^2 = 1 \,, \quad ab + cd = 0 \,.$$

Since a is a number between -1 and 1, we can find a unique $\theta \in [0, 2\pi)$ such that $a = \cos\theta,\ c = \sin\theta$. Then either

$$b = -\sin\theta,\ d = \cos\theta$$

or

$$b = \sin\theta, d = -\cos\theta,$$

so (a) or (b) must hold. $\square$

In the following we consider the three dimensional case. Denote by $R_z(\theta)$ the rotation around the z-axis by an angle θ:

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, rotation around the z-axis.

Similarly we define

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$, rotation around the x-axis,

and

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$, rotation around the y-axis.

Also we denote reflection with respect to the xy-plane:

$$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$, reflection with respect to the xy-plane,

$$L_x = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, reflection with respect to the yz-plane,

and

$$L_y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, reflection with respect to the xz-plane.

Theorem 1.3.4

** In $\mathbb{R}^3$, every orthogonal matrix can be written as*

(a) $R_z(\alpha)R_x(\beta)R_z(\gamma)$, *or*

(b) $R_z(\alpha)R_x(\beta)R_z(\gamma)L_z$,

for some $\alpha, \beta,$ and γ.

Proof. The details are omitted.

Section 1.4: The Cross Product in $\mathbb{R}^3$

Definition 1.4.1 (Cross product)
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, define the cross product of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ to be

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1) \\ &\equiv \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right). \end{aligned}$$

- The cross product defines a new 3-vector by two given 3-vectors. Remember the negative sign in the second term.

- There is no such product in the general dimension. *The cross product is important due to its relevance in physics.*

- In particular, we have

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 \ .$$

Theorem 1.4.2 (Properties of cross product)

For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$,

(a) $(\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{w} = \alpha\mathbf{u} \times \mathbf{w} + \beta\mathbf{v} \times \mathbf{w}\ , \quad \forall \alpha, \beta \in \mathbb{R}.$

(b) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$. *In particular,* $\mathbf{u} \times \mathbf{u} = 0$.

(c) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ *is not always true.*

Proof. The proofs of Theorem (a) and (b) are straightforward from definition.

As for (c), which asserts that the associative law does not hold, an example suffices:

$$(\mathbf{e}_1 \times \mathbf{e}_2) \times \mathbf{e}_2 = -\mathbf{e}_1 \ , \quad \mathbf{e}_1 \times (\mathbf{e}_2 \times \mathbf{e}_2) = \mathbf{0}.$$

• We know that a vector is completely determined by its magnitude and direction. *Thus we consider the direction and magnitude of the cross product.*

•(**Geometric Property**) Using the definition of the cross product, one can verify directly that

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \ , \quad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0 \ ,$$

so

$$(\alpha \mathbf{u} + \beta \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = 0 \ ,$$

that is, *it is perpendicular to the two dimensional subspace spanned by the vectors* $\mathbf{u}$ *and* $\mathbf{v}$.

Moreover, we have:

• *The direction of* $\mathbf{u} \times \mathbf{v}$ *is determined by the* **right hand rule**. That is, with the thumb making a right angle with the other four fingers of your right hand, first point the four fingers along the direction of $\mathbf{u}$ and then move them to $\mathbf{v}$ in an angle less than π. The direction of $\mathbf{u} \times \mathbf{v}$ is where your thumb points to.

How about the magnitude of cross product ?

Theorem 1.4.3

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$*,*

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta, \quad \theta \in [0, \pi] \ ,$$

where θ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Proof. The proof depends on the identity

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \ .$$

Indeed, by brute force

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|^2 &= (u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2 \\ &= u_2^2 v_3^2 + u_3^2 v_2^2 + u_1^2 v_3^2 + u_3^2 v_1^2 + u_1^2 v_2^2 + u_2^2 v_1^2 \\ &\quad - 2u_2 v_3 u_3 v_2 - 2u_1 v_3 u_3 v_1 - 2u_1 v_2 u_2 v_1 . \end{aligned}$$

On the other hand,

$$\begin{aligned}
&|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2 \\
&= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2 \\
&= u_2^2v_3^2 + u_3^2v_2^2 + u_1^2v_3^2 + u_3^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 \\
&\qquad - 2u_2v_3u_3v_2 - 2u_1v_3u_3v_1 - 2u_1v_2u_2v_1,
\end{aligned}$$

whence the identity holds. Now, by the Cosine Law,

$$\begin{aligned}
|\mathbf{u}\times\mathbf{v}| &= \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2 - |\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta} \\
&= |\mathbf{u}||\mathbf{v}||\sin\theta| \\
&= |\mathbf{u}||\mathbf{v}|\sin\theta\ ,
\end{aligned}$$

as $\sin\theta \geq 0$ on $[0,\pi]$.

• In conclusion the magnitude and direction of the decomposition of the cross product is given by

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin\theta \ \mathbf{n} \ ,$$

where $\mathbf{n}$ is the unit vector determined by the right hand rule (when $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, that is, when $\sin\theta \neq 0$).

Corollary 1.4.4

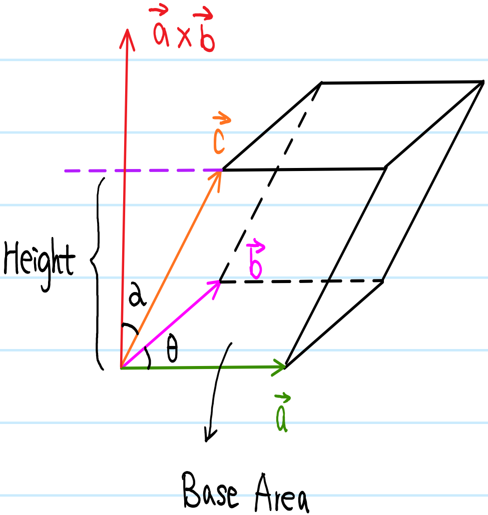
Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$, *then it holds that*

(1) *The area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$ *is equal to* $|\mathbf{a} \times \mathbf{b}|$.

(2) *The area of the triangle with two sides given by* $\mathbf{a}$ *and* $\mathbf{b}$ *is equal to* $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

(3) *The volume of the parallelepiped spanned by* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *is equal to*

$$V = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})| \ .$$


$\vec{a} \times \vec{b}$
$\vec{c}$
$\vec{b}$
Height
α
θ
$\vec{a}$
Base Area

Proof. (1) follows immediately from Theorem 1.5 and (2) from (1).

To prove (3), we may assume $\mathbf{a}$ and $\mathbf{b}$ lie on the xy-plane after a rotation. The volume of the parallelepiped is given by the product of the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ with its height. Now $|\mathbf{a} \times \mathbf{b}|$ is equal to the area of this parallelogram. On the other hand, its height is given by $|\mathbf{c} \cdot \mathbf{e}_3|$. Therefore, letting α be the angle between $\mathbf{c}$ and the z-axis,

$$\begin{aligned} |\mathbf{w} \cdot (\mathbf{a} \times \mathbf{b})| &= |\mathbf{c}||\mathbf{a} \times \mathbf{b}||\cos \alpha| \\ &= |\mathbf{a} \times \mathbf{b}|\ |\mathbf{c} \cdot \mathbf{e}_3| \\ &= V\ . \end{aligned}$$

$\square$

Example 1.4.5

Determine if the four points

$$A(1,0,1),\ B(2,4,-6),\ C(3,-1,5),\ D(1,-9,19)\ ,$$

lie on the same plane in $\mathbb{R}^3$.

Sol: Well, they lie on the same plane if and only if the parallelepiped formed by these vectors has zero volume. We compute the volume using this corollary after subtracting the first vector from the last three vectors (to make sure that the vectors are based at the origin):

$$\begin{aligned}(1,4,-7)\cdot\big((2,-1,4)\times(0,-9,18)\big) &= (1,4,-7)\cdot(18,-36,-18)\\ &= 0\ ,\end{aligned}$$

so they lie on the same plane. $\square$

Chapter 2:
Lines, Planes, and Quadratic Surfaces

Section 2.1: Hyperplanes

Definition 2.1.1 (Zero set)

Given a function f in n-many variables, we let

$$\Sigma = \{\mathbf{x} \in \mathbb{R}^n : \ f(\mathbf{x}) = 0\},$$

be its **zero set** of f. It is also called the **solution set** of f as one can also view $f(\mathbf{x}) = 0$ as solving an equation.

- When expressed in the form

$$\Sigma_c = \{\mathbf{x} \in \mathbb{R}^n : \ f(\mathbf{x}) = c\},$$

which corresponds to the zero set of the function $f - c$, this set is called the **level set** of the function f at c.

Now we consider the linear equation in $\mathbb{R}^n$, $n \geq 1$,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \ , \quad a_1, a_2, \cdots, a_n, b \in \mathbb{R} \ .$$

Here it is implicitly assumed at least one of the coefficients a_j's is non-zero.

• It is called a **homogeneous linear equation** when $b = 0$ and a **non-homogeneous linear equation** when $b \neq 0$.

• Using the dot product, we can write a linear equation in the form

$$\mathbf{a} \cdot \mathbf{x} = b \ , \quad \mathbf{a} \in \mathbb{R}^n \ , \ b \in \mathbb{R} \ .$$

Definition 2.1.2 (Hyperplane)

Let $\mathbf{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$, then the solution set

$$H = \{\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = b\},$$

is called the **hyperplane** associated to the equation.

- Generally, the vector **a** is named as **normal vector**.

- When $n = 2$, the hyperplane is called the **straight line** or simply the line associated to the equation.

- When $n = 3$, it is called the **plane** associated to the equation.

How to write down the equation of a hyperplane ?

• *A hyperplane is completely determined when its normal vector and a point on it are known.* In other words, letting $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ be a vector that is perpendicular to the plane and $\mathbf{p}$ a point on the plane, the equation of the hyperplane is given by

$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{p}) = 0.$$

Example 2.1.3

Find the equation of the plane which is parallel to the plane $2x - 7y + z = 0$ and passing through the point $(1, 2, 3)$.

Sol: By parallel we mean these two planes have the same normal direction. Therefore, the sought-after plane has normal $(2, -7, 1)$ and, as it passes through $(1, 2, 3)$, $b = (2, -7, 1) \cdot (1, 2, 3) = -9$. The equation for the plane is given by

$$(2, -7, 1) \cdot ((x, y, z) - (1, 2, 3)) = 0 \ ,$$

that is, $2x - 7y + z = -9$.

Example 2.1.4

Find the straight line passing through $(-1, 2)$ and is perpendicular to the straight line $2x + 5y = -9$.

Sol: The normal direction of the line $2x + 5y = -9$ is $(2, 5)$, so the normal direction of our straight line is $(5, -2)$ (you may choose $(-5, 2)$ as well). Therefore, the equation of our straight line is

$$(-5, 2) \cdot ((x, y) - (-1, 2)) = 0, \quad \text{or} \; -5x + 2y = 9 \;.$$

• For $n = 3$, it is apparent three points determine a plane uniquely unless they are collinear.

How to find the equation of the hyperplane passing certain points ?

• In higher dimension($n \geq 4$), it is more tedious to describe the conditions that n points determined a hyperplane in geometric terms.

Generally, given linearly independent points $\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n \in \mathbb{R}^n$, in order to determine the equation of the hyperplane passing through these points *it suffices to determine its normal direction* $\mathbf{a}$, which should satisfy the linear system

$$(\mathbf{p}_1 - \mathbf{p}_n) \cdot \mathbf{a} = 0\,, \quad (\mathbf{p}_2 - \mathbf{p}_n) \cdot \mathbf{a} = 0\,, \cdots, \quad (\mathbf{p}_{n-1} - \mathbf{p}_n) \cdot \mathbf{a} = 0\,.$$

• This is a system of n unknowns and $n-1$ equations, so it is always solvable.

• When the $n-1$ points $\mathbf{p}_j - \mathbf{p}_n, j = 1, \cdots, n-1$, are linearly independent, it is known that the solution is one dimensional, that is, it is spanned by a single vector, and we can take it to be the normal. *Therefore, the problem of determining the hyperplane is reduced to solving a linear system.*

When $n = 3$, we can take advantage of the cross product. Now we need to solve $(\mathbf{p}_1 - \mathbf{p}_3) \cdot \mathbf{a} = 0$ and $(\mathbf{p}_2 - \mathbf{p}_3) \cdot \mathbf{a} = 0$. We see that a normal direction is given by $(\mathbf{p}_1 - \mathbf{p}_3) \times (\mathbf{p}_2 - \mathbf{p}_3)$.

Theorem 2.1.5

The equation for the plane passing three non-collinear points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ is given by

$$\Big((\mathbf{p}_1 - \mathbf{p}_3) \times (\mathbf{p}_2 - \mathbf{p}_3)\Big) \cdot \big((x, y, z) - \mathbf{p}_3\big) = 0 \ .$$

Example 2.1.6

Find the equation of the plane passing through the points

$$\mathbf{p}_1(0,1,1),\ \mathbf{p}_2(2,3,0),\ \mathbf{p}_3(2,3,4)\ .$$

Sol: A direct calculation shows

$$\begin{aligned}
&(\mathbf{p}_1-\mathbf{p}_3)\times(\mathbf{p}_2-\mathbf{p}_3)\\
&=((0,1,1)-(2,3,4))\times((2,3,0)-(2,3,4))\\
&=(-2,-2,-3)\times(0,0,-4)\\
&=(8,-8,0),
\end{aligned}$$

so the equation is given by

$$(8,-8,0)\cdot((x,y,z)-(2,3,4))=8(x-2)-8(y-3)+0(z-4)=0,$$

that is, $x-y+1=0$.

Example 2.1.7

Find the equation of the plane passing through the points

$$\mathbf{p}_1(1,1,1),\ \mathbf{p}_2(2,-1,0),\ \mathbf{p}_3(0,-3,4)\ .$$

Sol: Although the cross product approach may be used, let us follow the general approach. First, we bring $(1,1,1)$ to the origin.

$$\mathbf{p}_2 - \mathbf{p}_1 = (2,-1,0) - (1,1,1) = (1,-2,-1),$$

$$\mathbf{p}_3 - \mathbf{p}_1 = (0,-3,4) - (1,1,1) = (-1,-4,3).$$

The normal direction of the plane (a, b, c) is perpendicular to these two vectors,

$$(1, -2, -1) \cdot (a, b, c) = 0, \quad (-1, -4, 3) \cdot (a, b, c) = 0 \ ,$$

which gives

$$a - 2b - c = 0, \quad a + 4b - 3c = 0 \ .$$

Using c as the parameter, we solve this system to get $a = 5c/3$ and $b = c/3$. That is, the vector $(5/3, 1/3, 1)c$ is perpendicular to the plane. Taking $c = 3$, our plane satisfies the equation

$$(5, 1, 3) \cdot ((x, y, z) - (1, 1, 1)) = 0, \qquad \text{i.e., } 5x + y + 3z = 9 \ .$$

□

Let H be a hyperplane and $\mathbf{p}$ a point outside the hyperplane, then the distance from $\mathbf{p}$ to H is defined as

$$dist(\mathbf{p}, H) = \min_{\mathbf{x} \in H} |\mathbf{p} - \mathbf{x}|.$$

Now we derive a formula for the **distance from a point to a hyperplane**.

Theorem 2.1.8 (Distance from point to hyperplane)

Let $\mathbf{p} \in \mathbb{R}^n$ *and* $H = \{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{a} \cdot \mathbf{x} = b\}$ *be a hyperplane in* $\mathbb{R}^n$. *The distance from* $\mathbf{p}$ *to the hyperplane is given by*

$$dist(\mathbf{p}, H) = \frac{|\mathbf{a} \cdot \mathbf{p} - b|}{|\mathbf{a}|}.$$

Proof. We note that the distance is equal to the length of the line segment from $\mathbf{p}$ perpendicular to the hyperplane.

Let $\mathbf{q}$ be the point on the plane so that $\mathbf{p}-\mathbf{q}$ is perpendicular to the hyperplane. We have two equations, namely,

$$\mathbf{a}\cdot\mathbf{q}=b\ ,\quad \mathbf{p}-\mathbf{q}=\lambda\mathbf{a}\ ,\quad \lambda\in\mathbb{R}\ .$$

The first equation means $\mathbf{q}$ is a point on the plane and the second equation means $\mathbf{p}-\mathbf{q}$ points to the normal direction of the hyperplane. We plug $\mathbf{q}=\mathbf{p}-\lambda\mathbf{a}$ in the first equation to get

$$\mathbf{a}\cdot(\mathbf{p}-\lambda\mathbf{a})=b\ ,$$

which yields

$$\lambda=\frac{\mathbf{a}\cdot\mathbf{p}-b}{|\mathbf{a}|^2}\ .$$

It follows that

$$\mathbf{q} = \mathbf{p} - \lambda\mathbf{a} = \mathbf{p} - \frac{\mathbf{a}\cdot\mathbf{p} - b}{|\mathbf{a}|^2}\mathbf{a},$$

$$\Big(\textbf{projection} \text{ of } \mathbf{p} \text{ onto the hyperplane}\Big).$$

As the distance from $\mathbf{p}$ to the hyperplane is given by $|\mathbf{p} - \mathbf{q}|$, we conclude that it is given by

$$dist(\mathbf{p}, H) = \frac{|\mathbf{a}\cdot\mathbf{p} - b|}{|\mathbf{a}|}\ .$$

$\square$

Corollary 2.1.9

Let $\mathbf{a} \cdot \mathbf{x} = b$ *and* $\mathbf{a} \cdot \mathbf{x} = c$ *be two parallel planes. The distance between them is given by* $|b - c|/|\mathbf{a}|$*.*

Proof. The distance from a point $\mathbf{p}$ on the second plane to the first one is given by $|\mathbf{a} \cdot \mathbf{p} - b|/|\mathbf{a}|$, and the formula follows after noting $\mathbf{a} \cdot \mathbf{p} = c$.

Example 2.1.10

Let $H: \ 2x + 5y - z + w = 2$ be a hyperplane in $\mathbb{R}^4$ and $P(1, 2, 0, -3)$ a point lying outside the hyperplane.

(a) Find the point on the hyperplane that realizes the distance between p and H.

(b) Find the distance from P to H.

Sol: We apply the formulas:

$$\mathbf{a} \cdot \mathbf{p} = (2, 5, -1, 1) \cdot (1, 2, 0, -3) = 9,$$

and

$$\lambda = \frac{\mathbf{a} \cdot \mathbf{p} - b}{|a|^2} = \frac{9-2}{31} = \frac{7}{31} \ ,$$

hence

$$\mathbf{q} = \mathbf{p} - \lambda \mathbf{a} = (1, 2, 0, -3) - \frac{7}{31}(2, 5, -1, 1) = \frac{1}{31}(17, 27, 7, -100) \ ,$$

which is the answer to (a) and

$$|\mathbf{p} - \mathbf{q}| = \frac{7}{\sqrt{31}} \ ,$$

is the answer to (b). $\square$

Section 2.2: Straight Lines

A straight line is determined by its **direction** and **a point it passes through**.

Definition 2.2.1 (Straight Lines)

Given $\mathbf{p} \in \mathbb{R}^n$ and a non-zero $\boldsymbol{\xi} \in \mathbb{R}^n$, a **straight line** passing through $\mathbf{p}$ along the direction determined by $\boldsymbol{\xi}$ is given by the set of points

$$\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{x} = \mathbf{p} + \boldsymbol{\xi} t \ , \quad t \in \mathbb{R}\}.$$

• $\boldsymbol{\xi}$ can not be zero, otherwise the straight line would degenerate into a single point;

• In some old texts, the equation of a straight line is expressed as

$$\frac{x_1 - p_1}{\xi_1} = \frac{x_2 - p_2}{\xi_2} = \cdots = \frac{x_n - p_n}{\xi_n} \ ,$$

which is an alternate description that the straight line passing $\mathbf{p}$ with slope $\boldsymbol{\xi}$.

For $\mathbb{R}^3$, we have the following theorem:

Theorem 2.2.2

- *Any straight line is the intersection of two linearly independent planes in* $\mathbb{R}^3$.
- *Conversely, the solution set of two linear equations with different normal directions is a straight line for some* $\mathbf{p}$ *and* $\boldsymbol{\xi}$.

Example 2.2.3

Find the expression for the straight lines which is the intersection of the planes

$$\begin{cases} x + y + z = 1, \\ 2x - y + 6z = 5, \end{cases} .$$

Sol: We may take z as the "time parameter" and write the system as

$$\begin{cases} x + y = 1 - z, \\ 2x - y = 5 - 6z. \end{cases}$$

Solve this equation to get

$$x = \frac{1}{3}(6 - 7z) \ , \quad y = \frac{1}{3}(-3 + 4z) \ .$$

Writing $t = z$, the straight line is given by

$$(x, y, z) = \left(\frac{1}{3}(6 - 7t), \frac{1}{3}(-3 + 4t), t\right)$$
$$= (2, -1, 0) + \left(-\frac{7}{3}, \frac{4}{3}, 1\right) t \ , t \in \mathbb{R}.$$

It passes through $(2, -1, 0)$ at $t = 0$ with constant velocity $(-7/3, 4/3, 1)$.

Alternatively, we can take y as the time parameter. We write

$$\begin{cases} x + z = -y + 1, \\ 2x + 6z = 5 + y, \end{cases}$$

Alternatively, we can take y as the time parameter. We write

$$\begin{cases} x + z = -y + 1, \\ 2x + 6z = 5 + y, \end{cases}$$

which gives

$$x = \frac{1}{4}(1 - 7z) \ , \quad z = \frac{1}{4}(3 + 3y) \ ,$$

so the straight line can be described as

$$(x, y, z) = \left(\frac{1}{4}, 0, \frac{3}{4}\right) + \left(-\frac{7}{4}, 1, \frac{3}{4}\right) t \ , \quad t \in \mathbb{R} \ .$$

Observing

$$\left(-\frac{7}{4}, 1, \frac{3}{4}\right) = \frac{3}{4}\left(-\frac{7}{3}, \frac{4}{3}, 1\right) \ ,$$

we see that they represent the same set.

Only now the particle starts at $(1/4, 0, 3/4)$ with constant velocity $(-7/4, 1, 3/4)$. Although in these two formulas the motions are different, the geometry is the same. $\square$

Remark It is not hard to see that either x, y or z can be chosen to be the time parameter as long as the 2×2-matrix obtained after moving the chosen variable to the other side is non-singular.

• Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The straight line passing through $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} + t(\mathbf{y} - \mathbf{x}) = (1 - t)\mathbf{x} + t\mathbf{y}, \;\; t \in \mathbb{R}.$$

• The segment connecting $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is given by

$$\mathbf{x} + t(\mathbf{y} - \mathbf{x}) = (1 - t)\mathbf{x} + t\mathbf{y}, \;\; t \in [0, 1].$$

Example 2.2.4

Consider the triangle whose vertices are $A(0,0), \;\; B(2,0), \;\; C(1,1)$. Find

(1) Its medium from A,

(2) Its height from A.

(3) * Its bisector from A.

Sol: (1) The midpoint of the side $\overline{BC}$ is given by

$$((2,0)+(1,1))/2=(3,1)/2.$$

The vector $(3,1)$ points to the direction of the median. As the median passes $A(0,0)$, the median is given by the set

$$\left\{(3,1)t: \quad t\in\left[0,\frac{1}{2}\right]\right\}.$$

(2) Let $\overline{AD}$ be the height from A where D is on the side BC. Let D be

$$(2,0) + t((1,1) - (2,0)) = (2-t, t)$$

where t is to be specified. The direction of $\overrightarrow{BC}$ points in

$$(1,1) - (2,0) = (-1,1).$$

Noting $\overrightarrow{AD} \perp \overrightarrow{BC}$, we have

$$(-1,1) \cdot (2-t, t) = 0 \ ,$$

which is readily solved to get $t = 1$. We conclude that $D = C$ and the height coincides with AC. In other words, this is a perpendicular triangle with the right angle at C.

(3)* Let $\theta = \angle CAB$. The lengths of AB and AC are given by 2 and $\sqrt{2}$ respectively. By the Cosine Law,

$$\cos\theta = \frac{(2,0)\cdot(1,1)}{2\sqrt{2}} = \frac{\sqrt{2}}{2}\ .$$

Using the half angle formula,

$$\cos\frac{\theta}{2} = \left(\frac{1+\cos\theta}{2}\right)^{1/2} = a\ , \quad a = \frac{\sqrt{2+\sqrt{2}}}{2}\ .$$

The direction of the bisector is given by

$$\left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\right) = (a,b), \quad b = \sqrt{1-a^2} = \frac{\sqrt{2-\sqrt{2}}}{2}\ .$$

On the other hand, BC is given by

$$(2,0)+((1,1)-(2,0))s=(2-s,s),\ \ s\in[0,1].$$

The line $(0,0)+t(a,b)$ hits $\overline{BC}$ at

$$t(a,b)=(2-s,s).$$

Solving for t and s, we get

$$t=2/(a+b), \qquad s=2b/(a+b).$$

We conclude that the bisector at A is given by

$$\left\{(a,b)t: \quad t\in\left[0,\frac{2}{a+b}\right]\right\}.$$

$\square$

Section 2.3: Quadratic hypersurfaces

Definition 2.3.1 (Quadratic hypersurfaces)

A **quadric hypersurface** is defined as the **zero set** Σ of a quadratic equation

$$\sum_{j,k=1}^{n} a_{jk} x_j x_k + \sum_{j=1}^{n} b_j x_j + c = 0,$$

where not all a_{jk}'s are zero.

Now we consider the quadratic curves in $\mathbb{R}^2$. Then we write the quadratic equation as

$$ax^2 + 2bxy + cy^2 + dx + ey = f \ , \tag{2.1}$$

and denotes it solution set by γ.

• We can also express the above quadratic equation in the form

$$(x, y) \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + dx + ey = f \ .$$

• To study the geometry of γ we simplify the equation by rotating the coordinates which do not change the shape of γ.

Theorem 2.3.2

For any symmetric 2×2-matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} ,$$

there is a rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

such that

$$R'AR = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} ,$$

where λ and μ are eigenvalues of the symmetric matrix.

continued.

Consequently, letting

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ,$$

then we have

$$ax^2 + 2bxy + cy^2 = \lambda u^2 + \mu v^2 .$$

Proof.

$$R'AR = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta & b\cos 2\theta + \frac{c-a}{2}\sin 2\theta \\ b\cos 2\theta + \frac{c-a}{2}\sin 2\theta & a\sin^2\theta + c\cos^2\theta - b\sin 2\theta \end{bmatrix}.$$

We can always choose some $\theta_0 \in [0, \pi)$ such that

$$b\cos 2\theta_0 + \frac{c-a}{2}\sin 2\theta_0 = 0,$$

so that $R'AR = D$ where D is a diagonal matrix

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}.$$

From

$$R'AR\,\mathbf{e}_1 = \lambda \mathbf{e}_1, \quad R'AR\,\mathbf{e}_2 = \mu \mathbf{e}_2 \ ,$$

we see that

$$A\mathbf{x} = \lambda \mathbf{x}, \quad A\mathbf{y} = \mu \mathbf{y} \ ,$$

where

$$\mathbf{x} = R\mathbf{e}_1, \quad \mathbf{y} = R\mathbf{e}_2 \ .$$

It shows that λ and μ are in fact the eigenvalues of A. $\square$

By introducing the new variables u, v as described in this theorem, our quadratic equation turns into another quadratic equation

$$\lambda u^2 + \mu v^2 + du + ev = f \ , \tag{2.2}$$

for different d and e. *Since the shape of γ remains unchanged under rotations, it suffices to study quadratic equation with the form (2.2).*

Theorem 2.3.3

Consider equation (2.2)*. It holds that*

(1). *If* λ *and* μ *are of the same sign, there is a Euclidean motion under which the equation assumes the form*

$$|\lambda|x^2 + |\mu|y^2 = c \ , \quad c \in \mathbb{R}.$$

Consequently, γ *is either an ellipse* ($c > 0$)*, a point* ($c = 0$) *or an empty set* ($c < 0$)*.*

(2). *If* λ *and* μ *are of different sign, there is a Euclidean motion under which the equation assumes the form*

$$|\lambda|x^2 - |\mu|y^2 = c \ , \quad c \in \mathbb{R} \ .$$

Consequently, γ *is either a hyperbola* ($c \neq 0$)*, or the union of two intersecting straight lines* ($c = 0$)*.*

continued.

(3). *If one of λ, μ is zero, there is a Euclidean motion under which the equation assumes the form*

$$|\lambda| x^2 + ay = c \ , \quad a, c \in \mathbb{R} \ .$$

Consequently, γ is either a parabola ($a \neq 0$), two parallel straight lines ($a = 0, c > 0$), the empty set ($a = 0, c < 0$) or a straight line ($a = c = 0$).

Remark We note that

$$\lambda\mu = ac - b^2 = \det A \ .$$

- λ and μ are of the same sign iff $ac - b^2 > 0$.
- They are of opposite sign iff $ac - b^2 < 0$.
- One of λ, μ vanishes iff $ac - b^2 = 0$.

Proof. **(1)**. If λ and μ are of the same sign. By multiplying -1 to this equation if necessary, we may assume they are positive. By completing square, it becomes

$$\lambda\left(x+\frac{d}{2\lambda}\right)^2+\mu\left(y+\frac{e}{2\mu}\right)^2=g\ ,\quad g=f+\frac{d^2}{2\lambda}+\frac{e^2}{4\mu}\ .$$

Therefore, after a translation

$$u=x+\frac{d}{2\lambda}\ ,\quad v=y+\frac{e}{2\mu}\ ,$$

we achieve at $\lambda u^2+\mu v^2=g$. When $g>0$, this is the standard form for an ellipse. When $g=0$, it degenerates into a single point. When $g<0$, this equation has no solution, so γ is an empty set.

(2). If λ and μ are of opposite sign. By multiplying -1 to this equation if necessary, we may assume λ is positive and μ is negative. Following the discussion in the first case, we arrive at $|\lambda|u^2 - |\mu|v^2 = g$. When $g \neq 0$, γ is a hyperbola. When $g = 0$, it is the union of the straight lines defined by

$$\sqrt{|\lambda|}u + \sqrt{|\mu|}v = 0\ , \quad \sqrt{|\lambda|}u - \sqrt{|\mu|}v = 0\ .$$

(3). If one of λ, μ is zero, by switching the x- and y-axis if necessary, we may assume $\lambda > 0$ and $\mu = 0$ so that the equation becomes

$$\lambda x^2 + dx + ey + f = 0 \ ,$$

for some new f. A partial completing square yields

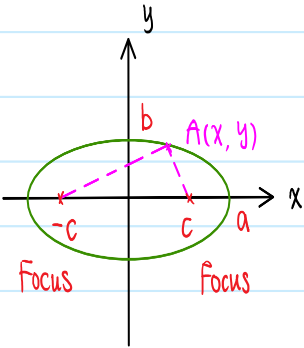
$$\lambda \left(x + \frac{d}{2\lambda} \right)^2 + ey + f - \frac{d^2}{4\lambda} = 0 \ .$$

Hence, after a horizontal translation, the equation becomes $\lambda x^2 + ey = f$. It is a parabola as long as $e \neq 0$. When $e = 0$ and $f > 0$, γ consists of two vertical lines $x = \pm\sqrt{f}$. It is empty when $e = 0$ and $f < 0$. It is the y-axis when $e = f = 0$.

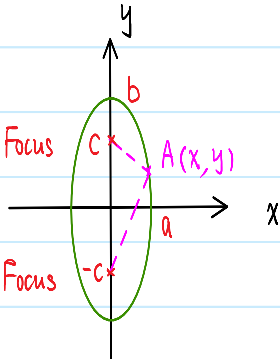
Therefore we have completely classified the curves defined by quadratic equations of two variables. $\square$

Ellipse

Standard form: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $(a>0,\ b>0)$



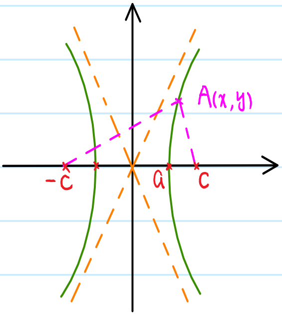
(i). $a>b$

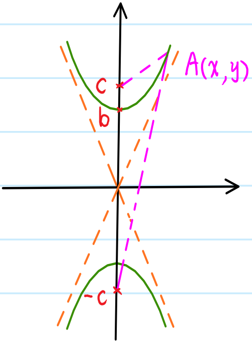


(ii). $a<b$

Hyperbola

Standard form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$, $(a>0,\ b>0)$





$\boxed{(i).\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$

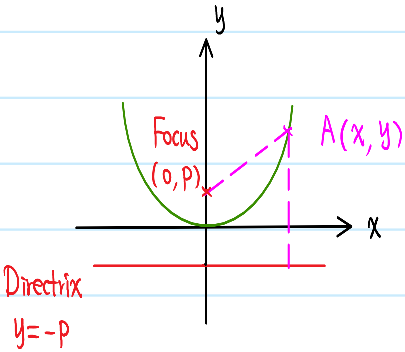
"left-right branches"

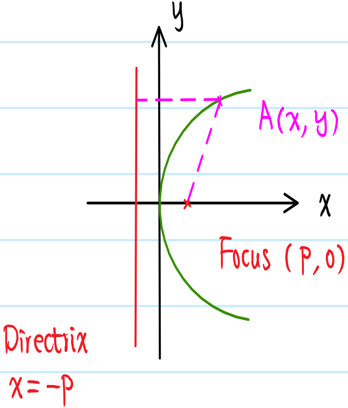
$\boxed{(ii).\ \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1}$

"top-bottom branches"

Parabola

Standard form: $x^2 = 4py$ or $y^2 = 4px$, $(p \neq 0)$





$\boxed{\text{(i). } x^2 = 4py \ (p>0)}$ $\boxed{\text{(ii). } y^2 = 4px \ (p>0)}$

Steps of transforming a quadratic equation into the "standard form" are:

Step 1. Solve the characteristic equation

$$\det \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix} = 0 \ ,$$

to determine the two eigenvalues λ_1 and λ_2 including multiplicity.

Step 2. Solve the linear systems

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

to obtain two orthogonal unit eigenvectors (u_1, v_1) and (u_2, v_2).

Step 3. The change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

will convert the equation in x, y into one in u, v without mixed term uv.

Step 4. Completing square to bring it into the standard form.

Example 2.3.4

Transform the equation

$$2xy - x + 3y = 1$$

to the standard form and determine its solution set.

Sol: We have $a = c = 0$ and $b = 1$ so $ac - b^2 = -1 < 0$ and there are two eigenvalues with opposite sign. In fact, the characteristic polynomial is $\lambda^2 - 1 = 0$ so the two eigenvalues are 1 and -1 with corresponding eigenvector $(1, 1)$ and $(-1, 1)$ so that

$$R = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

This is the rotation by $45°$. Note that the factor $\sqrt{2}/2$ is for normalization.

Letting

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} ,$$

that is,

$$x = \frac{\sqrt{2}}{2}(u - v), \quad y = \frac{\sqrt{2}}{2}(u + v) ,$$

$$\begin{aligned} 2xy - x + 3y - 1 &= u^2 - v^2 - \frac{\sqrt{2}}{2}(u - v) + \frac{3\sqrt{2}}{2}(u + v) - 1 \\ &= u^2 - v^2 + \sqrt{2}u + 2\sqrt{2}v - 1 \\ &= (u + \frac{\sqrt{2}}{2})^2 - (v - \sqrt{2})^2 + \frac{1}{2} , \end{aligned}$$

where in the last step we complete square. Letting $x' = u + \sqrt{2}$ and $y' = v + \sqrt{2}$, the equation finally achieves the standard form $x^2 - y^2 = -\frac{1}{2}$ which is a hyperbola after replacing (x', y') by (x, y).

The situation for all other dimensions is similar. Indeed, we need the following basic result in linear algebra.

Theorem 2.3.5

For any $n \times n$ symmetry matrix A, there is an orthogonal matrix R such that

$$R'AR = D$$

where D is a diagonal matric whose diagonal elements are precisely the eigenvalues of A (counting multiplicity)

For simplicity, we only consider quadratic equation in $\mathbb{R}^3$. Using this result, a suitable Euclidean motion would bring the general quadratic equation into

$$\lambda x^2 + \mu y^2 + \nu z^2 + dx + ey + fz = g \ . \tag{2.3}$$

and further classification according to the sign of the eigenvalues can be carried out as in the two variable case.

Theorem 2.3.6

(1). *If λ, μ, ν are of the same sign, there is a Euclidean motion to transform* (2.3) *to*

$$|\lambda|x^2 + |\mu|y^2 + |\nu|z^2 = g\ , \quad g \in \mathbb{R}\ . \quad \textbf{(ellipsoid)}$$

(2). *If two of λ, μ, ν are of the same sign and one in opposite sign, there is a Euclidean motion to transform* (2.3) *to*

$$|\lambda|x^2 + |\mu|y^2 - |\nu|z^2 = g\ , \quad g \in \mathbb{R}\ .$$

*(***hyperboloid of one sheet** $g > 0$, **elliptical cone** $g = 0$, **hyperboloid of two sheets** $g < 0$*).*

(3). If two of λ, μ, ν are of the same sign and the third one is zero, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2+|\mu|y^2+fz = g\ , \quad f, g \in \mathbb{R}(\textbf{elliptical paraboloid})$$

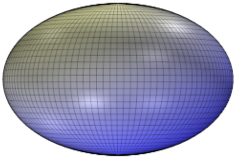
(4). If one of λ, μ, ν is zero and the other two are in opposite sign, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2-|\mu|y^2+fz = g\ , \quad f, g \in \mathbb{R}\ (\textbf{hyperbolic paraboloid})$$

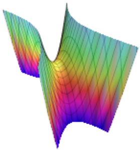
(5). If exactly two of λ, μ, ν are zero, there is a Euclidean motion to transform (2.3) to

$$|\lambda|x^2 + ey = g\ , \quad e, g \in \mathbb{R}\ (\textbf{paraboloid})$$

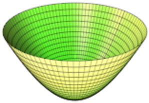
Quadratic Surfaces



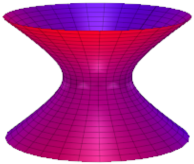
Ellipsoid



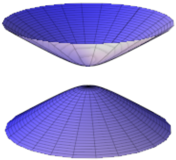
Hyperbolic paraboloid



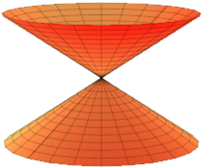
Elliptic paraboloid



Hyperboloid of one sheet



Hyperboloid of two sheets



Cone

Quadratic Surfaces

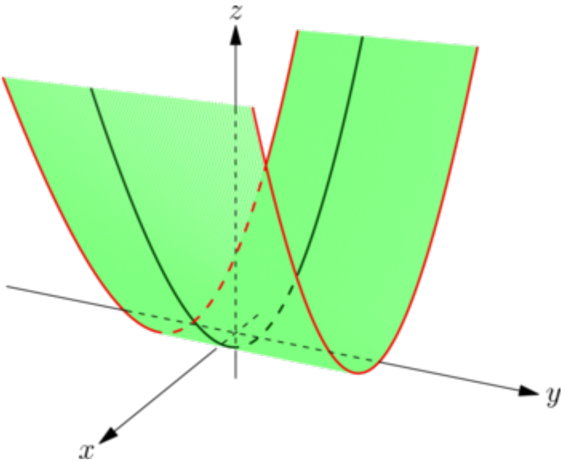


Figure: Parabloid

The End!