1. Suppose c is a cluster point of A.
By definition, for each $n \in \mathbb{N}$, $(A \cap (-\frac{1}{n}+c, \frac{1}{n}+c)) \setminus \{c\} \neq \phi$.
Define x_n to be a point in $(A \cap (-\frac{1}{n}+c, \frac{1}{n}+c)) \setminus \{c\}$, then given $\varepsilon > 0$, if we choose $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$, then $\forall n \geq N$,

$$|x_n - c| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} x_n = c$.

Now, if there exists $\{a_n\} \subseteq A \setminus \{c\}$ s.t. $\lim_{n\to\infty} a_n = c$, then for each $n \in \mathbb{N}$, by definition, $\exists N_n$ s.t. $\forall m \geq N_n$,

$$|x_m - c| < \frac{1}{n} .$$

Now, given $\varepsilon > 0$, if we choose n s.t. $\frac{1}{n} < \varepsilon$, then

$$|x_{N_n} - c| < \frac{1}{n} < \varepsilon , \text{ that is, } x_{N_n} \in (A \cap (-\tfrac{1}{n}+c, \tfrac{1}{n}+c)) \setminus \{c\}.$$

2. Let $c \in \mathbb{R} \setminus \mathbb{N}$, then $\varepsilon_c := \min\{c - [c], c - [c] + 1\} > 0$, where $[c]$ is the integral part of c, $[c] = \min\{n \in \mathbb{Z} : n \leq c\}$.
Finally, $\mathbb{N} \cap (-\varepsilon_c + c, \varepsilon_c + c) = \phi$

As for $c \in \mathbb{N}$, $\mathbb{N} \cap (-1+c, 1+c) = \{c\}$.

3. $\lim_{x \to c} f(x) = L \iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in (c-\delta, c+\delta) \setminus \{c\}$,

$$|f(x) - L| < \varepsilon.$$

$\iff \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in (-\delta, \delta) \setminus \{0\}$,

$$|f(x+c) - L| < \varepsilon.$$

$\iff \lim_{x \to 0} f(x) = L$

4a. For $x \neq 2$, $\frac{x^2-4}{x-2} = x+2$.
Hence, for $\varepsilon > 0$, if we chose $\delta = \varepsilon$, then $\forall x \in (2-\delta, 2+\delta)$,

$$|(x+2) - 4| = |x-2| < \delta = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

b. For $x \neq 0$, $\frac{x^2}{|x|} = |x|$.

Hence for $\varepsilon > 0$, if we chose $\delta = \varepsilon$, then $\forall x \in (-\delta, \delta)$,

$$\left| \frac{x^2}{|x|} - 0 \right| = |x| < \delta = \varepsilon.$$

5. For $x \neq 0$, $\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$.

Now, let $c \in \mathbb{R} \setminus \{1\}$, then $\varepsilon_c := |c-1| > 0$.
So, $\forall \delta > 0$, $\forall x \in (0, 0+\delta)$, $\left|\frac{|x|}{x} - c\right| = |c-1| > \frac{\varepsilon_c}{2}$.
This shows the ε-δ condition does not hold for $\varepsilon = \frac{\varepsilon_c}{2}$.

For $c = 1$, $\forall \delta > 0$, $\forall x \in (-\delta, 0)$, $\left|\frac{|x|}{x} - c\right| = 2 > 1$
This shows that the ε-δ condition does not hold for $\varepsilon = 1$.

6. Set $f(x) = g(x) = \begin{cases} \frac{|x-c|}{x-c} & \text{if } x \neq c \\ 0 & \text{if } x = c \end{cases}$, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$
do not exist but $\lim_{x \to c} (fg)(x) = 1$.

7. Set $f(x) = x - c$, $g(x) = \begin{cases} 0 & \text{if } x = c \\ \frac{1}{x-c} & \text{if } x \neq c \end{cases}$, then
$\lim_{x \to c} f(x) = 0$, $\lim_{x \to c} (fg)(x) = 1$ but $\lim_{x \to c} g(x)$ does not exist.

8. We make use of the uniqueness of limits.
Set $x_n = \frac{1}{n\pi}$ and $y_n = (2n\pi + \frac{\pi}{2})^{-1}$, then

$\lim_{n \to \infty} \sin \frac{1}{x_n} = 0$ and $\lim_{n \to \infty} \sin \frac{1}{y_n} = 1$.

9. Let $\varepsilon>0$, set $\delta=\varepsilon$, then $\forall\ \mathbb{Q}\cap(-\delta+c,\delta+c)\setminus\{c\}$,

$$|f(x)-c|=|x-c|<\delta=\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, $\lim_{x\to c} f(x)=c$.

10. Let $c\in\mathbb{R}\setminus\{0\}$. Fix any $L\in\mathbb{R}$, fix $\varepsilon_c=|L|$, then $\forall\delta>0$,

$$|f(x)-L|=\begin{cases}|x-L| & ; x\in\mathbb{Q}\cap(-\delta+c,\delta+c)\\ |x+L| & , x\in(\mathbb{R}\setminus\mathbb{Q})\cap(-\delta+c,\delta+c)\end{cases}$$

Now, if $|f(x)-L|<\varepsilon_c$, then $x\in(-|L|,0)\cap(0,|L|)=\phi$.
This is a contradiction and so $\lim_{x\to c} f(x)$ does not exist.

For $c=0$, let $\varepsilon>0$ and set $\delta=\frac{\varepsilon}{2}$, then $\forall x\in(-\delta,\delta)$,

$$|f(x)-0|=|f(x)|=|x|<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, $\lim_{x\to 0} f(x)=0$.

11. Let $\varepsilon>0$, choose δ from the definition of $\lim_{x\to 0} f(x)=L$ s.t. $\forall x\in(-\delta,\delta)\setminus\{0\}$,

$$|f(x)-L|<\varepsilon.$$

Set $\delta_a=\frac{\delta}{a}$, then $\forall x\in(-\delta_a,\delta_a)\setminus\{0\}$, $ax\in(-\delta,\delta)$ and so

$$|g(x)-L|=|f(ax)-L|<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, $\lim_{x\to 0} g(x)=0$.

12. Let $(A \setminus \{c\}) \cap (-\delta_f + c, \delta_f + c)$ be a neighborhood s.t. f is bounded, say,

$$|f(x)| \le M \quad \forall x \in ((-\delta_f + c, \delta_f + c) \cap A) \setminus \{c\}).$$

Let $\varepsilon > 0$ and $\delta_{g,\varepsilon} > 0$ s.t. $\forall x \in (A \cap (c - \delta_{g,\varepsilon}, c + \delta_{g,\varepsilon})) \setminus \{c\}$,

$$|g(x)| < \frac{\varepsilon}{M}.$$

Now, set $\delta := \min\{\delta_{g,\varepsilon}, \delta_f\}$, then $\forall x \in (A \cap (c-\delta, c+\delta)) \setminus \{c\}$,

$$|(fg)(x)| \le M|g(x)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{x \to c} (fg)(x) = 0$.

13. By induction, we have $f(x) = \frac{1}{n} f(nx)$.
Let $\delta_1 > 0$ s.t. $|f(x) - L| < 1 \quad \forall x \in (-\delta_1, \delta_1) \setminus \{0\}$; i.e.

$$|f(x)| < |L| + 1 \quad \forall x \in (-\delta_1, \delta_1) \setminus \{0\}.$$

Now, let $\varepsilon > 0$ and choose $\delta_\varepsilon > 0$ s.t. $|f(x) - L| < \frac{\varepsilon}{2} \quad \forall x \in (-\delta_\varepsilon, \delta_\varepsilon) \setminus \{0\}$.
Choose $N \in \mathbb{N}$ s.t. $\frac{1}{N}(|L| + 1) < \frac{\varepsilon}{2}$.

Finally, $\forall x \in ((-\delta, \delta) \cap (-\frac{\delta_1}{N}, \frac{\delta_1}{N})) \setminus \{0\}$,

$$\begin{aligned} |L| &\le |f(x) - L| + |f(x)| \\ &< \frac{\varepsilon}{2} + \frac{1}{N}|f(Nx)| \\ &< \frac{\varepsilon}{2} + \frac{1}{N}(|L| + 1) \\ &< \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $L = 0$.

13. In general, $\lim_{x\to c} f(x) = \lim_{x\to 0} f(x+c)$

$= \lim_{x\to 0} f(x) + f(c)$

$= f(c).$

14. Let $\varepsilon>0$ and set $\delta_1 = \sqrt{\frac{\varepsilon}{M}}$ and $\delta_2 = \frac{\varepsilon}{M}$, then

$|f(x)| \leq M|x|^2 < \varepsilon \quad \forall x \in (-\delta_1, \delta_1)$

and

$\left|\frac{f(x)}{x}\right| \leq M|x| < \varepsilon \quad \forall x \in (-\delta_2, \delta_2)\setminus\{0\}.$

Since $\varepsilon>0$ is arbitrary, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f(x)}{x} = 0.$

15. Claim: Let $f, g, h: A \to \mathbb{R}$ be functions where $A \subseteq \mathbb{R}$ has the property that $(a,\infty)\cap A \neq \phi$ $\forall a \in \mathbb{R}$ (or $(-\infty, a)\cap A \neq \phi$ $\forall a \in \mathbb{R}$). Suppose $g(x) \leq f(x) \leq h(x)$ $(a,\infty)\cap A$ for some $a\in\mathbb{R}$ (or $(-\infty, a)\cap A$)

and $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x)$ (or $\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} h(x)$), then

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x)$ (or $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} h(x)$).

Proof: Define $L = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x)$. Let $\varepsilon>0$, then $\exists\, a_g, a_h$ s.t.

$|g(x) - L| < \varepsilon \quad \forall x \in (a_g, \infty)$

and

$|h(x) - L| < \varepsilon \quad \forall x \in (a_h, \infty).$

Then, $\forall x \in (a_f, \infty)$, $a_f := \max\{a_g, a_h, a\}$,

$L - \varepsilon < g(x) \leq f(x) \leq h(x) < L + \varepsilon$, or

$|f(x) - L| < \varepsilon.$

15. Since $\varepsilon > 0$ is arbitrary, $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x)$.

16. Definition: Let $L \in \mathbb{R}$, $\lim_{x \to 0^+} f(x) = L$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in (0, a) \cap (0, \delta),$$
$$|f(x) - L| < \varepsilon.$$

17. Suppose $\lim_{x \to \infty} f(x) = +\infty$.
Let $\varepsilon > 0$ and $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.
Let $\delta > 0$ s.t. $\forall x \in (c, c+\delta)$, $|f(x)| > N$, then
$$\forall x \in (c, c+\delta), \quad \left|\frac{1}{f(x)}\right| < \frac{1}{N} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lim_{x \to c} \frac{1}{f(x)} = 0$.

Suppose $\lim_{x \to c} \frac{1}{f(x)} = 0$.

Let $M > 0$. Choose $N \in \mathbb{N}$ s.t. $N > M$.
Let $\delta > 0$ s.t. $\forall x \in (c, c+\delta)$, $\left|\frac{1}{f(x)}\right| < \frac{1}{N}$, then

$$\forall x \in (c, c+\delta), \quad |f(x)| > N > M.$$

Since $M > 0$ is arbitrary, $\lim_{x \to c} f(x) = +\infty$.