1. If $a, b \in \mathbb{R}$, prove the following.
   (a) If $a + b = 0$, then $b = -a$,
   (b) $-(-a) = a$,
   (c) $(-1)a = -a$,
   (d) $(-1)(-1) = 1$,
   (e) $-(a + b) = (-a) + (-b)$,
   (f) $(-a) \cdot (-b) = a \cdot b$,
   (g) $1/(-a) = -(1/a)$,
   (h) $-(a/b) = (-a)/b$ if $b \neq 0$.

2. If $a \in \mathbb{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.

3. If $a \neq 0$ and $b \neq 0$, show that $1/(ab) = (1/a)(1/b)$.

4. If $a, b$ and $c$ are real numbers, prove that
   (a) If $a < b$ and $c \leq d$, prove that $a + c < b + d$.
   (b) If $0 < a < b$ and $0 \leq c \leq d$, prove that $0 \leq ac \leq bd$.

5. (a) Show that if $a > 0$, then $1/a > 0$ and $1/(1/a) = a$.
   (b) Show that if $a < b$, then $a < 1/2(a + b) < b$.

6. (a) Prove there is no $n \in \mathbb{N}$ such that $0 < n < 1$. (Use the Well-Ordering Property of $\mathbb{N}$.)
   (b) Prove that no natural number can be both even and odd.

7. Let $S_1 := \{x \in \mathbb{R} : x \geq 0\}$. Show in detail that the set $S_1$ has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.

8. Let $S_2 := \{x \in \mathbb{R} : x > 0\}$. Does $S_2$ have lower bounds? Does $S_2$ have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.

9. Let $S_3 := \{1/n : n \in \mathbb{N}\}$. Show that $\sup S_3 = 1$ and $\inf S_3 = 0$.

10. Let $S_4 := \{1 - (-1)^n : n \in \mathbb{N}\}$. Find $\sup S_4$ and $\inf S_4$.

11. Let $S_5 := \{r \in \mathbb{Q} : r > 0\}$. Find $\inf S_5$.

12. Let $S$ be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$.

13. Show that if $A$ and $B$ are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$. (Remark: Can it be generalized to a finite / an infinite collection of bounded subsets of $\mathbb{R}$?)
14. Let \( S \subseteq \mathbb{R} \) and suppose that \( s^* := \sup S \) belongs to \( S \). If \( u \notin S \), show that \( \sup(S \cup \{u\}) = \sup\{s^*, u\} \). (Remark: Special case of the previous question with suitable modification.)

15. Show that a nonempty finite set \( \sup S \subseteq \mathbb{R} \) contains its supremum. (Hint: Use Mathematical Induction and the preceding exercise.)

16. If \( S := \{1/n - 1/m : n, m \in \mathbb{N}\} \), find \( \inf S \) and \( \sup S \).

17. Let \( A \) and \( B \) be bounded nonempty subsets of \( \mathbb{R} \), and let \( A + B := \{a + b : a \in A, b \in B\} \). Prove that \( \sup A + B = \sup A + \sup B \) and \( \inf A + B = \inf A + \inf B \).

18. If \( y > 0 \), show that there exists \( n \in \mathbb{N} \) such that \( 1/2^n < y \).

19. If \( u > 0 \) is any real number and \( x < y \), show that there exists a rational number \( r \) such that \( x < ru < y \). (Hence the set \( \{ry : r \in \mathbb{Q}\} \) is dense in \( \mathbb{R} \).)

20. Suppose \( a \) and \( b \) are positive real numbers. Show that there exists \( n \in \mathbb{N} \) such that \( a/n < b \).

21. Let \( I_n := [0, 1/n] \) for \( n \in \mathbb{N} \). Prove that \( \bigcap_{n=1}^{\infty} I_n = \{0\} \).

22. Let \( J_n := (0, 1/n) \) for \( n \in \mathbb{N} \). Prove that \( \bigcap_{n=1}^{\infty} J_n = \phi \).

23. Let \( K_n := (n, \infty) \) for \( n \in \mathbb{N} \). Prove that \( \bigcap_{n=1}^{\infty} K_n = \{0\} \).

24. Suppose \( I_n, n \in \mathbb{N}, \) is a nested sequence of closed bounded intervals. Prove that \( I_{2n}, n \in \mathbb{N}, \) is also a nested sequence of closed bounded intervals. Furthermore, if \( \xi \in \mathbb{R} \) such that \( \xi \in I_{2n} \) for any \( n \in \mathbb{N} \), show that \( \xi \in I_n \) for any \( n \in \mathbb{N} \).