1. Show that if \( f : A \to B \) and \( E, F \) are subsets of \( A \), then \( f(E \cup F) = f(E) \cup f(F) \) and \( f(E \cap F) \subseteq f(E) \cap f(F) \).

2. Show that if \( f : A \to B \) and \( G, H \) are subsets of \( B \), then \( f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H) \) and \( f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H) \).

3. (a) Show that if \( f : A \to B \) is injective and \( E \subseteq A \), then \( f^{-1}(f(E)) = E \). Give an example to show that equality need not hold if \( f \) is not injective.

   (b) Show that if \( f : A \to B \) is surjective and \( H \subseteq B \), then \( f(f^{-1}(H)) = H \). Give an example to show that equality need not hold if \( f \) is not surjective.

4. (a) Suppose that \( f \) is an injection. Show that \( f^{-1} \circ f(x) = x \) for all \( x \in D(f) \) and that \( f \circ f^{-1}(y) = y \) for all \( y \in R(f) \).

   (b) If \( f \) is a bijection of \( A \) onto \( B \), show that \( f^{-1} \) is a bijection of \( B \) onto \( A \).

5. Prove that if \( f : A \to B \) is bijective and \( g : B \to C \) is bijective, then the composite \( g \circ f \) is a bijective map of \( A \) onto \( C \).

6. Let \( f : A \to B \) and \( g : B \to C \) be functions.

   (a) Show that if \( g \circ f \) is injective, then \( f \) is injective.

   (b) Show that if \( g \circ f \) is surjective, then \( g \) is surjective.

7. Let \( f, g \) be functions such that \((g \circ f)(x) = x\) for all \( x \in D(f) \) and \((f \circ g)(y) = y\) for all \( y \in D(g) \).

   Prove that \( g = f^{-1} \).

8. Prove that \( \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \) for all \( n \in \mathbb{N} \).

9. Prove the second version of Principle of Mathematical Induction:

   Let \( n_0 \in \mathbb{N} \) and let \( P(n) \) be a statement for each natural number \( n \geq n_0 \). Suppose that
   
   - The statement \( P(n_0) \) is true.
   - For all \( k \geq n_0 \), the truth of \( P(k) \) implies the truth of \( P(k+1) \).

   Then \( P(n) \) is true for all \( n \geq n_0 \).

10. Prove the strong version of Principle of Mathematical Induction:

    Let \( S \) be a subset of \( \mathbb{N} \) such that

    - \( 1 \in S \).
    - For every \( k \in \mathbb{N} \), if \( \{1, 2, \cdots, k\} \subseteq S \), then \( k + 1 \in S \).

    Then \( S = \mathbb{N} \).
11. Prove a variation of Principle of Mathematical Induction:

Let $S$ be a subset of $\mathbb{N}$ such that

- $2^k \in S$ for all $k \in \mathbb{N}$.
- If $k \in S$ and $k \geq 2$, then $k - 1 \in S$.

Then $S = \mathbb{N}$.

12. Show that the set $S = \{n \in \mathbb{N} : n \geq 2015\}$ is countably infinite.

13. Prove that if $S$ and $T$ are countably infinite, then $S \cup T$ is countably infinite.

14. Prove that if $S$ is countably infinite and $T$ is finite, then $S/T$ is countably infinite.

15. Suppose that $f : S \to T$ is an injective function, where $S$ is an infinite set. Prove that $T$ is an infinite set.

16. Suppose that $f : S \to T$ is an surjective function, where $T$ is a countably infinite set. Is $S$ an infinite set? Why?

17. Let $S$ be a set. $\mathcal{P}(S)$ is defined to be the collection of all subsets of $S$.

   (a) Write down $\mathcal{P}(S)$ explicitly if $S = \{1, 2, 3\}$. How many elements does $\mathcal{P}(S)$ contain?

   (b) Use mathematical induction to prove that if the set $S$ has $n$ elements, then $\mathcal{P}(S)$ has $2^n$ elements.