§ 2 Real Numbers

2.1 Construction of \( \mathbb{R} \)

Construction of \( \mathbb{R} \):

1) Completion of \( \mathbb{Q} \).

Regard \( \mathbb{Q} \) as a metric space with \( d(x,y) = |x-y| \), define \( \mathbb{R} \) to be the completion of \( \mathbb{Q} \).

(Refer to DJ)

2) Axiomatic Approach:

(Roughly speaking: impose/assume properties we need)

(i) Field axioms / Algebraic properties

Field axioms / Algebraic properties:

\((\mathbb{R}, +, \cdot, 0, 1)\) equips with additions + and multiplication \( \cdot \) that satisfy:

(A1) (Commutative law) \( a + b = b + a \) for all \( a, b \in \mathbb{R} \).

(A2) (Associative law) \( (a + b) + c = a + (b + c) \) for all \( a, b, c \in \mathbb{R} \).

(A3) (Existence of 0) there exists \( 0 \in \mathbb{R} \) such that \( a + 0 = 0 + a \) for all \( a \in \mathbb{R} \).

(A4) (Existence of additive inverse) for all \( a \in \mathbb{R} \), there exists \( b \in \mathbb{R} \) such that \( a + (-a) = 0 \).

(I1) (Commutative law) \( a \cdot b = b \cdot a \) for all \( a, b \in \mathbb{R} \).

(I2) (Associative law) \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \) for all \( a, b, c \in \mathbb{R} \).

(I3) (Existence of 1) there exists \( 1 \in \mathbb{R} \) such that \( a \cdot 1 = 1 \cdot a \) for all \( a \in \mathbb{R} \).

(I4) (Existence of multiplicative inverse) for all \( a \in \mathbb{R} \setminus \{0\} \), there exists \( b \in \mathbb{R} \) such that \( a \cdot b = b \cdot a = 1 \).

(D) (Distributive law) \( a \cdot (b + c) = a \cdot b + a \cdot c \) and \( (b + c) \cdot a = b \cdot a + c \cdot a \) for all \( a, b, c \in \mathbb{R} \).

Idea: Forget everything you learned before, start from those axioms (things accepted to be true) and prove things you suspect to be true.
Theorem:

a) (Uniqueness of additive inverse)

If \( b, c \in \mathbb{R} \) are additive inverses of \( a \in \mathbb{R} \), then \( b = c \).

(Therefore we denote it by \(-a\))

b) (Uniqueness of multiplicative inverse)

If \( b, c \in \mathbb{R} \setminus \{0\} \) are multiplicative inverses of \( a \in \mathbb{R} \setminus \{0\} \), then \( b = c \).

(Therefore we denote it by \( a^* \) or \( \frac{1}{a} \))

Proof:

(a) By assumption, \( a + b = b + a = 0 \) and \( a + c = c + a = 0 \).

Now, \( c = a + c \) \hspace{1cm} (A3)

\[ \begin{align*}
& = (b + a) + c \\
& = b + (a + c) \hspace{1cm} (A2) \\
& = b + 0 \\
& = b \hspace{1cm} (A3)
\end{align*} \]

\( \therefore b = c \)

(b) Exercise!

Theorem:

a) (Uniqueness of \( 0 \))

If \( z \in \mathbb{R} \) such that \( z + a = 0 \) for all \( a \in \mathbb{R} \), then \( z = 0 \).

b) (Uniqueness of \( 0 \))

If \( u \in \mathbb{R} \) such that \( u \cdot a = a \) for all \( a \in \mathbb{R} \), then \( u = 1 \).

c) If \( a \in \mathbb{R} \), \( a \cdot 0 = 0 \)

Proof of (c):

\( a + a \cdot 0 = a \cdot 1 + a \cdot 0 \)

\[ \begin{align*}
& = a \cdot (1 + 0) \\
& = a \cdot 1 \\
& = a
\end{align*} \]

\[ (-a) + a + a \cdot 0 = -a + a \]

\( \therefore a \cdot 0 = 0 \)

(Think: What do we use in each step?)
Theorem:
If \( a \cdot b = 0 \), then either \( a = 0 \) or \( b = 0 \).

Proof:
It suffices to show if \( a \neq 0 \), then \( b = 0 \).
By assumption, \( a \cdot b = 0 \)
\[
\frac{1}{a} \cdot (a \cdot b) = \frac{1}{a} \cdot 0
\]
\[
(\frac{1}{a} \cdot a) \cdot b = 0 \quad \text{(M2 and previous theorem)}
\]
\[
1 \cdot b = 0 \quad \text{(M4)}
\]
\[
b = 0 \quad \text{(M3)}
\]

Exercise: Show that \((-1) \cdot (-1) = 1\)

Definition:
- (Subtraction)
  If \( a, b \in \mathbb{R} \), \( a - b \) is defined as \( a + (-b) \).
- (Division)
  If \( a \in \mathbb{R} \) and \( b \in \mathbb{R} \setminus \{0\} \), \( \frac{a}{b} \) is defined as \( a \cdot (\frac{1}{b}) \).

(ii) Order properties of \( \mathbb{R} \)

Order properties of \( \mathbb{R} \):
There is a subset \( \mathbb{P} \subseteq \mathbb{R} \), called the set of positive real numbers, that satisfies
- \( a, b \in \mathbb{P} \Rightarrow a + b \in \mathbb{P} \)
- \( a, b \in \mathbb{P} \Rightarrow a \cdot b \in \mathbb{P} \)
- (Trichotomy property) If \( a \in \mathbb{R} \), then exactly one of the following holds:
  \( \mathbb{P} \), \( a = 0 \), \( -a \notin \mathbb{P} \).

Theorem:
\( 1 \in \mathbb{P} \).

Proof:
By the last property and the fact that \( 0 \neq 1 \), we have either \( 1 \in \mathbb{P} \) or \( -1 \in \mathbb{P} \).
However, if \( -1 \in \mathbb{P} \), then \( 1 = (\cdot 1) \cdot (-1) \in \mathbb{P} \) (Contradiction)

(Why? Exercise !)
Definition:
If \( a, b \in \mathbb{R} \),
- if \( a - b \in \mathbb{P} \), then we write \( a > b \) (a is greater than b) or \( b < a \) (b is less than a).
- if \( a - b \in \mathbb{P} \cup \{0\} \), then we write \( a \geq b \) or \( b \leq a \).

Trichotomy property can be reformulated as:
If \( a, b \in \mathbb{R} \), then exactly one of the following holds:
\( a - b \in \mathbb{P} \), \( a - b = 0 \), \( -(a - b) = b - a \in \mathbb{P} \)

(Why? Exercise!)

i.e. \( a \geq b, a = b, a \leq b \)

Theorem:
Let \( a, b, c \in \mathbb{R} \), then
- \( a > b \) and \( b > c \) \( \Rightarrow \) \( a > c \)
- \( a > b \) \( \Rightarrow \) \( a + c > b + c \)
- \( a > b \) and \( c > 0 \) \( \Rightarrow \) \( ca > cb \)
- \( a > b \) and \( c < 0 \) \( \Rightarrow \) \( ca < cb \)

proof: Exercise!

Theorem:
a) If \( a \in \mathbb{R} \setminus \{0\} \), then \( a^2 = a \cdot a > 0 \)
b) If \( n \in \mathbb{N} \), then \( n > 0 \)

proof of (b):
Mathematical Induction.

Exercise:
1) If \( a \in \mathbb{P} \), show that \( \frac{1}{a} \in \mathbb{P} \).
2) If \( a > 1 \), show that \( \frac{1}{a} < 1 \).
3) If \( a, b \in \mathbb{P} \) and \( a > b^2 \), show that \( a > b \).
Theorem:

If $a \in \mathbb{R}$ such that $0 \leq a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

Proof:

Suppose $a > 0$.

Idea:

\[ 0 < a \]

\[ \text{take } \varepsilon = \frac{a}{2} > 0 \text{, then we get contradiction.} \]

Main issues: Why $a > \frac{1}{2}a$?

\[ 1 - 0 = 1 \in \mathbb{R} \]
\[ 1 > 0 \]
\[ 2 = 1 + 1 > 0 + 1 = 1 \]
\[ 2 > 1 \]
\[ 1 > \frac{1}{2} \quad \text{(By previous exercise)} \]
\[ a > \frac{1}{2} \cdot a \]

Theorem:

If $ab > 0$, then either $a, b > 0$ or $a, b < 0$.

(iii) Completeness properties of $\mathbb{R}$

Definition:

Let $S$ be a nonempty subset of $\mathbb{R}$.

- $S$ is said to be bounded above (below) if there exists $u \in \mathbb{R}$ ($\ell \in \mathbb{R}$) such that $s \leq u$ ($s \geq \ell$) for all $s \in S$. Each $u$ ($\ell$) is called an upper bound (a lower bound) of $S$.

- $S$ is said to be bounded if it is both bounded above and below.

- If $S$ is bounded above (below), then $u \in \mathbb{R}$ ($\ell \in \mathbb{R}$) is said to be a supremum (infimum) or a least upper bound (greatest lower bound) of $S$ if it satisfies

  (i) $u$ ($\ell$) is an upper (a lower) bound of $S$,

  (ii) if $u'$ ($\ell'$) is an upper (a lower) bound of $S$, then $u \leq u'$ ($\ell \leq \ell'$).

We denote $u$ and $\ell$ by $\sup S$ and $\inf S$ (By showing the uniqueness of them).
Lemma: (Alternative definition of supremum)
Let $S$ be a nonempty subset of $\mathbb{R}$.
$\sup S = u$ if and only if $u$ satisfies
(i) $s \leq u$ for all $s \in S$ (i.e. $u$ is an upper bound)
(ii) If $v < u$, then there exists $s' \in S$ such that $v < s'$
(Remark: How about $\inf S$?)
Idea:

proof:
(i) $\iff$ (i) trivial
(ii) $\iff$ (ii) contrapositive

Lemma: (Another alternative definition)
Let $S$ be a nonempty subset of $\mathbb{R}$.
$\sup S = u$ if and only if $u$ satisfies
(i) $s \leq u$ for all $s \in S$ (i.e. $u$ is an upper bound)
(ii) for all $\varepsilon > 0$, there exists $s' \in S$ such that $u - \varepsilon < s'$
proof:
(ii) $\Rightarrow$ (i): Consider $\varepsilon = u - v$ (i.e. $v = u - \varepsilon$)

Example:
Let $S = \{x \in \mathbb{R} : x \geq 1\}$, show that $\sup S = 1$.
(i) Clearly, $1$ is an upper bound of $S$.
(ii) Let $\varepsilon > 0$, then take $s = 1 - \frac{\varepsilon}{2} \in S$, we have $1 - \varepsilon < s$
$\therefore \sup S = 1$.
(Note $1 \notin S$, i.e. $\sup S$ is NOT necessary in $S$.)

Completeness properties of $\mathbb{R}$:
If $S$ is a nonempty subset of $\mathbb{R}$ and it is bounded above, then $\sup S$ exists.
Axiomatic Construction of $\mathbb{R}$:

The set of all real numbers $\mathbb{R}$, that has (is assumed to have) the following properties:

1. Field axioms / Algebraic properties
2. Order properties
3. Completeness properties

Exercise:

Prove that

1) If $S$ is a nonempty subset of $\mathbb{R}$ and it is bounded below, then $\inf S$ exists.
   (Hint: Consider $-S = \{-s : s \in S\}$)

2) If $A$ and $B$ are nonempty subsets of $\mathbb{R}$ and $a \leq b$ for all $a \in A$ and $b \in B$,
   then $\sup A \leq \inf B$.

3) If $S$ is a nonempty subset of $\mathbb{R}$ and it is bounded above, then $\sup (a+S) = \sup S + a$
   where $a+S = \{a+s : s \in S\}$

2.2 Archimedian Property and Related Application

Theorem 1 (Archimedian property)

If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x < n_x$.

proof:

Suppose the contrary, $x \geq n$ for all $n \in \mathbb{N}$ (i.e. $x$ is an upper bound of $\mathbb{N}$).

By completeness property, there exists $u = \sup \mathbb{N}$.

\[\begin{array}{c}
  \underline{x} \quad \underline{u} \quad \underline{u+1} \\
  \uparrow \quad \uparrow \quad \uparrow \\
  \text{u-1 is not an upper bound} \\
  \downarrow \quad \downarrow \\
  \underline{m} \quad \underline{u} \quad \underline{u+1} \\
  \text{there exists } m \in \mathbb{N} \text{ such that } m > u - 1 \\
  \text{(i.e. } m+1 > u = \sup \mathbb{N} \text{) Contradiction!}
\end{array}\]
Corollary:
If $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$, then $\inf S = 0$.

Proof:
$S$ is nonempty and bounded below by 0.
By Archimedean property, for all $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ (i.e. $\frac{1}{n} < \varepsilon$)

$$0 \leq \frac{1}{n} < \varepsilon = 0 + \varepsilon$$

$$\therefore \inf S = 0.$$  

From the proof, we can also observe that:
If $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < \varepsilon$.

Corollary: (Refined statement of Archimedean property)
If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $ny - 1 < y < ny$.

Proof:
Let $E_y = \{ m \in \mathbb{N} \mid y < m \}$

Idea:

$$\lfloor y \rfloor, \lfloor y \rfloor + 1, \lfloor y \rfloor + 2, \ldots \in E_y$$

$L_x$ : floor function that gives the largest integer not greater than $x$ (i.e. $L_x \leq x$)

Main issue: How to show the existence of $n_y$?

Archimedean property $\Rightarrow E_y \neq \emptyset$

Well ordering property of $\mathbb{N}$ $\Rightarrow E_y$ has a least element $n_y$

Then $ny - 1 \notin E_y$ and so $ny - 1 < y$.

Existence of $n_y$: By Archimedean property

Existence of $n_y$: By well ordering property
Theorem: (Existence of \( \sqrt{2} \))

There exists a positive real number \( x \) such that \( x^2 = 2 \).

Proof:

Let \( S = \{ s \in \mathbb{R} : s > 0, s^2 < 2 \} \)
- \( \emptyset \neq S \neq \emptyset \)
- \( S \) is bounded above by \( 2 \). (If there exists \( s \in S \) such that \( s > 2 \), then \( s^2 > 4 \) !)

\[ \sup S \] exists and we define \( x = \sup S \).

Claim: \( x^2 = 2 \).

(By Trichotomy property, we only need to show \( x < 2 \) and \( x > 2 \) are NOT true !)

1. \( \text{Suppose } x^2 > 2 \) (\( \Rightarrow x > \sqrt{2} \))

   Idea: Show that there exists \( n \in \mathbb{N} \) such that \( \frac{x}{\sqrt{2}} \) is another upper bound of \( S \)
   (which leads contradiction)

   Find \( n \in \mathbb{N} \) such that \( (x - \frac{1}{n})^2 < 2 \)

   \[x^2 - \frac{2x}{n} + \frac{1}{n^2} < 2 \]

   \[ (x - 2) + \frac{1}{n} < \frac{2}{n} \]

   \[ \frac{x^2 - 2}{2n} < \frac{1}{n} \]

   \( \frac{x^2 - 2}{2n} > 0 \), by collary of Archimedean property, \( \frac{1}{n} \) has solution for \( n \in \mathbb{N} \).

   \( (x - \frac{1}{n})^2 > s^2 \) \( \forall s \in S \) and \( x - \frac{1}{n}, s > 0 \Rightarrow x - \frac{1}{n} > s \) (Contradiction !)

2. \( \text{Suppose } x^2 < 2 \) (\( \Rightarrow x < \sqrt{2} \))

   Exercise!
Theorem: \( \mathbb{Q} \) is dense in \( \mathbb{R} \).

If \( x, y \in \mathbb{R} \) and \( x < y \), then there exists \( r \in \mathbb{Q} \) such that \( x < r < y \).

proof:
Without loss of generality, assume \( x > 0 \).

Idea:
\[ \text{length} = y - x > \frac{1}{n} \quad \text{for some} \ n \in \mathbb{N} \]

\[ x \quad \text{y} \]

\[ \text{length} = ny - nx > 1 \]

\[ mx \quad m \quad ny \]

there exists \( m \in \mathbb{N} \) such that \( mx < m < ny \)

\[ x < \frac{m}{n} < y \]

\[ r \in \mathbb{Q} \quad \text{What we need!} \]

Similar result:

Theorem:
If \( x, y \in \mathbb{R} \) and \( x < y \), then there exists \( p \in \mathbb{R} \setminus \mathbb{Q} \) such that \( x < p < y \).

proof:

\[ \sqrt{2} \text{ is irrational (Why?)} \]

\[ \text{length} = \frac{1}{n} (y - x) > \frac{1}{n} \quad \text{for some} \ n \in \mathbb{N} \]

\[ x/\sqrt{2} \quad y/\sqrt{2} \]

\[ \text{length} = \frac{m}{n} (y - x) > 1 \]

\[ mx/\sqrt{2} \quad m \quad ny/\sqrt{2} \]

\[ x < \frac{m}{n} < y \]

\[ r \in \mathbb{R} \setminus \mathbb{Q} \]

Same trick!
23. Intervals

Notations:

If \( a < b \), then

\( (a, b) = \{ x \in \mathbb{R} : a < x < b \} \) \quad \text{open interval}

\[ [a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \} \quad \text{closed interval} \quad \text{finite interval}
\]

\( (a, b] = \{ x \in \mathbb{R} : a < x \leq b \} \) \quad \text{half open (closed) interval}

\( [a, \infty) = \{ x \in \mathbb{R} : a \leq x \} \)

\( (-\infty, b] = \{ x \in \mathbb{R} : x < b \} \) \quad \text{infinite open interval}

Note: \( \infty, -\infty \notin \mathbb{R} \), ONLY convention.

Characterization Theorem:

If \( S \subseteq \mathbb{R} \) that contains at least two points and has the property

\[ x, y \in S \text{ and } x < y \Rightarrow \exists z \in [x, y] \subseteq S \quad (\ast) \]

then \( S \) is an interval.

Idea: Can \( S \) be such a subset of \( \mathbb{R} \)?

\[ \begin{array}{c}
S \\
\underbrace{\vphantom{\sum} {x}}_{<} \underbrace{\vphantom{\sum} {y}}_{>}
\end{array} \]

No! There exists \( x, y \in S \) with \( x < y \) and \( a \notin [x, y] \) such that \( a \notin S \).

Property \( \ast \) governs that \( S \) must be "something nice" (an interval).

proof:

There are 4 cases:

1. \( S \) is bounded, let \( a = \inf S \) and \( b = \sup S \), then \( S = (a, b) \) or \([a, b] \) or \([a, b) \) or \((a, b] \)

2. \( S \) is bounded above but NOT bounded below, let \( b = \sup S \), then \( S = (a, b) \) or \((-\infty, b] \)

3. \( S \) is bounded below but NOT bounded above, let \( a = \inf S \), then \( S = (a, \infty) \) or \([a, \infty) \)

4. \( S \) is unbounded, then \( S = \mathbb{R} = (-\infty, \infty) \).

Exercise!
Definition:

A sequence of intervals $I_n$, $n \in \mathbb{N}$, is said to be nested if $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots$

(Here, $I_n$ is NOT necessary to be closed)

Examples:

1) If $I_n = [0, \frac{1}{n}]$, then the sequence is nested. Furthermore, $\bigcap_{n=1}^{\infty} I_n = \{0\}$.

2) If $I_n = (0, \frac{1}{n})$, then the sequence is nested. However, $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

Theorem: (Nested Interval Property)

If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequence of closed intervals, then there exists $p \in \mathbb{R}$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

Proof:

Let $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$.

Exercise: Show that $A$ is bounded above (by $b$).

- $B$ is bounded below (by $a$).

\[ a \leq a_n \leq a \quad \text{and} \quad b \geq b_n \geq b \]

\[ \sup A \leq a_n \leq \inf B \]

\[ a_n \rightarrow a, \quad b_n \rightarrow b \]

: $\sup A$ and $\inf B$ exist and $\sup A \geq \inf B$

Exercise: Show that for all $p \in [\sup A, \inf B]$, $p \in I_n$ for all $n \in \mathbb{N}$.

(As we can see, $p$ is NOT unique, but it is the case if we further impose a condition.)

Theorem:

Furthermore, if $\inf [b_n - a_n] = 0$, then there exists unique $p \in \mathbb{R}$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

Proof:

Suppose $p, q \in I_n$ for all $n \in \mathbb{N}$ and $p \leq q$, i.e., $a_n \leq p \leq q \leq b_n$.

Then $q - p \leq b_n - a_n$ for all $n \in \mathbb{N}$, i.e., $q - p$ is a lower bound of $[b_n - a_n : n \in \mathbb{N}]$.

Therefore, $0 \leq q - p \leq \inf [b_n - a_n] = 0$

\[ q - p = 0, \quad \text{i.e.,} \quad q = p. \]
Theorem: (Uncountability of \( \mathbb{R} \))

The set \( \mathbb{R} \) is uncountable.

proof:

It suffices to show \([0,1]\) is uncountable.

Suppose the contrary, \( I \) is countable and \( I = \{x_i, x_2, \ldots, x_n, \ldots\} \).

Construct a sequence of closed intervals:

Step 1: Choose \( I_1 \in [0,1] \) such that \( x_i \notin I_1 \) (How?)

Inductive step: Choose \( I_n \in I \) such that \( I_n \subseteq I_{n-1} \) and \( x_i \notin I_n \) (How?)

By the construction, \( I_n \) is a nested sequence of intervals,

\( \Rightarrow \) there exists \( p \in I \) such that \( p \notin I_n \) for all \( n \in \mathbb{N} \).

However \( p \in I \Rightarrow p = x_n \) for some \( n \in \mathbb{N} \)

(Contradicts to \( p \notin I_n \))

Binary Representation

Example:

\( 0.625 \in [0,1] \)

\[ x = \frac{1}{2} + \frac{1}{8} \]

\[ = 1 \times \frac{1}{2} + 0 \times \left( \frac{1}{2} \right)^3 + 1 \times \left( \frac{1}{2} \right)^3 + 0 \times \left( \frac{1}{2} \right)^3 + \cdots \]

\[ \therefore \ 0.625 \text{ has a binary representation } (10100\ldots) \] (Remark: 0.101\_4 in secondary school)

Theorem:

If \( x \in [0,1] \), then there exists a sequence \( \{a_n\} \) of zeros or ones such that

\[ \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} \leq x \leq \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n} + \frac{a_{n+1}}{2^{n+1}} \text{ for all } n \in \mathbb{N}. \]

In this case, we write \( x = (a_1a_2\ldots a_n\ldots)_2 \).
proof:

Idea: Bisection!

Let \( x \in [0,1] \)

Step 1:

Let \( I_1 = [0,1] \)

\( x \) lies on the right subinterval of \( I_1 \), then we take \( a_1 = 1 \)

Step 2:

Let \( I_2 = [ \frac{1}{2}, 1 ] \)

\( x \) lies on the left subinterval of \( I_2 \), then we take \( a_2 = 0 \)

\( \vdots \)

Repeating \( I_{2n} = [ \frac{q_n}{2} + \frac{q_{n+1}}{2}, \frac{q_n}{2} + \frac{q_{n+1}}{2} + \frac{q_{n+1}}{2} ] \)

and \( a_{2n} \) is determined by whether \( x \) lies on the left or right subinterval of \( I_{2n} \).

The result follows from the fact that \( x \in I_n \) for all \( n \in \mathbb{N} \).

Only trouble:

\[ x = 0.625 \]

\[ 0.625 = (101000 \ldots)_2 \text{ or } (100111 \ldots)_2 \]

\( 0.625 = 0 \text{ or } 1 \) ?

Consequence: \( 0.625 = (101000 \ldots)_2 \) or \( (100111 \ldots)_2 \)

\( \therefore \) Binary representation is Not unique.

However, conversely, given a sequence / representation \((a,a_2,a_3,\ldots)_2\), it corresponds to a unique real number in \([0,1]\).

Why ? Simply because of the nested property of \( \mathbb{R} \).

Exercise:

Figure out the decimal representation.

Remark: Think why \( 0.999 \ldots = 1 \) ?