1 Appendix

Definition (Closed Linear Operator) (1) The graph $G(T)$ of a linear operator T on the domain $\mathcal{D}(T) \subset X$ into Y is the set $(x, Tx) : x \in \mathcal{D}(T)\}$ in the product space $X \times Y$. Then T is closed if its graph $G(T)$ is a closed linear subspace of $X \times Y$, i.e., if $x_n \in \mathcal{D}(T)$ converges strongly to $x \in X$ and Tx_n converges strongly to $y \in Y$, then $x \in \mathcal{D}(T)$ and $y = Tx$. Thus the notion of a closed linear operator is an extension of the notion of a bounded linear operator.
(2) A linear operator T is said be closable if $x_n \in \mathcal{D}(T)$ converges strongly to 0 and Tx_n converges strongly to $y \in Y$, then $y = 0$.

For a closed linear operator T, the domain $\mathcal{D}(T)$ is a Banach space if it is equipped by the graph norm

$$|x|_{\mathcal{D}(T)} = (|x|_X^2 + |Tx|_Y^2)^{\frac{1}{2}}.$$

Example (Closed linear Operator) Let $T = \frac{d}{dt}$ with $X = Y = L^2(0,1)$ is closed and

$$dom\,(A) = H^1(0,1) = \{f \in L^2(0,1) : \text{absolutely continuous functions on } [0,1] \\ \text{with square integrable derivative}\}.$$

If $y_n = Tx_n$, then

$$x_n(t) = x_n(0) + \int_0^t y_n(s)\,ds.$$

If $x_n \in dom(T) \to x$ and $y_n \to y$ in $L^2(0,1)$, then letting $n \to \infty$ we have

$$x(t) = x(0) + \int_0^t y(s)\,ds,$$

i.e., $x \in dom\,(T)$ and $Tx = y$.

In general if for $\lambda\, I + T$ for some $\lambda \in R$ has a bounded inverse $(\lambda\, I + T)^{-1}$, then $T : dom\,(A) \subset X \to X$ is closed. In fact, $Tx_n = y_n$ is equivalent to

$$x_n = (\lambda\, I + T)^{-1}(y_n + \lambda\, x_n$$

Suppose $x_n \to x$ and $y_n \to y$ in X, letting $n \to \infty$ in this, we have $x \in dom\,(T)$ and $Tx = T(\lambda\, I + T)^{-1}(\lambda\, x + y) = y$.

Definition (Dual Operator) Let T be a linear operator X into Y with dense domain $\mathcal{D}(T)$. The dual operator of T^* of T is a linear operator on Y^* into X^* defined by

$$\langle y^*, Tx \rangle_{Y^* \times Y} = \langle T^* y^*, x \rangle_{X^* \times X}$$

for all $x \in \mathcal{D}(T)$ and $y^* \in \mathcal{D}(T^*)$.

In fact, for $y^* \in Y^*$ $x^* \in X^*$ satisfying

$$\langle y^*, Tx \rangle = \langle x^*, x \rangle \text{ for all } x \in \mathcal{D}(T)$$

is uniquely defined if and only if $\mathcal{D}(T)$ is dense. The only if part follows since if $\overline{\mathcal{D}(T)} \neq X$ then the Hahn-Banach theory there exits a nonzero $x_0^* \in X^*$ such that $\langle x_0^*, x \rangle = 0$ for all

$\mathcal{D}(T)$, which contradicts to the uniqueness assumption. If T is bounded with $\mathcal{D}(T) = X$ then T^* is bounded with $\|T\| = \|T^*\|$.

Examples Consider the gradient operator $T : L^2(\Omega) \to L^2(\Omega)^n$ as

$$Tu = \nabla u = (D_{x_1}u, \cdots D_{x_n}u)$$

with $\mathcal{D}(T) = H^1(\Omega)$. The, we have for $v \in L^2(\Omega)^n$

$$T^*v = -div\, v = -\sum D_{x_k}v_k$$

with domain $\mathcal{D}(T^*) = \{v \in L^2(\Omega)^n : div\, v \in L^2(\Omega) \text{ and } n \cdot v = 0 \text{ at the boundary } \partial\Omega\}$. In fact by the divergence theorem

$$(Tu, v) = \int_\Omega \nabla u \cdot v \int_{\partial\Omega} (n \cdot v)u\, ds - \int_\Omega u(div\, v)\, dx = (u, T^*v)$$

for all $v \in C^1(\Omega)$. First, let $u \in H^1_0(\Omega)$ we have $T^*v = -div\, v \in L^2(\Omega)$ since $H^1_0(\Omega)$ is dense in $L^2(\Omega)$. Thus, $n \cdot v \in L^(\partial\Omega)$ and $n \cdot v = 0$.

Definition (Hilbert space Adjoint operator) Let X, Y be Hilbert spaces and T be a linear operator X into Y with dense domain $\mathcal{D}(T)$. The Hilbert self adjoint operator of T^* of T is a linear operator on Y into X defined by

$$(y, Tx)_Y = (T^*y, x)_X$$

for all $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$. Note that if we let $T' : Y^* \to X^*$ is the dual operator of T, then

$$T^*R_{Y^*\to Y} = R_{X^*\to X}T'$$

where $R_{X^*\to X}$ and $R_{Y^*\to Y}$ are the Riesz maps.

Examples (self-adjoint operator) Let $X = L^2(\Omega)$ and T be the Laplace operator

$$Tu = \Delta u = \sum_{k=1}^n D_{x_kx_k}u$$

with domain $\mathcal{D}(T) = H^2(\Omega) \cap H^1_0(\Omega)$. Then T is sel-adjoint, i.e., $T^* = T$. In fact

$$(Tu, v)_X = \int_\Omega \Delta u\, v\, dx = \int_{\partial\Omega} ((n \cdot \nabla u)v - (n \cdot \nabla v)u)\, ds + \int_\Omega \Delta v\, u\, dx = (x, T^*v)$$

for all $v \in C^1(\Omega)$.

Let us denote by $F : X \to X^*$, the duality mapping of X, i.e.,

$$F(x) = \{x^* \in X^* : \langle x, x^* \rangle = |x|^2 = |x^*|^2\}.$$

By Hahn-Banach theorem, $F(x)$ is non-empty. In general F is multi-valued. Therefore, when X is a Hilbert space, $\langle \cdot, \cdot \rangle$ coincides with its inner product if X^* is identified with X and $F(x) = x$.

Let H be a Hilbert space with scalar product (ϕ, ψ) and X be a real, reflexive Banach space and $X \subset H$ with continuous dense injection. Let X^* denote the strong dual space of X. H is identified with its dual so that $X \subset H = H^* \subset X^*$. The dual product $\langle \phi, \psi \rangle$ on $X \times X^*$ is the continuous extension of the scalar product of H restricted to $X \times H$.

Theorem (Alligned Element) Let X be a normed space. For each $x_0 \in X$ there exists an $f \in X^*$ such that

$$f(x_0) = |f|_{X^*}|x_0|_X.$$

Proof: Let $S = \{\alpha\, x_0 : \alpha \in R\}$ and define $f(\alpha\, x_0) = \alpha\, |x_0|_X$. By Hahn-Banach theorem there exits an extension $F \in X^*$ of f such that $F(x) \leq |x|$ for all $x \in X$. Since

$$-F(x) = F(-x) \leq |-x| = |x|,$$

we have $|F(x)| \leq |x|$, in particular $|F|_{X^*} \leq 1$. On the other hand, $F(x_0) = f(x_0) = |x_0|$, thus $|F|_{X^*} = 1$ and $F(x_0) = f(x_0) = |F||x_0|$. $\square$

The following proposition contains some further important properties of the duality mapping F.

Theorem (Duality Mapping) (a) $F(x)$ is a closed convex subset.
(b) If X^* is strictly convex (i.e., balls in X^* are strictly convex), then for any $x \in X$, $F(x)$ is single-valued. Moreover, the mapping $x \to F(x)$ is demicontinuous, i.e., if $x_n \to x$ in X, then $F(x_n)$ converges weakly star to $F(x)$ in X^*.
(c) Assume X be uniformly convex (i.e., for each $0 < \epsilon < 2$ there exists $\delta = \delta(\epsilon) > 0$ such that if $|x| = |y| = 1$and $|x - y| > \epsilon$, then $|x + y| \leq 2(1 - \delta)$). If x_n converges weakly to x and $\limsup_{n\to\infty} |x_n| \leq |x|$, then x_n converges strongly to x in X.
(d) If X^* is uniformly convex, then the mapping $x \to F(x)$ is uniformly continuous on bounded subsets of X.

Proof: (a) Closeness of $F(x)$ is an easy consequence of the follows from the continuity of the duality product. Choose $x_1^*, x_2^* \in F(x)$ and $\alpha \in (0, 1)$. For arbitrary $z \in X$ we have (using $|x_1^*| = |x_2^*| = |x|$) $\langle z, \alpha x_1^* + (1-\alpha)x_2^* \rangle \leq \alpha|z|\,|x_1^*| + (1-\alpha)|z|\,|x_2^*| = |z|\,|x|$, which shows $|\alpha x_1^* + (1-\alpha)x_2^*| \leq |x|$. Using $\langle x, x^* \rangle = \langle x, x_1^* \rangle = |x|^2$ we get $\langle x, \alpha x_1^* + (1-\alpha)x_2^* \rangle = \alpha\langle x, x_1^* \rangle + (1-\alpha)\langle x, x_2^* \rangle = |x|^2$, so that $|\alpha x_1^* + (1-\alpha)x_2^*| = |x|$. This proves $\alpha x_1^* + (1-\alpha)x_2^* \in F(x)$.
(b) Choose $x_1^*,\ x_2^* \in F(x)$, $\alpha \in (0, 1)$ and assume that $|\alpha x_1^* + (1-\alpha)x_2^*| = |x|$. Since X^* is strictly convex, this implies $x_1^* = x_2^*$. Let $\{x_n\}$ be a sequence such that $x_n \to x \in X$. From $|F(x_n)| = |x_n|$ and the fact that closed balls in X^* are weakly star compact we see that there exists a weakly star accumulation point x^* of $\{F(x_n)\}$. Since the closed ball in X^* is weakly star closed, thus

$$\langle x, x^* \rangle = |x|^2 \geq |x^*|^2.$$

Hence $\langle x, x^* \rangle = |x|^2 = |x^*|^2$ and thus $x^* = F(x)$. Since $F(x)$ is single-valued, this implies $F(x_n)$ converges weakly to $F(x)$.
(c) Since $\liminf |x_n| \leq |x|$, thus $\lim_{n\to\infty} |x_n| = |x|$. We set $y_n = x_n/|x_n|$ and $y = x/|x|$. Then y_n converges weakly to y in X. Suppose y_n does not converge strongly to y in X. Then there exists an $\epsilon > 0$ such that for a subsequence $y_{\tilde{n}}$ $|y_{\tilde{n}} - y| > \epsilon$. Since X^* is uniformly convex there exists a $\delta > 0$ such that $|y_{\tilde{n}} + y| \leq 2(1 - \delta)$. Since the norm is weakly lower semicontinuos, letting $\tilde{n} \to \infty$ we obtain $|y| \leq 1 - \delta$, which is a contradiction.

(d) Assume F is not uniformly continuous on bounded subsets of X. Then there exist constants $M > 0$, $\epsilon > 0$ and sequences $\{u_n\}, \{v_n\}$ in X satisfying

$$|u_n|, |v_n| \le M, \quad |u_n - v_n| \to 0, \quad \text{and} \quad |F(u_n) - F(v_n)| \ge \epsilon.$$

Without loss of the generality we can assume that, for a constant $\beta > 0$, we have in addition $|u_n| \ge \beta$, $|v_n| \ge \beta$. We set $x_n = u_n/|u_n|$ and $y_n = v_n/|v_n|$. Then we have

$$|x_n - y_n| = \frac{1}{|u_n|\,|v_n|}\,||v_n|u_n - |u_n|v_n|$$

$$\le \frac{1}{\beta^2}\left(|v_n||u_n - v_n| + ||v_n| - |u_n||\,|v_n|\right) \le \frac{2M}{\beta^2}|u_n - v_n| \to 0 \quad \text{as } n \to \infty.$$

Obviously we have $2 \ge |F(x_n) + F(y_n)| \ge \langle x_n, F(x_n) + F(y_n)\rangle$ and this together with

$$\langle x_n, F(x_n) + F(y_n)\rangle = |x_n|^2 + |y_n|^2 + \langle x_n - y_n, F(y_n)\rangle$$

$$= 2 + \langle x_n - y_n, F(y_n)\rangle \ge 2 - |x_n - y_n|$$

implies

$$\lim_{n\to\infty} |F(x_n) + F(y_n)| = 2.$$

Suppose there exists an $\epsilon_0 > 0$ and a subsequence $\{n_k\}$ such that $|F(x_{n_k}) - F(y_{n_k})| \ge \epsilon_0$. Observing $|F(x_{n_k})| = |F(y_{n_k})| = 1$ and using uniform convexity of X^* we conclude that there exists a $\delta_0 > 0$ such that

$$|F(x_{n_k}) + F(y_{n_k})| \le 2(1 - \delta_0),$$

which is a contradiction to the above. Therefore we have $\lim_{n\to\infty} |F(x_n) - F(y_n)| = 0$. Thus

$$|F(u_n) - F(v_n)| \le |u_n|\,|F(x_n) - F(y_n)| + ||u_n| - |v_n||\,|F(y_n)|$$

which implies $F(u_n)$ converges strongly to $F(v_n)$. This contradiction proves the result. $\Box$

Problem Let $X = C[0,1]$ be the space of continuous functions with sup norm. Then show that $X^* = BV(0,1) =$ the space of (right continuous) bounded variation functions on $[0,1]$, i.e. for every $f \in X^*$ there exists $\nu \in BV(0,1)$ such that $f(x) = \int_0^1 x(t)\,d\nu(t)$ (Riemann Stieltjes integral) for all $x \in X$. $\delta_{t_0} \in X^*$ (i.e. $\delta_{x_0}(\phi) = \phi(t_0)$ for $\phi \in X$). and $\delta_{t_0} \in F(x)$ for $t_0 \in [0,1]$ satisfying $x(t_0) = \max_{t\in[0,1]} |x(t)|$.

Problem Let A be a closed linear operator on a Banach space. $D(A) = dom\,(A)$ is a Banach space with the graph norm

$$|x|_{D(A)} = |x|_X + |Ax|_X.$$

Problem Let $c \in L^\infty(0,1)$. Define the linear operators $A_1 u = -(c(x)u)_x$ in $X = L^1(0,1)$. and $A_2 u = c(x)u_x$ in $X = L^p(0,1)$.

(a) Find $dom(A_2)$ so that A_2 is ω-dissipative. — Hint: $c' \le M$ (bounded above) if $p > \infty$. If $p = \infty$, then no condition is necessary. Inflow $c(0) > 0$ and Outflow $c(0) \le 0$.

Find $dom(A_1)$ so that A_1 is ω-dissipative. — Hint Assume $c > 0$. Since $cu \in C[0,1]$ one can decompose $[0,1]$ the sub intervals (t_i, t_{i+1}) on which $cu > 0$ or $cu < 0$ and $cu(t_i) = 0$ and let $u^* = sign_0(cu) = sign_0(u)$. Thus, we have

$$(A_1 u, u^*) = \int_0^1 (-(cu)_x u^*(x))\, dx = c(0)|u(0)| - c(1)|u(1)|$$

(c) In general show that $dom(A_1)$ and $dom(A_2)$ are different (Hint: piecewise constant)

1.1 Dissipativity

In order to obtain the useful equivalent conditions for the dissipativity, we consider the derivatives of the norm $|\cdot|$ of X, which define pairs in some way analogous to the inner product on a Hilbert space.

Definition 1.2 We define the functions $\langle\cdot,\cdot\rangle_+,\ \langle\cdot,\cdot\rangle_- : X \times X \to R$ by

$$\langle y, x\rangle_+ = \lim_{\alpha\to 0^+} \frac{|x+\alpha\, y| - |x|}{\alpha}$$

$$\langle y, x\rangle_- = \lim_{\alpha\to 0^+} \frac{|x| - |x-\alpha\, y|}{\alpha}$$

Also, we defines the functions $\langle\cdot,\cdot\rangle_s,\ \langle\cdot,\cdot\rangle_i : X \times X \to R$ by

$$\langle y, x\rangle_s = \lim_{\alpha\to 0^+} \frac{|x+\alpha\, y|^2 - |x|^2}{2\alpha}$$

$$\langle y, x\rangle_i = \lim_{\alpha\to 0^+} \frac{|x|^2 - |x-\alpha\, y|^2}{2\alpha}$$

Here, we note that $\alpha^{-1}(|x+\alpha\, y| - |x|)$ is an increasing function. In fact, if $0 < \alpha < \beta$ then

$$(\beta-\alpha)\,|x| = |(\beta x + \alpha\beta y) - (\alpha x + \alpha\beta y)| \geq \beta\,|x+\alpha\, y| - \alpha\,|x+\beta\, y|$$

and thus

$$\beta^{-1}(|x+\beta\, y| - |x|) \geq \alpha^{-1}(|x+\alpha\, y| - |x|).$$

Moreover, since $\alpha^{-1}(|x+\alpha\, y| - |x|) \geq -|y|$, this function is bounded below. Hence, $\lim_{\alpha\to 0^+} = \inf_{\alpha>0}$ exists for all $x,\ y \in X$. From the definition we have

$$(1.2) \qquad \langle y, x\rangle_- = -\langle -y, x\rangle_+ \quad \text{and} \quad \langle y, x\rangle_i = -\langle -y, x\rangle_s$$

Since the norm is continuous, it follows that

$$(1.3) \qquad \langle y, x\rangle_s = |x|\,\langle y, x\rangle_+ \quad \text{and} \quad \langle y, x\rangle_i = |x|\,\langle y, x\rangle_-.$$

Also, from $2\,|x| \leq |x+\alpha\, y| + |x-\alpha\, y|$, we have

$$\alpha^{-1}(|x| - |x-\alpha\, y|) \leq \alpha^{-1}(|x+\alpha\, y| - |x|).$$

Thus,

$$(1.4) \qquad \langle y, x\rangle_- \le \langle y, x\rangle_+ \quad \text{and} \quad \langle y, x\rangle_i \le \langle y, x\rangle_s$$

Moreover, we have the following lemma.

Lemma 1.1 Let $x,\ y \in X$.

(1) There exists an element f^+ such that

$$\langle y, x\rangle_s = \sup\{Re\,\langle y, f\rangle : f \in F(x)\} = Re\,\langle y, f^+\rangle$$

(2) There exists an element f^- such that

$$\langle y, x\rangle_i = \inf\{Re\,\langle y, f\rangle : f \in F(x)\} = Re\,\langle y, f^-\rangle$$

(3) $\langle \alpha\, x + y, x\rangle_q = \alpha\,|x| + \langle y, x\rangle_q$ for $\alpha \in R$ where q is either $+$ or $-$.

(4) For $z \in X$

$$\langle y + z, x\rangle_- \ge \langle y, x\rangle_- + \langle z, x\rangle_- \quad \text{and} \quad \langle y + z, x\rangle_+ \le \langle y, x\rangle_+ + \langle z, x\rangle_+$$

and thus

$$\langle y, x\rangle_- - \langle z, x\rangle_+ \le \langle y - z, x\rangle_- \le \langle y, x\rangle_+ - \langle z, x\rangle_-$$

(5) $\langle \cdot, \cdot\rangle_- : X \times X \to R$ is lower semicontinuous and $\langle \cdot, \cdot\rangle_+ : X \times X \to R$ is upper semicontinuous.

Proof: (3) and (4) are obvious from the definition. For (5) since for each $\alpha > 0$

$$\alpha^{-1}(|x + \alpha\, y| - |x|)$$

is a continuous function of $X \times X \to R$, the upper continuity of $\langle \cdot, \cdot\rangle_+$ follows from its definition. Since $\langle y, x\rangle_- = -\langle -y, x\rangle_+$, $\langle \cdot, \cdot\rangle_- : X \times X \to R$ is lower semicontinuous. □

Now, the following theorem gives the equivalent conditions for the dissipativeness of A.

Theorem 1.2 Let $x,\ y \in X$. The following statements are equivalent.

(i) $Re\,\langle y, x^*\rangle \le 0.$ for some $x^* \in F(x)$.

(ii) $|x - \lambda\, y| \ge |x|$ for all $\lambda > 0$.

(iii) $\langle y, x\rangle_- \le 0$

(iv) $\langle y, x\rangle_i \le 0$.

Proof: $(i) \to (ii)$. By the definition of F, we have

$$|x|^2 = \langle x, x^*\rangle \le Re\,\langle x - \lambda\, y, x^*\rangle \le |x - \lambda\, y|\,|x^*|$$

for all $\lambda > 0$. Thus, (ii) holds.

$(ii) \to (i)$. For each $\lambda > 0$ let $f_\lambda \in F(x - \lambda\, y)$. Then $|f_\lambda| \ne 0$ and we set $g_\lambda = |f_\lambda|^{-1}\, f_\lambda$. Since the unit sphere of the dual space X^* is compact in the weak-star topology of X^*, we may assume that

$$\lim_{\lambda \to 0}\,\langle u, g_\lambda\rangle = \langle u, g\rangle \quad \text{for all } u \in X$$

where g is some element in X^*. Next, since

$$|x| \le |x - \lambda\, y| = Re\,\langle x - \lambda\, y, g_\lambda\rangle \le |x| - \lambda\, Re\,\langle y, g_\lambda\rangle \tag{1.7}$$

for all $\lambda > 0$, it follows that $Re\,\langle y, g_\lambda\rangle \le 0$ for all $\lambda > 0$, and letting $\lambda \to 0^+$, $Re\,\langle y, g\rangle \le 0$. Note that (1.7) also implies $|x| \le Re\,\langle x, g\rangle$ and thus $\langle x, g\rangle = |x|$. This implies that $|x|\, g \in F(x)$ and hence (i) holds.

Since $\alpha \to \alpha^{-1}(|x - \alpha\, y| - |x|)$ is an decreasing function (2) and (3) are equivalent by the definition of $\langle\cdot,\cdot\rangle_-$. $\square$

1.2 Lax-Milgram Theory and Applications

Let H be a Hilbert space with scalar product (ϕ, ψ) and X be a Hilbert space and $X \subset H$ with continuous dense injection. Let X^* denote the strong dual space of X. H is identified with its dual so that $X \subset H = H^* \subset X^*$ (i.e., H is the pivoting space). The dual product $\langle \phi,\, \psi\rangle$ on $X^* \times X$ is the continuous extension of the scalar product of H restricted to $H \times X$. This framework is called the Gelfand triple.

Let σ is a bounded coercive bilinear form on $X \times X$. Note that given $x \in X$, $F(y) = \sigma(x, y)$ defines a bounded linear functional on X. Since given $x \in X$, $y \to \sigma(x, y)$ is a bounded linear functional on X, say $x^* \in X^*$. We define a linear operator A from X into X^* by $x^* = Ax$. Equation $\sigma(x, y) = F(y)$ for all $y \in X$ is equivalently written as an equation

$$Ax = F \in X^*.$$

Here,

$$\langle Ax, y\rangle_{X^*\times X} = \sigma(x, y), \quad x,\ y \in X,$$

and thus A is a bounded linear operator. In fact,

$$|Ax|_{X^*} \le \sup_{|y|\le 1} |\sigma(x, y)| \le M\, |x|.$$

Let R be the Riesz operator $X^* \to X$, i.e.,

$$|Rx^*|_X = |x^*| \text{ and } (Rx^*, x)_X = \langle x^*, x\rangle \text{ for all } x \in X,$$

then $\hat{A} = RA$ represents the linear operator $\hat{A} \in \mathcal{L}(X, X)$. Moreover, we define a linear operator $\tilde{A}$ on H by

$$\tilde{A}x = Ax \in H$$

with

$$dom\,(\tilde{A}) = \{x \in X : |\sigma(x, y)| \le c_x\, |y|_H \text{ for all } y \in X\}.$$

That is, $\tilde{A}$ is a restriction of A on $dom\,(\tilde{A})$. We will use the symbol A for all three linear operators as above in the lecture note and its use should be understood by the underlining context.

Lax-Milgram Theorem Let X be a Hilbert space. Let σ be a (complex-valued) sesquilinear form on $X \times X$ satisfying

$$\sigma(\alpha\, x_1 + \beta\, x_2, y) = \alpha\, \sigma(x_1, y) + \beta\, \sigma(x_2, y)$$

$$\sigma(x, \alpha\, y_1 + \beta\, y_2) = \bar{\alpha}\, \sigma(x, y_1) + \bar{\beta}\, \sigma(x, y_2),$$

$$|\sigma(x,y)| \leq M\,|x||y| \quad \text{for all } \ x, y \in X \quad \text{(Bounded)}$$

and

$$Re\, \sigma(x,x) \geq \delta\,|x|^2 \ \text{ for all } x \in X \text{ and } \delta > 0 \quad \text{(Coercive).}$$

Then for each $f \in X^*$ there exist a unique solution $x \in X$ to

$$\sigma(x,y) = \langle f, y \rangle_{X^* \times X} \quad \text{for all } \ y \in X$$

and

$$|x|_X \leq \delta^{-1}\,|f|_{X^*}.$$

Proof: Let us define the linear operator S from X^* into X by

$$Sf = x, \quad f \in X^*$$

where $x \in X$ satisfies

$$\sigma(x,y) = \langle f, y \rangle \quad \text{for all } \ y \in X.$$

The operator S is well defined since if $x_1,\ x_2 \in X$ satisfy the above, then $\sigma(x_1 - x_2, y) = 0$ for all $y \in X$ and thus $\delta\,|x_1 - x_2|_X^2 \leq Re\,\sigma(x_1 - x_2, x_1 - x_2) = 0$.

Next we show that dom(S) is closed in X^*. Suppose $f_n \in dom(S)$, i.e., there exists $x_n \in X$ satisfying $\sigma(x_n, y) = \langle f_n, y \rangle$ for all $y \in X$ and $f_n \to f$ in X^* as $n \to \infty$. Then

$$\sigma(x_n - x_m, y) = \langle f_n - f_m, y \rangle \quad \text{for all } \ y \in X$$

Setting $y = x_n - x_m$ in this we obtain

$$\delta\,|x_n - x_m|_X^2 \leq Re\,\sigma(x_n - x_m, x_n - x_m) \leq |f_n - f_m|_{X^*}|x_n - x_m|_X.$$

Thus $\{x_n\}$ is a Cauchy sequence in X and so $x_n \to x$ for some $x \in X$ as $n \to \infty$. Since σ and the dual product are continuous, thus $x = Sf$.

Now we prove that dom(S) $= X^*$. Suppose dom(S) $\neq X^*$. Since dom(S) is closed there exists a nontrivial $x_0 \in X$ such that $\langle f, x_0 \rangle = 0$ for all $f \in$dom(S). Consider the linear functional $F(y) = \sigma(x_0, y),\ y \in X$. Then since σ is bounded $F \in X^*$ and $x_0 = SF$. Thus $F(x_0) = 0$. But since $\sigma(x_0, x_0) = \langle F, x_0 \rangle = 0$, by the coercivity of σ $x_0 = 0$, which is a contradiction. Hence dom(S) $= X^*$. □

Assume that σ is coercive. By the Lax-Milgram theorem A has a bounded inverse $S = A^{-1}$. Thus,

$$dom\,(\tilde{A}) = A^{-1}H.$$

Moreover $\tilde{A}$ is closed. In fact, if

$$x_n \in dom\,(\tilde{A}) \to x \text{ and } f_n = Ax_n \to f \text{ in } H,$$

then since $x_n = Sf_n$ and S is bounded, $x = Sf$ and thus $x \in dom\,(\tilde{A})$ and $\tilde{A}x = f$.

If σ is symmetric, $\sigma(x,y) = (x,y)_X$ defines an inner product on X. and SF coincides with the Riesz representation of $F \in X^*$. Moreover,

$$\langle Ax, y\rangle = \langle Ay, x\rangle \text{ for all } x, y \in X.$$

and thus $\tilde{A}$ is a self-adjoint operator in H.

Example (Laplace operator) Consider $X = H_0^1(\Omega)$, $H = L^2(\Omega)$ and

$$\sigma(u,\phi) = (u,\phi)_X = \int_\Omega \nabla u \cdot \nabla \phi\, dx.$$

Then,

$$Au = -\Delta u = -(\frac{\partial^2}{\partial x_1^2}u + \frac{\partial^2}{\partial x_2^2}u)$$

and

$$dom\,(\tilde{A}) = H^2(\Omega) \cap H_0^1(\Omega).$$

for Ω with C^1 boundary or convex domain Ω.

For $\Omega = (0,1)$ and $f \in L^2(0,1)$

$$\int_0^1 \frac{d}{dx}y\frac{d}{dx}u\, dt = \int_0^1 f(x)y(x)\, dx$$

is equivalent to

$$\int_0^1 \frac{d}{dx}y\,(\frac{d}{dx}u + \int_x^1 f(s)\, ds)\, dx = 0$$

for all $y \in H_0^1(0,1)$. Thus,

$$\frac{d}{dx}u + \int_x^1 f(s)\, ds = c \text{ (a constant)}$$

and therefore $\frac{d}{dx}u \in H^1(0,1)$ and

$$Au = -\frac{d^2}{dx^2}u = f \text{ in } L^2(0,1).$$

Example (Elliptic operator) Consider a second order elliptic equation

$$\mathcal{A}u = -\nabla\cdot(a(x)\nabla u) + b(x)\cdot\nabla u + c(x)u(x) = f(x), \quad \frac{\partial u}{\partial \nu} = g \text{ at } \Gamma_1 \quad u = 0 \text{ at } \Gamma_0$$

where Γ_0 and Γ_1 are disjoint and $\Gamma_0 \cup \Gamma_1 = \Gamma$. Integrating this against a test function ϕ, we have

$$\int_\Omega \mathcal{A}u\phi\, dx = \int_\Omega (a(x)\nabla u\cdot\nabla\phi + b(x)\cdot\nabla u\phi + c(x)u\phi)\, dx - \int_{\Gamma_1} g\phi\, ds_x = \int_\Omega f(x)\phi(x)\, dx,$$

for all $\phi \in C^1(\Omega)$ vanishing at Γ_0. Let $X = H^1_{\Gamma_0}(\Omega)$ is the completion of $C^1(\Omega)$ vanishing at Γ_0 with inner product

$$(u, \phi) = \int_\Omega \nabla u \cdot \nabla \phi \, dx$$

i.e.,

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u|_{\Gamma_0} = 0\}$$

Define the bilinear form σ on $X \times X$ by

$$\sigma(u, \phi) = \int_\Omega (a(x)\nabla u \cdot \nabla \phi + b(x) \cdot \nabla u \phi + c(x) u \phi.$$

Then, by the Green's formula

$$\sigma(u, u) = \int_\Omega (a(x)|\nabla u|^2 + b(x) \cdot \nabla(\frac{1}{2}|u|^2) + c(x)|u|^2)\, dx$$

$$= \int_\Omega (a(x)|\nabla u|^2 + (c(x) - \frac{1}{2}\nabla \cdot b)\, |u|^2)\, dx + + \int_{\Gamma_1} \frac{1}{2} n \cdot b |u|^2 \, ds_x.$$

If we assume

$$0 < \underline{a} \le a(x) \le \bar{a}, \quad c(x) - \frac{1}{2}\nabla \cdot b \ge 0, \quad n \cdot b \ge 0 \text{ at } \Gamma_1,$$

then σ is bounded and coercive with $\delta = \underline{a}$.

The Banach space version of Lax-Milgram theorem is as follows.

Banach-Necas-Babuska Theorem Let V and W be Banach spaces. Consider the linear equation for $u \in W$

$$a(u, v) = f(v) \quad \text{for all } v \in V \tag{1.1}$$

for given $f \in V^*$, where a is a bounded bilinear form on $W \times V$. The problem is well-posed in if and only if the following conditions hold:

$$\begin{gathered} \inf_{u \in W} \sup_{v \in V} \frac{a(u, v)}{|u|_W |v|_V} \ge \delta > 0 \\ a(u, v) = 0 \text{ for all } u \in W \text{ implies } v = 0 \end{gathered} \tag{1.2}$$

Under conditions we have the unique solution $u \in W$ to (1.1) satisfies

$$|u|_W \le \frac{1}{\delta}\, |f|_{V^*}.$$

Proof: Let A be a bounded linear operator from W to V^* defined by

$$\langle Au, v \rangle = a(u, v) \text{ for all } u \in W, \ v \in V.$$

The inf-sup condition is equivalent to for any w?W

$$|Aw|_{V*} \ge \delta |u|_W,$$

and thus the range of A, $R(A)$, is closed in V^* and $N(A)=0$. But since V is reflexive and

$$\langle Au,v\rangle_{V^*\times V}=\langle u,A^*v\rangle_{W\times W^*}$$

from the second condition $N(A^*)=\{0\}$. It thus follows from the closed range and open mapping theorems that A^{-1} is bounded. □

Next, we consider the generalized Stokes system. Let V and Q be Hilbert spaces. We consider the mixed variational problem for $(u,p)\in V\times Q$ of the form

$$a(u,v)+b(p,v)=f(v), \quad b(u,q)=g(q) \tag{1.3}$$

for all $v\in V$ and $q\in Q$, where a and b is bounded bilinear form on $V\times V$ and $V\times Q$. If we define the linear operators $A\in\mathcal{L}(V,V^*)$ and $B\in\mathcal{L}(V,Q^*)$ by

$$\langle Au,v\rangle=a(u,v) \quad \text{and} \quad \langle Bu,q\rangle=b(u,q)$$

then it is equivalent to the operator form:

$$\begin{pmatrix} A & B^* \\ \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ \\ p \end{pmatrix}=\begin{pmatrix} f \\ \\ g \end{pmatrix}.$$

Assume the coercivity on a

$$a(u,u)\geq\delta\,|u|_V^2 \tag{1.4}$$

and the inf-sup condition on b

$$\inf_{q\in P}\sup_{u\in V}\frac{b(u,q)}{|u|_V|q|_Q}\geq\beta>0 \tag{1.5}$$

Note that inf-sup condition that for all q there exists $u\in V$ such that $Bu=q$ and $|u|_V\leq\frac{1}{\beta}|q|_Q$. Also, it is equivalent to $|B^*p|_{V^*}\geq\beta\,|p|_Q$ for all $p\in Q$.

Theorem (Mixed problem) Under conditions (1.4)-(1.5) there exits a unique solution $(u,p)\in V\times Q$ to (1.3) and

$$|u|_V+|p|_Q\leq c\,(|f|_{V^*}+|g|_{Q^*})$$

Proof: For $\epsilon>0$ consider the penalized problem

$$\begin{aligned} &a(u_\epsilon,v)+b(v,P_\epsilon)=f(v), \quad \text{for all } v\in V \\ \\ &-b(u_\epsilon,q)+\epsilon(p_\epsilon,q)_Q=-g(q) \quad \text{for all } q\in Q. \end{aligned} \tag{1.6}$$

By the Lax-Milgram theorem for every $\epsilon>0$ there exists a unique solution. From the first equation

$$\beta\,|p_\epsilon|_Q\leq|f-Au_\epsilon|_{V^*}\leq|f|_{V^*}+M\,|u_\epsilon|_Q.$$

Letting $v=u_\epsilon$ and $q=p_\epsilon$ in the first and second equation, we have

$$\delta\,|u_\epsilon|_V^2+\epsilon\,|p_\epsilon|_Q^2\leq|f|_{V^*}||u_\epsilon|_V+|p_\epsilon|_Q|g|_{Q^*}\leq C\,(|f|_{V^*}+|g|_{Q^*})|u_\epsilon|_V),$$

and thus $|u_\epsilon|_V$ and $|p_\epsilon|_Q$ as well, are bounded uniformly in $\epsilon>0$. Thus, (u_ϵ,p_ϵ) has a weakly convergent subspace to (u,p) in $V\times Q$ and (u,p) satisfies (1.3). □

1.3 Distribution and Generalized Derivatives

In this section we introduce the distribution (generalized function). The concept of distribution is very essential for defining a generalized solution to PDEs and provides the foundation of PDE theory. Let $\mathcal{D}(\Omega)$ be a vector space of all infinitely many continuously differentiable functions $C_0^\infty(\Omega)$ with compact support in Ω. For any compact set K of Ω, let $\mathcal{D}_K(\Omega)$ be the set of all functions $f \in C_0^\infty(\Omega)$ whose support are in K. Define a family of seminorms on $\mathcal{D}(\Omega)$ by

$$p_{K,m}(f) = \sup_{x\in K} \sup_{|s|\le m} |D^s f(x)|$$

where

$$D^s = \left(\frac{\partial}{\partial x_1}\right)^{s_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{s_n}$$

where $s = (s_1, \cdots, s_n)$ is nonnegative integer valued vector and $|s| = \sum s_k \le m$. Then, $\mathcal{D}_K(\Omega)$ is a locally convex topological space.

Definition (Distribution) A linear functional T defined on $C_0^\infty(\Omega)$ is a distribution if for every compact subset K of Ω, there exists a positive constant C and a positive integer k such that

$$|T(\phi)| \le C\, sup_{|s|\le k,\, x\in K} |D^s\phi(x)| \text{ for all } \phi \in \mathcal{D}_K(\Omega).$$

Definition (Generalized Derivative) A distribution S defined by

$$S(\phi) = -T(D_{x_k}\phi) \text{ for all } \phi \in C_0^\infty(\Omega)$$

is called the distributional derivative of T with respect to x_k and we denote $S = D_{x_k}T$.

In general we have

$$S(\phi) = D^sT(\phi) = (-1)^{|s|}\, T(D^s\phi) \text{ for all } \phi \in C_0^\infty(\Omega).$$

This definition is naturally followed from that for f is continuously differentiable

$$\int_\Omega D_{x_k} f\phi\, dx = -\int_\Omega f \frac{\partial}{\partial x_k}\phi\, dx$$

and thus $D_{x_k}f = D_{x_k}T_f = T_{\frac{\partial}{\partial x_k}} f$. Thus, we let D^sf denote the distributional derivative of T_f.

Example (Distribution) (1) For f is a locally integrable function on Ω, one defines the corresponding distribution by

$$T_f(\phi) = \int_\Omega f\phi\, dx \text{ for all } \phi \in C_0^\infty(\Omega).$$

since

$$|T_f(\phi)| \le \int_K |f|\, dx \sup_{x\in K} |\phi(x)|.$$

(2) $T(\phi) = \phi(0)$ defines the Dirac delta δ_0 at $x = 0$, i.e.,

$$|\delta_0(\phi)| \le \sup_{x\in K} |\phi(x)|.$$

(3) Let H be the Heaviside function defined by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

Then,

$$D_{T_H}(\phi) = -\int_{-\infty}^{\infty} H(x)\phi'(x)\,dx = \phi(0)$$

and thus $DT_H = \delta_0$ is the Dirac delta function at $x = 0$.

(4) The distributional solution for $-D^2 u = \delta_{x_0}$ satisfies

$$-\int_{-\infty}^{\infty} u\phi''\,dx = \phi(x_0)$$

for all $\phi \in C_0^\infty(R)$. That is, $u = \frac{1}{2}|x - x_0|$ is the fundamental solution, i.e.,

$$-\int_{-\infty}^{\infty} |x - x_0|\phi''\,dx = \int_{\infty}^{x_0} \phi'(x)\,dx - \int_{x_0}^{\infty} \phi'(x)\,dx = 2\phi(x_0).$$

In general for $d \geq 2$ let

$$G(x, x_0) = \begin{cases} \frac{1}{4\pi}\log|x - x_0| & d = 2 \\ \\ c_d\,|x - x_0|^{2-d} & d \geq 3. \end{cases}$$

Then

$$\Delta G(x, x_0) = 0, \quad x \neq x_0.$$

and $u = G(x, x_0)$ is the fundamental solution to to $-\Delta$ in R^d,

$$-\Delta u = \delta_{x_0}.$$

In fact, let $B_\epsilon = \{|x - x_0| \leq \epsilon\}$ and $\Gamma = \{|x - x_0| = \epsilon\}$ be the surface. By the divergence theorem

$$\int_{R^d \setminus B_\epsilon(x_0)} G(x, x_0)\Delta\phi(x)\,dx = \int_\Gamma \frac{\partial}{\partial\nu}\phi(G(x, x_0) - \frac{\partial}{\partial\nu}G(x, x_0)\phi(s))\,ds$$

$$= \int_\Gamma (\epsilon^{2-d}\frac{\partial\phi}{\partial\nu} - (2 - d)\epsilon^{1-d}\phi(s))\,ds \to \frac{1}{c_d}\phi(x_0)$$

That is, $G(x, x_0)$ satisfies

$$-\int_{R^d} G(x, x_0)\Delta\phi\,dx = \phi(x_0).$$

In general let $\mathcal{L}$ be a linear diffrenrtial operator and $\mathcal{L}^*$ denote the formal adjoint operator of $\mathcal{L}$ An locally integrable function u is said to be a distributional solution to $\mathcal{L}u = T$ where $\mathcal{L}$ with a distribution T if

$$\int_\Omega u(\mathcal{L}^*\phi)\,dx = T(\phi)$$

for all $\phi \in C_0^\infty(\Omega)$.

Definition (Sovolev space) For $1 \le p < \infty$ and $m \ge 0$ the Sobolev space is

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega) : D^s f \in L^p(\Omega),\ |s| \le m\}$$

with norm

$$|f|_{W^{m,p}(\Omega)} = \left(\int_\Omega \sum_{|s| \le m} |D^s f|^p \, dx \right)^{\frac{1}{p}}.$$

That is,

$$|D^s f(\phi)| \le c\, |\phi|_{L^q} \text{ with } \frac{1}{p} + \frac{1}{q} = 1.$$

Remark (1) $X = W^{m,p}(\Omega)$ is complete. In fact If $\{f_n\}$ is Cauchy in X, then $\{D^s f_n\}$ is Cauchy in $L^p(\Omega)$ for all $|s| \le m$. Since $L^p(\Omega)$ is complete, $D^s f_n \to g^s$ in $L^p(\Omega)$. But since

$$\lim_{n\to\infty} \int_\Omega f_n D^s \phi \, dx = \int_\Omega f D^s \phi \, dx = \int g^s \phi \, dx,$$

we have $D^s f = g^s$ for all $|s| \le m$ and $|f_n - f|_X \to 0$ as $n \to \infty$.
(2) $H^{m,p} \subset W^{1,p}(\Omega)$. Let $H^{m,p}(\Omega)$ be the completion of $C^m(\Omega)$ with respect to $W^{I,p}(\Omega)$ norm. That is, $f \in H^{m,p}(\Omega)$ there exists a sequence $f_n \in C^m(\Omega)$ such that $f_n \to f$ and $D^s f_n \to g^s$ strongly in $L^p(\Omega)$ and thus

$$D^s f_n(\phi) = (-1)^{|s|} \int_\Omega D_s f_n \phi \, dx \to (-1)^{|s|} \int_\Omega g^s \phi \, dx$$

which implies $g^s = D^s f$ and $f \in W^{1,p}(\Omega)$.
(3) If Ω has a Lipschitz continuous boundary, then

$$W^{m,p}(\Omega) = H^{m,p}(\Omega).$$

2 Minty–Browder Theorem

Definition (Monotone Mapping)
(a) A mapping $A \subset X \times X^*$ be given. is called *monotone* if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \ge 0 \quad \text{for all } [x_1, y_1], [x_2, y_2] \in A.$$

(b) A monotone mapping A is called *maximal monotone* if any monotone extension of A coincides with A, i.e., if for $[x, y] \in X \times X^*$, $\langle x - u, y - v \rangle \ge 0$ for all $[u, v] \in A$ then $[x, y] \in A$.
(c) The operator A is called *coercive* if for all sequences $[x_n, y_n] \in A$ with $\lim_{n\to\infty} |x_n| = \infty$ we have

$$\lim_{n\to\infty} \frac{\langle x_n, y_n \rangle}{|x_n|} = \infty.$$

(d) Assume that A is single-valued with $\mathrm{dom}(A) = X$. The operator A is called hemicontinuous on X if for all $x_1, x_2, x \in X$, the function defined by

$$t \in R \to \langle x, A(x_1 + t x_2)\rangle$$

is continuous on R.

For example, let F be the duality mapping of X. Then F is monotone, coercive and hemicontinuous. Indeed, for $[x_1, y_1], [x_2, y_2] \in F$ we have

$$\langle x_1 - x_2, y_1 - y_2\rangle = |x_1|^2 - \langle x_1, y_2\rangle - \langle x_2, y_1\rangle + |x_2|^2 \geq (|x_1| - |x_2|)^2 \geq 0, \qquad (2.1)$$

which shows monotonicity of F. Coercivity is obvious and hemicontinuity follows from the continuity of the duality product.

<u>**Lemma 1**</u> Let X be a finite dimensional Banach space and A be a hemicontinuous monotone operator from X to X^*. Then A is continuous.

Proof: We first show that A is bounded on bounded subsets. In fact, otherwise there exists a sequence $\{x_n\}$ in X such that $|Ax_n| \to \infty$ and $x_n \to x_0$ as $n \to \infty$. By monotonicity we have

$$\langle x_n - x, \frac{Ax_n}{|Ax_n|} - \frac{Ax}{|Ax_n|}\rangle \geq 0 \quad \text{for all } \ x \in X.$$

Without loss of generality we can assume that $\frac{Ax_n}{|Ax_n|} \to y_0$ in X^* as $n \to \infty$. Thus

$$\langle x_0 - x, y_0\rangle \geq 0 \quad \text{for all } \ x \in X$$

and therefore $y_0 = 0$. This is a contradiction and thus A is bounded. Now, assume $\{x_n\}$ converges to x_0 and let y_0 be a cluster point of $\{Ax_n\}$. Again by monotonicity of A

$$\langle x_0 - x, y_0 - Ax\rangle \geq 0 \quad \text{for all } \ x \in X.$$

Setting $x = x_0 + t\,(u - x_0)$, $t > 0$ for arbitrary $u \in X$, we have

$$\langle x_0 - u, y_0 - A(x_0 + t\,(u - x_0)) \geq 0\rangle \quad \text{for all } \ u \in X.$$

Then, letting limit $t \to 0^+$, by hemicontinuity of A we have

$$\langle x_0 - u, y_0 - Ax_0\rangle \geq 0 \quad \text{for all } \ u \in X,$$

which implies $y_0 = Ax_0$. $\square$

<u>**Lemma 2**</u> Let X be a reflexive Banach space and $A : X \to X^*$ be a hemicontinuous monotone operator. Then A is maximumal monotone.

Proof: For $[x_0, y_0] \in X \times X^*$

$$\langle x_0 - u, y_0 - Au\rangle \geq 0 \quad \text{for all } \ u \in X.$$

Setting $u = x_0 + t\,(x - x_0)$, $t > 0$ and letting $t \to 0^+$, by hemicontinuity of A we have

$$\langle x_0 - x, y_0 - Ax_0\rangle \geq 0 \quad \text{for all } \ x \in X.$$

Hence $y_0 = Ax_0$ and thus A is maximum monotone. $\Box$

The next theorem characterizes maximal monotone operators by a range condition.

Minty–Browder Theorem Assume that X, X^* are reflexive and strictly convex. Let F denote the duality mapping of X and assume that $A \subset X \times X^*$ is monotone. Then A is maximal monotone if and only if

$$Range\,(\lambda F + A) = X^*$$

for all $\lambda > 0$ or, equivalently, for some $\lambda > 0$.

Proof: Assume that the range condition is satisfied for some $\lambda > 0$ and let $[x_0, y_0] \in X \times X^*$ be such that

$$\langle x_0 - u, y_0 - v\rangle \geq 0 \quad \text{for all } [u, v] \in A.$$

Then there exists an element $[x_1, y_1] \in A$ with

$$\lambda F x_1 + y_1 = \lambda F x_0 + y_0. \tag{2.2}$$

From these we obtain, setting $[u, v] = [x_1, y_1]$,

$$\langle x_1 - x_0, Fx_1 - Fx_0\rangle \leq 0.$$

By monotonicity of F we also have the converse inequality, so that

$$\langle x_1 - x_0, Fx_1 - Fx_0\rangle = 0.$$

From (2.1) this implies that $|x_1| = |x_0|$ and $\langle x_1, Fx_0\rangle = |x_1|^2$, $\langle x_0, Fx_1\rangle = |x_0|^2$. Hence $Fx_0 = Fx_1$ and

$$\langle x_1, Fx_0\rangle = \langle x_0, Fx_0\rangle = |x_0|^2 = |Fx_0|^2.$$

If we denote by F^* the duality mapping of X^* (which is also single-valued), then the last equation implies $x_1 = x_0 = F^*(Fx_0)$. This and (2.2) imply that $[x_0, y_0] = [x_1, y_1] \in A$, which proves that A is maximal monotone. $\Box$

In stead of the detailed proof of "only if' part of Theorem, we state the following results. $\Box$

Corollary Let X be reflexive and A be a monotone, everywhere defined, hemicontinous operator. If A is coercive, then $R(A) = X^*$.

Proof: Suppose A is coercive. Let $y_0 \in X^*$ be arbitrary. By the Appland's renorming theorem, we may assume that X and X^* are strictly convex Banach spaces. It then follows from Theorem that every $\lambda > 0$, equation

$$\lambda\, Fx_\lambda + Ax_\lambda = y_0$$

has a solution $x_\lambda \in X$. Multiplying this by x_λ,

$$\lambda\, |x_\lambda|^2 + \langle x_\lambda, Ax_\lambda\rangle = \langle y_0, x_\lambda\rangle.$$

and thus

$$\frac{\langle x_\lambda, Ax_\lambda\rangle}{|x_\lambda|_X} \le |y_0|_{X^*}$$

Since A is coercive, this implies that $\{x_\lambda\}$ is bounded in X as $\lambda \to 0^+$. Thus, we may assume that x_λ converges weakly to x_0 in X and Ax_λ converges strongly to y_0 in X^* as $\lambda \to 0^+$. Since A is monotone

$$\langle x_\lambda - x, y_0 - \lambda\, Fx_\lambda - Ax\rangle \ge 0,$$

and letting $\lambda \to 0^+$, we have

$$\langle x_0 - x, y_0 - Ax\rangle \ge 0,$$

for all $x \in X$. Since A is maximal monotone, this implies $y_0 = Ax_0$. Hence, we conclude $R(A) = X^*$. $\square$

Theorem (Galerkin Approximation) Assume X is a reflexive, separable Banach space and A is a bounded, hemicontinuous, coercive monotone operator from X into X^*. Let $X_n = span\{\phi\}_{i=1}^n$ satisfies the density condition: for each $\psi \in X$ and any $\epsilon > 0$ there exists a sequence $\psi_n \in X_n$ such that $|\psi - \psi_n| \to 0$ as $n \to \infty$. The x_n be the solution to

$$\langle \psi, Ax_n\rangle = \langle \psi, f\rangle \quad \text{for all } \ \psi \in X_n, \tag{2.3}$$

then there exists a subsequence of $\{x_n\}$ that converges weakly to a solution to $Ax = f$.

Proof: Since $\langle x, Ax\rangle/|x|_X \to \infty$ as $|x|_X \to \infty$ there exists a solution x_n to (2.3) and $|x_n|_X$ is bounded. Since A is bounded, thus Ax_n bounded. Thus there exists a subsequence of $\{n\}$ (denoted by the same) such that x_n converges weakly to x in X and Ax_n converges weakly in X^*. Since

$$\lim_{n\to\infty} \langle \psi, Ax_n\rangle = \lim_{n\to\infty} \left(\langle \psi_n, f\rangle + \langle \psi - \psi_n, Ax_n\rangle\right) = \langle \psi, f\rangle$$

Ax_n converges weakly to f. Since A is monotone

$$\langle x_n - u, Ax_n - Au\rangle \ge 0 \quad \text{for all } \ u \in X$$

Note that

$$\lim_{n\to\infty} \langle x_n, Ax_n\rangle = \lim_{n\to\infty} \langle x_n, f\rangle = \langle x, f\rangle.$$

Thus taking limit $n \to \infty$, we obtain

$$\langle x - u, f - Au\rangle \ge 0 \text{ for all } \ u \in X.$$

Since A is maximum monotone this implies $Ax = f$. $\square$

The main theorem for monotone operators applies directly to the model problem involving the p-Laplace operator

$$-div(|\nabla u|^{p-2}\nabla u) = f \text{ on } \Omega$$

(with appropriate boundary conditions) and

$$-\Delta u + c\,u = f, \quad -\frac{\partial}{\partial n}u \in \beta(u). \text{ at } \partial\Omega$$

with β maximal monotone on R. Also, nonlinear problems of non-variational form are applicable, e.g.,

$$Lu + F(u) = f \text{ on } \Omega$$

where

$$L(u) = -div(\sigma(\nabla u) - \vec{b}\,u)$$

and we are looking for a solution $u \in W_0^{1,p}(\Omega),\ 1 < p < \infty$. We assume the following conditions:
(i) Monotonicity for the principle part $L(u)$:

$$(\sigma(\xi) - \sigma(\eta), \xi - \eta)_{R^n} \geq 0 \text{ for all } \xi,\ \eta \in R^n.$$

(ii) Monotonicity for $F = F(u)$:

$$(F(u) - F(v), u - v) \geq 0 \text{ for all } u,\ v \in R.$$

(iii) Coerciveness and Growth condition: for some $c,\ d > 0$

$$(\sigma(\xi), \sigma) \geq c\,|\xi|^p, \quad |\sigma(\xi)| \leq d\,(1 + |\xi|^{p-1})$$

hold for all $\xi \in R^n$.

3 Convex Functional and Subdifferential

Definition (Convex Functional) (1) A proper convex functional on a Banach space X is a function φ from X to $(-\infty, \infty]$, not identically $+\infty$ such that

$$\varphi((1-\lambda)\,x_1 + \lambda\,x_2) \leq (1-\lambda)\,\varphi(x_1) + \lambda\,\varphi(x_2)$$

for all $x_1,\ x_2 \in X$ and $0 \leq \lambda \leq 1$.
(2) A functional $\varphi : X \to R$ is said to be lower-semicontinuous if

$$\varphi(x) \leq \liminf_{y \to x}\ \varphi(y) \quad \text{for all } \ x \in X.$$

(3) A functional $\varphi : X \to R$ is said to be weakly lower-semicontinuous if

$$\varphi(x) \leq \liminf_{n \to \infty}\ \varphi(x_n)$$

for all weakly convergent sequence $\{x_n\}$ to x.
(4) The subset $D(\varphi) = \{x \in X; \varphi(x) < \infty\}$ of X is called the domain of φ.
(5) The epigraph of φ is defined by $epi(\varphi) = \{(x, c) \in X \times R : \varphi(x) \leq c\}$.

Lemma 3 A convex functional φ is lower-semicontinuous if and only if it is weakly lower-semicontinuous on X.

Proof: Since the level set $\{x \in X : \varphi(x) \leq c\}$ is a closed convex subset if φ is lower-semicontinuous. Thus, the claim follows the fact that a convex subset of X is closed if and only if it is weakly closed.

__**Lemma 4**__ If φ be a proper lower-semicontinuous, convex functional on X, then φ is bounded below by an affine functional, i.e., there exist $x^* \in X^*$ and $c \in R$ such that

$$\varphi(x) \geq \langle x^*, x\rangle + \beta, \quad x \in X.$$

Proof: Let $x_0 \in X$ and $\beta \in R$ be such that $\varphi(x_0) > c$. Since φ is lower-semicontinuous on X, there exists an open neighborhood $V(x_0)$ of X_0 such that $\varphi(x) > c$ for all $x \in V(x_0)$. Since the ephigraph $epi(\varphi)$ is a closed convex subset of the product space $X \times R$. It follows from the separation theorem for convex sets that there exists a closed hyperplane $H \subset X \times R$;

$$H = \{(x,r) \in X \times R : \langle x_0^*, x\rangle + r = \alpha\} \quad \text{with } \ x_0^* \in X^*, \ \alpha \in R,$$

that separates $epi(\varphi)$ and $V(x_0) \times (-\infty, c)$. Since $\{x_0\} \times (-\infty, c) \subset \{(x,r) \in X \times R : \langle x_0^*, x\rangle + r < \alpha\}$ it follows that

$$\langle x_0^*, x\rangle + r > \alpha \quad \text{for all } \ (x,c) \in epi(\varphi)$$

which yields the desired estimate.

Theorem C.6 If $F : X \to (-\infty, \infty]$ is convex and bounded on an open set U, then F is continuous on U.

Proof: We choose $M \in R$ such that $F(x) \leq M - 1$ for all $x \in U$. Let $\hat{x}$ be any element in U. Since U is open there exists a $\delta > 0$ such that the open ball $\{x \in X : |x - \hat{x}| < \delta$ is contained in U. For any $epsilon \in (0,1)$, let $\theta = \dfrac{\epsilon}{M - F(\hat{x})}$. Then for $x \in X$ satisfying $|x - \hat{x}| < \theta\,\delta$

$$|\frac{x - \hat{x}}{\theta} + \hat{x} - \hat{x}| = \frac{|x - \hat{x}|}{\theta} < \delta$$

Hence $\dfrac{x - \hat{x}}{\theta} + \hat{x} \in U$. By the convexity of F

$$F(x) \leq (1-\theta)F(\hat{x}) + \theta\, F(\frac{x - \hat{x}}{\theta} + \hat{x}) \leq (1-\theta)F(\hat{x}) + \theta\, M$$

and thus

$$F(x) - F(\hat{x}) < \theta(M - F(\hat{x}) = \epsilon$$

Similarly, $\dfrac{\hat{x} - x}{\theta} + \hat{x} \in U$ and

$$F(\hat{x}) \leq \frac{\theta}{1+\theta}\, F(\frac{\hat{x} - x}{\theta} + \hat{x}) + \frac{1}{1+\theta}\, F(x) < \frac{\theta M}{1+\theta} + \frac{1}{1+\theta}\, F(x)$$

which implies

$$F(x) - F(\hat{x}) > -\theta(M - F(\hat{x}) = -\epsilon$$

Therefore $|F(x) - F(\bar{x})| < \epsilon$ if $|x - \hat{x}| < \theta\delta$ and F is continuous in U. $\square$

__**Definition (Subdifferential)**__ Given a proper convex functional φ on a Banach space X the subdifferential of $\partial\varphi(x)$ is a subset in X^*, defined by

$$\partial\varphi(x) = \{x^* \in X^* : \varphi(y) - \varphi(x) \geq \langle x^*, y - x\rangle \ \text{ for all } \ y \in X\}.$$

Since for $x_1^* \in \partial\varphi(x_1)$ and $x_2^* \in \partial\varphi(x_2)$,

$$\varphi(x_1) - \varphi(x_2) \leq \langle x_2^*, x_1 - x_2 \rangle$$

$$\varphi(x_2) - \varphi(x_1) \leq \langle x_1^*, x_2 - x_1 \rangle$$

it follows that $\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq 0$. Hence $\partial\varphi$ is a monotone operator from X into X^*.

Example 1 Let φ be Gateaux differentiable at x. i.e., there exists $w^* \in X^*$ such that

$$\lim_{t \to 0^+} \frac{\varphi(x + t\,v) - \varphi(x)}{t} = \langle w^*, h \rangle \quad \text{for all } \ h \in X$$

and w^* is the Gateaux differential of φ at x and is denoted by $\varphi'(x)$. If φ is convex, then φ is subdifferentiable at x and $\partial\varphi(x) = \{\varphi'(x)\}$. Indeed, for $v = y - x$

$$\frac{\varphi(x + t\,(y - x)) - \varphi(x)}{t} \leq \varphi(y) - \varphi(x), \quad 0 < t < 1$$

Letting $t \to 0^+$ we have

$$\varphi(y) - \varphi(x) \geq \langle \varphi'(x), y - x \rangle \quad \text{for all } \ y \in X,$$

and thus $\varphi'(x) \in \partial\varphi(x)$. On the other hand if $w^* \in \partial\varphi(x)$ we have for $y \in X$ and $t > 0$

$$\frac{\varphi(x + t\,y) - \varphi(x)}{t} \geq \langle w^*, y \rangle.$$

Taking limit $t \to 0^+$, we obtain

$$\langle \varphi'(x) - w^*, y \rangle \geq 0 \quad \text{for all } \ y \in X.$$

This implies $w^* = \varphi'(x)$.

Example 2 If $\varphi(x) = \frac{1}{2}\,|x|^2$ then we will show that $\partial\varphi(x) = F(x)$, the duality mapping. In fact, if $x^* \in F(x)$, then

$$\langle x^*, x - y, \rangle = |x|^2 - \langle y, x^* \rangle \geq \frac{1}{2}\,(|x|^2 - |y|^2) \quad \text{for all } \ y \in X.$$

Thus $x^* \in \partial\varphi(x)$. Conversely, if $x^* \in \partial\varphi(x)$, then

$$\frac{1}{2}\,(|y|^2 - |x|^2) \geq \langle x^*, y - x \rangle \quad \text{for all } \ y \in X \tag{3.1}$$

We let $y = t\,x,\ 0 < t < 1$ and obtain

$$\frac{1 + t}{2}\,|x|^2 \leq \langle x, x^* \rangle$$

and thus $|x|^2 \leq \langle x, x^* \rangle$. Similarly, if $t > 1$, then we conclude $|x|^2 \geq \langle x, x^* \rangle$ and therefore $|x|^2 = \langle x, x^* \rangle$ and $|x^*| \geq |x|$. On the other hand, letting $y = x + \lambda\,u,\ \lambda > 0$ in (3.1), we have

$$\lambda\,\langle x^*, u \rangle \leq \frac{1}{2}\,(|x + \lambda\,u|^2 - |x|^2) \leq \lambda\,|u||x| + \lambda\,|u|^2,$$

which implies $\langle x^*, u\rangle \le |u||x|$. Hence $|x^*| \le |x|$ and we obtain $|x|^2 = |x^*|^2 = \langle x^*, x\rangle$.

Example 3 Let K be a closed convex subset of X and I_K be the indicator function of K, i.e.,

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{otherwise.} \end{cases}$$

Obviously, I_K is convex and lower-semicontinuous on X. By definition we have for $x \in K$

$$\partial I_K(x) = \{x^* \in X^* : \langle x^*, x - y\rangle \ge 0 \ \text{ for all } \ y \in K\}$$

Thus $D(I_K) = D(\partial I_K) = K$ and $\partial_K(x) = \{0\}$ for each interior point of K. Moreover, if x lies on the boundary of K, then $\partial I_K(x)$ coincides with the cone of normals to K at x.

Note that $\partial F(x)$ is closed and convex and may be empty.

Theorem C.10 If a convex function F is continuous at $\bar{x}$ then $\partial F(\bar{x})$ is non empty.

Proof: Since F is continuous at x for any $\epsilon > 0$ there exists a neighborhood U_ϵ of $\bar{x}$ such that

$$F(x) \le F(\bar{x}) + \epsilon, \quad x \in U_\epsilon.$$

Then $U_\epsilon \times (F(\bar{x}) + \epsilon, \infty)$ is an open set in $X \times R$ and is contained in epi F. Hence (epi $F)^o$ is non empty. Since F is convex epi F is convex and (epi $F)^o$ is convex. For any neighborhood of O of $(\bar{x}, F(\bar{x}))$ there exists a $t < 1$ such that $(\bar{x}, tF(\bar{x})) \in O$. But, $tF(\bar{x}) < F(\bar{x})$ and so $(\bar{x}, tF(\bar{x})) \notin$ epi F. Thus $(\bar{x}, F(\bar{x})) \notin$ (epi $F)^o$. By the Hahn Banach separation theorem, there exists a closed hyperplane $S = \{(x, a) \in X \times R : \langle x^*, x\rangle + \alpha\, a = \beta\}$ for nontrivial $(x^*, \alpha) \in X^* \times R$ and $\beta \in R$ such that

$$\begin{aligned} &\langle x^*, x\rangle + \alpha\, a > \beta \quad \text{for all } \ (x, a) \in (\text{epi } F)^o \\ &\langle x^*, \bar{x}\rangle + \alpha\, F(\bar{x}) = \beta. \end{aligned} \tag{3.2}$$

Since $\overline{(\text{epi } F)^o} = \overline{\text{epi } F}$ every neighborhood of $(x, a) \in$ epi F contains an element of (epi $\varphi)^o$. Suppose $\langle x^*, x\rangle + \alpha\, a < \beta$. Then

$$\{(x', a') \in X \times R : \langle x^*, x'\rangle + \alpha\, a' < \beta\}$$

is an neighborhood of (x, a) and contains an element of (epi $F)^o$, which contradicts to (3.2). Hence

$$\langle x^*, x\rangle + \alpha\, a \ge \beta \quad \text{for all } \ (x, a) \in \text{epi } F. \tag{3.3}$$

Suppose $\alpha = 0$. For any $u \in U_\epsilon$ there is an $a \in R$ such that $F(u) \le a$. Then from (3.3)

$$\langle x^*, u\rangle = \langle x^*, u\rangle + \alpha\, a \ge \beta$$

and thus

$$\langle x^*, u - \bar{x}\rangle \ge 0 \text{ for all } u \in U_\epsilon.$$

Choose a δ such that $|u - \bar{x}| \le \delta$ implies $u \in U$. For any nonzero element $x \in X$ let $t = \frac{\delta}{|x|}$. Then $|(tx + \bar{x}) - \bar{x}| = |tx| = \delta$ so that $tx + \bar{x} \in U_\epsilon$. Hence

$$\langle x^*, x\rangle = \langle x^*, (tx + \bar{x}) - \bar{x}\rangle / t \ge 0.$$

Similarly, $-t\,x+\bar{x}\in U_\epsilon$ and

$$\langle x^*,x\rangle=\langle x^*,(-tx+\bar{x})-\bar{x}\rangle/(-t)\leq 0.$$

Thus, $\langle x^*,x\rangle$ and $x^*=0$, which is a contradiction. Therefore α is nonzero. It now follows from (3.2)–(3.3) that

$$\langle -\frac{x^*}{\alpha},x-\bar{x}\rangle+F(\bar{x})\leq F(x)$$

for all $x\in X$ and thus $-\frac{x^*}{\alpha}\in\partial F(\bar{x})$. $\square$

Definition (Lower semi-continuous) (1) A functional F is lower-semi continuous if

$$\liminf_{n\to\infty}F(x_n)\geq F(\lim_{n\to\infty}x_n)$$

(2) A functional F is weakly lower-semi continuous if

$$\liminf_{n\to\infty}F(x_n)\geq F(w-\lim_{n\to\infty}x_n)$$

Theorem (Lower-semicontinuous) (1) Norm is weakly lower-semi continuous.
(2) A convex lower-semicontinuous functional is weakly lower-semi continuous.

Proof: Assume $x_n\to x$ weakly in X. Let $x^*\in F(x)$, i.e., $\langle x^*,x\rangle=|x^*||x|$. Then, we have

$$|x|^2=\lim_{n\to\infty}\langle x^*,x_n\rangle$$

and

$$|\langle x^*,x_n\rangle|\leq|x_n||x^*|.$$

Thus,

$$\liminf_{n\to\infty}\,|x_n|\geq|x|.$$

(2) Since F is convex,

$$F(\sum_k\,t_kx_k)\leq\sum_k\,t_k\,F(x_k)$$

for all convex combination of x_k, i.e., $\sum\sum_k t_k=1,\ t_k\geq 0$. By the Mazur lemma there exists a sequence of convex combination of weak convergent sequence $(\{x_k\},\{F(x_k)\})$ to $(x,F(x))$ in $X\times R$ that converges strongly to $(x,F(x))$ and thus

$$F(x)\leq\liminf n\to\infty\,F(x_n).\square$$

Theorem (Weierstrass) If $\varphi(x)$ is a lower-semicontinuous proper convex functional on a reflexible Banach X satisfying the coercivity $\lim_{|x|\to\infty}\varphi(x)=\infty$. Then there exists a minimizer $x^*\in X$ of φ. A minimizer x^* satisfies the (necessary) condition

$$0\in\partial\varphi(x^*).$$

Proof: Since $\varphi(x_0)$ is coercive there exist a bounden minimizing sequence $\{x_n\}$ such that $\lim_{n\to\infty}\varphi(x_n)=\eta=\inf_{x\in X}\varphi(x)=0$. Since X is reflexible, there exits a weakly convergent subsequence x_{n_k} to $x^* \in X$. Since if the convex functional is lower-semicontinuous, then is weakly lower-semicontinuous. Thus, $\eta = \varphi(x^*)$. Since $\varphi(x) - \varphi(x^*) \geq 0$ for all $x \in X$, $0 \in \partial\varphi(x^*)$. $\square$

Theorem(Rockafellar) Let X be real Banach space. If φ is lower-semicontinuous proper convex functional on X, then $\partial\varphi$ is a maximal monotone operator from X into X^*.

Proof: We prove the theorem when X is reflexive. By Apuland theorem we can assume that X and X^* are strictly convex. By Minty-Browder theorem $\partial\varphi$ it suffices to prove that $R(F + \partial\varphi) = X^*$. For $x_0^* \in X^*$ we must show that equation $x_0^* \in Fx + \partial\varphi(x)$ has at least a solution x_0 Define the proper convex functional on X by

$$f(x) = \frac{1}{2}\,|x|_X^2 + \varphi(x) - \langle x_0^*, x\rangle.$$

Since f is lower-semicontinuous and $f(x) \to \infty$ as $|x| \to \infty$ there exists $x_0 \in D(f)$ such that $f(x_0) \leq f(x)$ for all $x \in X$. Since F is monotone

$$\varphi(x) - \varphi(x_0) \geq \langle x_0^*, x - x_0, \rangle - \langle x - x_0, F(x)\rangle.$$

Setting $x_t = x_0 + t\,(u - x_0)$ and since φ is convex, we have

$$\varphi(u) - \varphi(x_0) \geq \frac{1}{t}(\varphi(x_t) - \varphi(x_0)) \geq \langle x_0^*, u - x_0, \rangle - \langle F(x_t), u - x_0\rangle.$$

Taking limit $t \to 0^+$, we obtain

$$\varphi(u) - \varphi(x_0) \geq \langle x_0^*, u - x_0\rangle - \langle F(x_0), u - x_0\rangle,$$

which implies $x_0^* - F(x_0) \in \partial\varphi(x_0)$. $\square$

We have the perturbation result.

Theorem Assume that X is a real Hilbert space and that A is a maximal monotone operator on X. Let φ be a proper, convex and lower semi-continuous functional on X satisfying $dom(A) \cap dom(\partial\varphi)$ is not empty and

$$\varphi((I + \lambda\,A)^{-1}x) \leq \varphi(x) + \lambda\,M, \qquad \text{for all } \lambda > 0,\ x \in D(\varphi),$$

where M is some non-negative constant. Then the operator $A + \partial\varphi$ is maximal monotone.

We use the following lemma.

Lemma Let A and B be m-dissipative operators on X. Then for every $y \in X$ the equation

$$y \in -Ax - B_\lambda x \tag{3.4}$$

has a unique solution $x \in dom\,(A)$.

Proof: Equation (3.4) is equivalent to $y = x_\lambda - w_\lambda - B_\lambda x_\lambda$ for some $w_\lambda \in A(x_\lambda)$. Thus,

$$x_\lambda - \frac{\lambda}{\lambda+1}w_\lambda = \frac{\lambda}{\lambda+1}y + \frac{1}{\lambda+1}(x_\lambda + \lambda\,B_\lambda x_\lambda)$$

$$= \frac{\lambda}{\lambda+1}y + \frac{1}{\lambda+1}(I - \lambda\,B)^{-1}.$$

Since A is m-dissipative, we conclude that (3.4) is equivalent to that x_λ is the fixed point of the operator

$$\mathcal{F}_\lambda x = (I - \frac{\lambda}{\lambda+1}\,A)^{-1}(\frac{\lambda}{\lambda+1}\,y + \frac{1}{\lambda+1}\,(I - \lambda\,B)^{-1}x).$$

By m-dissipativity of the operators A and B their resolvents are contractions on X and thus

$$|\mathcal{F}_\lambda x_1 - \mathcal{F}_\lambda x_2| \leq \frac{\lambda}{\lambda+1}\,|x_1 - x_2| \text{ for all } \lambda > 0, \;\; x_1,\; x_2 \in X.$$

Hence, $\mathcal{F}_\lambda$ has the unique fixed point x_λ and $x_\lambda \in dom\,(A)$ solves (3.4). $\square$

Proof of Theorem: From Lemma there exists x_λ for $y \in X$ such that

$$y \in x_\lambda - (-A)_\lambda x_\lambda + \partial\varphi(x_\lambda)$$

Moreover, one can show that $|x_\lambda|$ is bounded uniformly. Since

$$y - x_\lambda + (-A)_\lambda x_\lambda \in \partial\varphi(x_\lambda)$$

for $z \in X$

$$\varphi(z) - \varphi(x_\lambda) \geq (z - x_\lambda, y - x_\lambda + (-A)_\lambda x_\lambda)$$

Letting $\lambda(I + \lambda A)^{-1}x$, so that $z - x_\lambda = \lambda(-A)_\lambda x_\lambda$ and we obtain

$$(\lambda(-A)_\lambda x_\lambda, y - x_\lambda + (-A)_\lambda x_\lambda) \leq \varphi((I + \lambda\,A)^{-1}) - \varphi(x_\lambda) \leq \lambda\,M,$$

and thus

$$|(-A)_\lambda x_\lambda|^2 \leq |(-A)_\lambda x_\lambda||y - x_\lambda| + M.$$

Since $x_\lambda|$ is bounded and so that $|(-A)_\lambda x_\lambda|$. $\square$