Take Home Final, Friday, April 29 and Due May 10th

Problem 1 Consider the beam equation

$$\rho(x)\,\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}(EI(x)\frac{\partial^2 u}{\partial x^2} + Cd(x)\frac{\partial^3 u}{\partial t \partial x^2}) = 0, \quad x \in (0, L)$$

where the Young moduli $0 < \bar{c} \leq EI(x) \leq \bar{c}$ and the damping moduli $Cd(x) \geq 0$ is bounded and the mass $0 < \bar{\rho} \leq \rho(x) \leq \overline{\rho}$. Consider the following boundary conditions.
(a) (simply supported) $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t)$ and $\frac{\partial^3 u}{\partial x^3}(0,t) = \frac{\partial u^3}{\partial x^3}(L,t) = 0$.
(b) (fixed-free ends) $u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u^2}{\partial x^2}(L,t) = \frac{\partial u^3}{\partial x^3}(L,t) = 0$.
Use the Gelfand triple formulation in Example, page 24 to prove the well-posedness of the equation.

Problem 2 Consider the nonlinear heat equation

$$\frac{\partial}{\partial t}u = \Delta u - u^3, \quad u = 0 \text{ at } \partial\Omega$$

Show the convergence of the successive iterate (see, (1.4), page 87):

$$\frac{u^n - u^{n-1}}{\lambda} = \Delta u^n - (u^{n-1})^2 u^n.$$

First show that $|u^n|_\infty \leq |u^0|_\infty$ and let $D = \{|u|_\infty \leq |u^0|_\infty\} = D_\alpha$. Then, use Theorem 5.0, page 87 to prove the convergence.