In this course we discuss the well-posededness of the evolution equations in Banach spaces. Such problems arise in PDEs dynamics and functional equations. We develop the linear and nonlinear theory for the corresponding solution semigroups. The lectures include for example the Hille-Yosiida theory, Lumer-Philiips theory for linear semigroup and Crandall-Liggett theory for nonlinear conrtractive semigroup and Crandall-Pazy theory for nonlinear evolution equations. Especially, (numerical) approximation theory for PDE solutions are discussed based on Trotter-Kato theory and Takahashi-Oharu theory, Chernoff theory and the operator splitting method. The theory and its applications are examined and demonstrated using many motivated PDE examples including linear dynamics (e.g. heat, wave and hyperbolic equations) and nonlinear dynamics (e.g. nonlinear diffusion, conservation law, Hamilton-Jacobi and Navier-Stokes equations). A new class of PDE examples are formulated and the detailed applications of the theory is carried out.

The lecture also covers the basic elliptic theory via Lax-Milgram, Minty-Browder theory and convex optimization. Functional analytic methods are also introduced for the basic PDEs theory.

The students are expected to have the basic knowledge in real and functional analysis and PDEs.

Lecture notes will be provided. Reference book: "Evolution equations and Approximation" K. Ito and F. Kappel, World Scientific.

1 Linear Cauchy problem and C_0-semigroup theory

In this section we discuss the Cauchy problem of the form

$$\frac{d}{dt}u(t) = Au(t) + f(t), \quad u(0) = u_0 \in X$$

in a Banach space X, where $u_0 \in X$ is the initial condition and $f \in L^1(0,T;X)$. Such problems arise in PDE dynamics and functional equations.

We construct the mild solution $u(t) \in C(0,T;X)$:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds \tag{1.1}$$

where a family of bounded linear operator $\{S(t),\ t \ge 0\}$ is C_0-semigroup on X.

Definition (C_0 semigroup) (1) Let X be a Banach space. A family of bounded linear operators $\{S(t),\ t \ge 0\}$ on X is called a strongly continuous (C_0) semigroup if

$$S(t+s) = S(t)S(s) \text{ for } t,\ s \ge 0 \text{ with } S(0) = I$$

$$|S(t)\phi - \phi| \to 0 \text{ as } t \to 0^+ \quad \text{for all } \phi \in X.$$

(2) A linear operator A in X defined by

$$A\phi = \lim_{t\to 0^+} \frac{S(t)\phi - \phi}{t} \tag{1.2}$$

with

$$dom(A) = \{\phi \in X : \text{the strong limit of } \lim_{t\to 0^+} \frac{S(t)\phi-\phi}{t} \text{ in } X \text{ exists}\}.$$

is called the infinitesimal generator of the C_0 semigroup $S(t)$.

In this section we present the basic theory of the linear C_0-semigroup on a Banach space X. The theory allows to analyze a wide class of the physical and engineering dynamics using the unified framework. We also present the concrete examples to demonstrate the theory. There is a necessary and sufficient condition (Hile-Yosida Theorem) for a closed, densely defined linear A in X to be the infinitesimal generator of the C_0 semigroup $S(t)$. Moreover, we will show that the mild solution $u(t)$ satisfies

$$\langle u(t), \psi\rangle = \langle u_0, \psi\rangle + \int (\langle x(s), A^*\psi\rangle + \langle f(s), \psi\rangle\, ds \tag{1.3}$$

for all $\psi \in dom\,(A^*)$.

Examples (1) For $A \in \mathcal{L}(X)$, define a sequence of linear operators in X

$$S_N(t) = \sum_k \frac{1}{k!}(At)^k.$$

Then

$$|S_N(t)| \le \sum \frac{1}{k!}(|A|t)^k \le e^{|A|\,t}$$

and

$$\frac{d}{dt}S_N(t) = AS_{N-1}(t)$$

Since

$$S(t) = e^{At} = \lim_{N\to\infty} S_N(t), \tag{1.4}$$

in the operator norm, we have

$$\frac{d}{dt}S(t) = AS(t) = S(t)A.$$

(2) Consider the the hyperbolic equation

$$u_t + u_x = 0, \quad u(0,x) = u_0(x) \text{ in } (0,1). \tag{1.5}$$

Define the semigroup $S(t)$ of translations on $X = L^2(0,1)$ by

$$[S(t)u_0](x) = \tilde{u}_0(x-t), \quad \text{where} \quad \tilde{u}_0(x) = 0, x \le 0, \quad \tilde{u}_0 = u_0 \text{ on } [0,1]. \tag{1.6}$$

Then, $\{S(t),\ t \ge 0\}$ is a C_0 semigroup on X. If we define $u(t,x) = [S(t)u_0](x)$ with $u_0 \in H^1(0,1)$ with $u_0(0) = 0$ satisfies (1.8) a.e.. The generator A is given by

$$A\phi = -\phi' \text{ with } dom(A) = \{\phi \in H^1(0,1) \text{ with } \phi(0) = 0\}.$$

In fact

$$\frac{S(t)u_0 - u_0}{t} = \frac{\tilde{u}_0(x-t) - \tilde{u}_0}{t} = -u_0'(x), \quad \text{a.e. } x \in (0,1).$$

if $u_0 \in dom(A)$. Thus, $u(t) = S(t)u_0$ satisfies the Cauchy problem $\frac{d}{dt}u(t) = Au(t)$ if $u_0 \in dom(A)$.

On the other hand if we apply the operator exponential formula (1.4) for this A,

$$u(t,x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} u_0^k(x) t^k = u_0(x-t)$$

for $u_0 \in C^\infty(0,1)$, which coincides with (1.6). That is, the solution semigroup $S(t)$ is the extension of the operator exponential formula.

(3) Let $X_t \in R^d$ is a Markov process, i.e.

$$E^x[g(X_{t+h})|\mathcal{F}_t] = E^{0,X_t}[g(X_h)].$$

for all $g \in X = L^2(R^n)$. Define the linear operator $S(t)$ by

$$(S(t)u_0)(x) = E^{0,x}[u_0(X_t)], \quad t \geq 0, \quad u_0 \in X.$$

The semigroup property of $S(t)$ follows from the Markov property, i.e.

$$S(t+s)u_0 = E^{0,x}[u_0(X_{t+s})] = E[E[u_0(X^{0,x}_{t+s})|\mathcal{F}_t] = E[E^{0,X^{0,x}_t} u_0(X_s)]] = E[(S(t)u_0)(X_s)] = S(s)(S(t)u_0).$$

The strong continuity follows from that $X^{0,x}_t - x$ is a.s for all $x \in R^n$. If $X_t = B_t$ is a Brownian motion, then the semigroup $S(t)$ is defined by

$$[S(t)u_0](x) = \frac{1}{(\sqrt{2\pi t}\sigma)^n} \int_{R^n} e^{-\frac{|x-y|^2}{2\sigma^2 t}} u_0(y)\, dy, \tag{1.7}$$

and $u(t) = S(t)u_0$ satisfies the heat equation.

$$u_t = \frac{\sigma^2}{2}\Delta u, \quad u(0,x) = u_0(x) \text{ in } L^2(R^n). \tag{1.8}$$

1.1 Finite difference in time

Let A be closed, densely defined linear operator $dom(A) \to X$. We use the finite difference method in time to construct the mild solution (1.1). For a stepsize $\lambda > 0$ consider a sequence $\{u^n\}$ in X generated by

$$\frac{u^n - u^{n-1}}{\lambda} = Au^n + f^{n-1}, \tag{1.9}$$

with

$$f^{n-1} = \frac{1}{\lambda} \int_{(n-1)\lambda}^{n\lambda} f(t)\, dt.$$

Assume that for $\lambda > 0$ the resolvent operator

$$J_\lambda = (I - \lambda\, A)^{-1}$$

is bounded. Then, we have the product formula:

$$u^n = J_\lambda^n u_0 + \sum_{k=0}^{n-1} J_\lambda^{n-k} f^k \,\lambda. \tag{1.10}$$

In order to $u^n \in X$ is uniformly bounded in n for all $u_0 \in X$ (with $f = 0$), it is necessary that

$$|J_\lambda^n| \leq \frac{M}{(1-\lambda\omega)^n} \text{ for } \lambda\omega < 1, \tag{1.11}$$

for some $M \geq 1$ and $\omega \in R$.

<u>Hille's Theorem</u> Define a piecewise constant function in X by

$$u_\lambda(t) = u^{k-1} \text{ on } [t_{k-1}, t_k)$$

Then,

$$\max_{t\in[0,T]} |u_\lambda - u(t)|_X \to 0$$

as $\lambda \to 0^+$ to the mild solution (1.1). That is,

$$S(t)x = \lim_{n\to\infty} (I - \frac{t}{n}A)^{[\frac{t}{n}]}x$$

exists for all $x \in X$ and $\{S(t),\ t \geq 0\}$ is the C_0 semigoup on X and its generator is A, where $[s]$ is the largest integer less than $s \in R$.

Proof: First, note that

$$|J_\lambda| \leq \frac{M}{1-\lambda\omega}$$

and for $x \in dom(A)$

$$J_\lambda x - x = \lambda J_\lambda A x,$$

and thus

$$|J_\lambda x - x| = |\lambda J_\lambda A x| \leq \frac{\lambda}{1-\lambda\omega}|Ax| \to 0$$

as $\lambda \to 0^+$. Since $dom(A)$ is dense in X it follows that

$$|J_\lambda x - x| \to 0 \text{ as } \lambda \to 0^+ \text{ for all } x \in X.$$

Define the linear operators $T_\lambda(t)$ and $S_\lambda(t)$ by

$$S_\lambda(t) = J_\lambda^k \text{ and } T_\lambda(t) = J_\lambda^{k-1} + \frac{t-t_k}{\lambda}(J_\lambda^k - J_\lambda^{k-1}), \text{ on } (t_{k-1}, t_k].$$

Then,

$$\frac{d}{dt}T_\lambda(t) = AS_\lambda(t), \text{ a.e. in } t \in [0,T].$$

Thus,

$$T_\lambda(t)u_0 - T_\mu(t)u_0 = \int_0^t \frac{d}{ds}(T_\lambda(s)T_\mu(t-s)u_0)\,ds = \int_0^t (S_\lambda(s)T_\mu(t-s) - T_\lambda(s)S_\mu(t-s))Au_0\,ds$$

Since

$$T_\lambda(s)u - S_\lambda(s)u = \frac{s-t_k}{\lambda} T_\lambda(t_{k-1})(J_\lambda - I)u \text{ on } s \in (t_{k-1}, t_k].$$

By the bounded convergence theorem

$$|T_\lambda(t)u_0 - T_\mu(t)u|_X \to 0$$

as $\lambda,\ \mu \to 0^+$ for all $u \in dom(A^2)$. Thus, the unique limit defines the linear operator $S(t)$ by

$$S(t)u_0 = \lim_{\lambda \to 0^+} S_\lambda(t)u_0. \tag{1.12}$$

for all $u_0 \in dom(A^2)$. Since

$$|S_\lambda(t)| \le \frac{M}{(1-\lambda\omega)^{[t/n]}} \le Me^{\omega t}$$

and $dom(A^2)$ is dense, (1.12) holds for all $u_0 \in X$. Moreover, we have

$$S(t+s)u = \lim_{\lambda \to 0^+} J_\lambda^{n+m} = J_\lambda^n J_\lambda^m u = S(t)S(s)u$$

and $\lim_{t\to 0^+} S(t)u = \lim_{t\to 0^+} J_t u = u$ for all $u \in X$. Thus, $S(t)$ is the C_0 semigroup on X. Moreover, $\{S(t), t \ge 0\}$ is in the class $G(M,\omega)$, i.e.,

$$|S(t)| \le Me^{\omega t}.$$

Note that

$$T_\lambda(t)u_0 - u_0 = A\int_0^t S_\lambda u_0\, ds.$$

Since $\lim_{\lambda\to 0^+} T_\lambda(t)u_0 = \lim_{\lambda\to 0^+} S_\lambda(t)u_0 = S(t)u_0$ and A is closed, we have

$$S(t)u_0 - u_0 = A\int_0^t S(s)u_0\, ds, \qquad \int_0^t S(s)u_0\, ds \in dom(A).$$

If B is a generator of $\{S(t),\ t \ge 0\}$, then

$$Bx = \lim_{t\to 0^+} \frac{S(t)x - x}{t} = Ax$$

if $x \in dom(A)$. Conversely, if $u_0 \in dom(B)$, then $u_0 \in dom(A)$ since A is closed and $t \to S(t)u$ is continuous at 0 for all $u \in X$ and thus

$$\frac{1}{t}A\int_0^t S(s)u_0\, ds = Au_0 \text{ as } t \to 0^+.$$

Hence

$$Au_0 = \frac{S(t)u_0 - u_0}{t} = Bu_0$$

That is, A is the generator of $\{S(t),\ t \ge 0\}$.

Similarly, we have

$$\sum_{k=0}^{n-1} J_\lambda^{n-k} f^k \; = \int_0^t S_\lambda(t-s) f(s)\,ds \to \int_0^t S(t-s) f(s)\,ds \text{ as } \lambda \to 0^+$$

by the Lebesgue dominated convergence theorem. □

The following theorem states the basic properties of C_0 semigroups:

Theorem (Semigroup) (1) There exists $M \geq 1$, $\omega \in R$ such that $S \in G(M,\omega)$ class, i.e.,

$$|S(t)| \leq M\, e^{\omega t}, \quad t \geq 0. \tag{1.13}$$

(2) If $x(t) = S(t)x_0,\ x_0 \in X$, then $x \in C(0,T;X)$
(3) If $x_0 \in dom\,(A)$, then $x \in C^1(0,T;X) \cap C(0,T;dom(A))$ and

$$\frac{d}{dt}x(t) = Ax(t) = AS(t)x_0.$$

(4) The infinitesimal generator A is closed and densely defined. For $x \in X$

$$S(t)x - x = A \int_0^t S(s)x\,ds. \tag{1.14}$$

(5) $\lambda > \omega$ the resolvent is given by

$$(\lambda\, I - A)^{-1} = \int_0^\infty e^{-\lambda s} S(s)\,ds \tag{1.15}$$

with estimate

$$|(\lambda\, I - A)^{-n}| \leq \frac{M}{(\lambda-\omega)^n}. \tag{1.16}$$

Proof: (1) By the uniform boundedness principle there exists $M \geq 1$ such that $|S(t)| \leq M$ on $[0,t_0]$ For arbitrary $t = k\,t_0 + \tau$, $k \in N$ and $\tau \in [0,t_0)$ it follows from the semigroup property we get

$$|S(t)| \leq |S(\tau)||S(t_0|^k \leq M e^{k\,log\,|S(t_0)|} \leq M e^{\omega\,t}$$

with $\omega = \frac{1}{t_0}\log|S(t_0)|$.
(2) It follows from the semigroup property that for $h > 0$

$$x(t+h) - x(t) = (S(h) - I)S(t)x_0$$

and for $t - h \geq 0$

$$x(t-h) - x(t) = S(t-h)(I - S(h))x_0$$

Thus, $x \in C(0,T;X)$ follows from the strong continuity of $S(t)$ at $t = 0$.
(3)–(4) Moreover,

$$\frac{x(t+h)-x(t)}{h} = \frac{S(h)-I}{h}S(t)x_0 = S(t)\frac{S(h)x_0 - x_0}{h}$$

and thus $S(t)x_0 \in dom(A)$ and

$$\lim_{h\to 0^+} \frac{x(t+h)-x(t)}{h} = AS(t)x_0 = Ax(t).$$

Similarly,

$$\lim_{h\to 0^+} \frac{x(t-h)-x(t)}{-h} = \lim_{h\to 0^+} S(t-h)\frac{S(h)\phi-\phi}{h} = S(t)Ax_0.$$

Hence, for $x_0 \in dom(A)$

$$S(t)x_0 - x_0 = \int_0^t S(s)Ax_0\,ds = \int_0^t AS(s)x_0\,ds = A\int_0^t S(s)x_0\,ds \tag{1.17}$$

If $x_n \in don(A) \to x$ and $Ax_n \to y$ in X, we have

$$S(t)x - x = \int_0^t S(s)y\,ds$$

Since

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t S(s)y\,ds = y$$

$x \in dom(A)$ and $y = Ax$ and hence A is closed. Since A is closed it follows from (1.17) that for $x \in X$

$$\int_0^t S(s)x\,ds \in dom(A)$$

and (1.14) holds. For $x \in X$ let

$$x_h = \frac{1}{h}\int_0^h S(s)x\,ds \in dom(A)$$

Since $x_h \to x$ as $h \to 0^+$, $dom(A)$ is dense in X.
(5) For $\lambda > \omega$ define $R_t \in \mathcal{L}(X)$ by

$$R_t = \int_0^t e^{-\lambda s}S(s)\,ds.$$

Since $A - \lambda\,I$ is the infinitesimal generator of the semigroup $e^{\lambda t}S(t)$, from (1.14)

$$(\lambda\,I - A)R_t x = x - e^{-\lambda t}S(t)x.$$

Since A is closed and $|e^{-\lambda t}S(t)| \to 0$ as $t \to \infty$, we have $R = \lim_{t\to\infty} R_t$ satisfies

$$(\lambda\,I - A)R\phi = \phi.$$

Conversely, for $\phi \in dom(A)$

$$R(A-\lambda\,I)\phi = \int_0^\infty e^{-\lambda s}S(s)(A-\lambda\,I)\phi = \lim_{t\to\infty} e^{-\lambda t}S(t)\phi - \phi = -\phi$$

Hence

$$R = \int_0^\infty e^{-\lambda s} S(s)\, ds = (\lambda\, I - A)^{-1}$$

Since for $\phi \in X$

$$|Rx| \le \int_0^\infty |e^{-\lambda s} S(s)x| \le M \int_0^\infty e^{(\omega-\lambda)s} |x|\, ds = \frac{M}{\lambda-\omega}|x|,$$

we have

$$|(\lambda\, I - A)^{-1}| \le \frac{M}{\lambda-\omega}, \quad \lambda > \omega.$$

Note that

$$(\lambda\, I - A)^{-2} = \int_0^\infty e^{-\lambda t} S(t)\, ds \int_0^\infty e^{\lambda\, s} S(s)\, ds = \int_0^\infty \int_0^\infty e^{-\lambda(t+s)} S(t+s)\, ds\, dt$$

$$= \int_0^\infty \int_t^\infty e^{-\lambda\sigma} S(\sigma)\, d\sigma\, dt = \int_0^\infty \sigma e^{-\lambda\sigma} S(\sigma)\, d\sigma.$$

By induction, we obtain

$$(\lambda\, I - A)^{-n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} S(t)\, dt. \tag{1.18}$$

Thus,

$$|(\lambda\, I - A)^{-n}| \le \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(\lambda-\omega)t}\, dt = \frac{M}{(\lambda-\omega)^n}.\square$$

We then we have the necessary and sufficient condition:

Hile-Yosida Theorem A closed, densely defined linear operator A on a Banach space X is the infinitesimal generator of a C_0 semigroup of class $G(M,\omega)$ if and only if

$$|(\lambda\, I - A)^{-n}| \le \frac{M}{(\lambda-\omega)^n} \quad \text{for all } \lambda > \omega \tag{1.19}$$

Proof: The sufficient part follows from the previous Theorem. In addition, we describe the Yosida construction. Define the Yosida approximation $A_\lambda \in \mathcal{L}(X)$ of A by

$$A_\lambda = \frac{J_\lambda - I}{\lambda} = AJ_\lambda. \tag{1.20}$$

Define the Yosida approximation:

$$S_\lambda(t) = e^{A_\lambda t} = e^{-\frac{t}{\lambda}} e^{J_\lambda \frac{t}{\lambda}}.$$

Since

$$|J_\lambda^k| \le \frac{M}{(1-\lambda\omega)^k}$$

we have

$$|S_\lambda(t)| \le e^{-\frac{t}{\lambda}} \sum_{k=0}^\infty \frac{1}{k!} |J_\lambda^k| (\frac{t}{\lambda})^k \le M e^{\frac{\omega}{1-\lambda\omega}t}.$$

Since

$$\frac{d}{ds}S_\lambda(s)S_{\hat\lambda}(t-s) = S_\lambda(s)(A_\lambda - A_{\hat\lambda})S_{\hat\lambda}(t-s),$$

we have

$$S_\lambda(t)x - S_{\hat\lambda}(t)x = \int_0^t S_\lambda(s)S_{\hat\lambda}(t-s)(A_\lambda - A_{\hat\lambda})x\,ds$$

Thus, for $x \in dom(A)$

$$|S_\lambda(t)x - S_{\hat\lambda}(t)x| \le M^2 te^{\omega t}\,|(A_\lambda - A_{\hat\lambda})x| \to 0$$

as $\lambda,\ \hat\lambda \to 0^+$. Since $dom(A)$ is dense in X this implies that

$$S(t)x = \lim_{\lambda\to 0^+} S_\lambda(t)x \text{ exist for all } x \in X,$$

defines a C_0 semigroup of $G(M,\omega)$ class. The necessary part follows from (1.18) □

Theorem (Mild solution) (1) If for $f \in L^1(0,T;X)$ define

$$x(t) = x(0) + \int_0^t S(t-s)f(s)\,ds,$$

then $x(t) \in C(0,T;X)$ and it satisfies

$$x(t) = A\int_0^t x(s)\,ds + \int_0^t f(s)\,ds. \tag{1.21}$$

(2) If $Af \in L^1(0,T;X)$ then $x \in C(0,T;dom(A))$ and

$$x(t) = x(0) + \int_0^t (Ax(s) + f(s))\,ds.$$

(3) If $f \in W^{1,1}(0,T;X)$, i.e. $f(t) = f(0) + \int_0^t f'(s)\,ds$, $\frac{d}{dt}f = f' \in L^1(0,T;X)$, then $Ax \in C(0,T;X)$ and

$$A\int_0^t S(t-s)f(s)\,ds = S(t)f(0) - f(t) + \int_0^t S(t-s)f'(s)\,ds. \tag{1.22}$$

Proof: Since

$$\int_0^t\int_0^\tau S(t-s)f(s)\,dsd\tau = \int_0^t (\int_s^t S(\tau-s)d\,\tau)f(s)\,ds,$$

and

$$A\int_0^t S(s)\,ds = S(t) - I$$

we have $x(t) \in dom(A)$ and

$$A\int_0^t x(s)\,ds = S(t)x - x + \int_0^t S(t-s)f(s)\,ds - \int_0^t f(s)\,ds.$$

and we have (1.21).
(2) Since for $h > 0$

$$\frac{x(t+h)-x(t)}{h} = \int_0^t S(t-s)\frac{S(h)-I}{h}f(s)\,ds + \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds$$

if $Af \in L^1(0,T;X)$

$$\lim_{h\to 0^+}\frac{x(t+h)-x(t)}{h} = \int_0^t S(t-s)Af(s)\,ds + f(t)$$

a.e. $t \in (0,T)$. Similarly,

$$\frac{x(t-h)-x(t)}{-h} = \int_0^{t-h} S(t-h-s)\frac{S(h)-I}{h}f(s)\,ds + \frac{1}{h}\int_{t-h}^{t} S(t-s)f(s)\,ds$$

$$\to \int_0^t S(t-s)Af(s)\,ds + f(t)$$

a.e. $t \in (0,T)$.
(3) Since

$$\frac{S(h)-I}{h}x(t) = \frac{1}{h}(\int_0^h S(t+h-s)f(s)\,ds - \int_t^{t+h} S(t+h-s)f(s)\,ds$$

$$+\int_0^t S(t-s)\frac{f(s+h)-f(s)}{h}\,ds,$$

letting $h \to 0^+$, we obtain(1.22). □

It follows from Theorems the mild solution

$$x(t) = S(t)x(0) + \int_0^t S(t-s)f(s)\,ds$$

satisfies

$$x(t) = x(0) + A\int_0^t x(s) + \int_0^t f(s)\,ds.$$

Note that the mild solution $x \in C(0,T;X)$ depends continuously on $x(0) \in X$ and $f \in L^1(0,T;X)$ with estimate

$$|x(t)| \le M(e^{\omega t}|x(0)| + \int_0^t e^{\omega(t-s)}|f(s)|\,ds).$$

Thus, the mild solution is the limit of a sequence $\{x_n\}$ of strong solutions with $x_n(0) \in dom(A)$ and $f_n \in W^{1,1}(0,T;X)$, i.e., since $dom(A)$ is dense in X and $W^{1,1}(0,T;X)$ is dense in $L^1(0,T;X)$,

$$|x_n(t) - x(t)|_X \to 0 \text{ uniformly on } [0,T]$$

for

$$|x_n(0)-x(0)|_X \to 0 \text{ and } |f_n - f|_{L^1(0,T;X)} \to 0 \text{ as } n \to \infty.$$

Moreover, the mild solution $x \in C(0,T:X)$ is a weak solution to the Cauchy problem

$$\frac{d}{dt}x(t) = Ax(t) + f(t) \tag{1.23}$$

in the sense of (1.3), i.e., for all $\psi \in dom(A^*)$ $\langle x(t),\psi\rangle_{X\times X^*}$ is absolutely continues and

$$\frac{d}{dt}\langle x(t),\psi\rangle = \langle x(t),\psi\rangle + \langle f(t),\psi\rangle \text{ a.e. in } (0,T).$$

If $x(0) \in dom(A)$ and $Af \in L^1(0,T;X)$, then $Ax \in C(0,T;X)$, $x \in W^{1,1}(0,T;X)$ and

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \text{ a.e. in } (0,T)$$

If $x(0) \in dom(A)$ and $f \in W^{1,1}(0,T;X)$, then $x \in C(0,T;dom(A)) \cap C^1(0,T;X)$ and

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \text{ everywhere in } [0,T].$$

1.2 Weak-solution and Ball's result

Let A be a densely defined, closed linear operator on a Banach space X. Consider the Cauchy equation in X:

$$\frac{d}{dt}u = Au + f(t), \tag{1.24}$$

where $u(0) = x \in X$ and $f \in L^1(0,\tau;X)$ is a weak solution to of (1.24) if for every $\psi \in$dom(A^*) the function $t \to \langle u(t),\psi\rangle$ is absolutely continuous on $[0,\tau]$ and

$$\frac{d}{dt}\langle u(t),\psi\rangle = \langle u(t),A^*\psi\rangle + \langle f(t),\psi\rangle, \quad \text{a.e. in } [0,\tau]. \tag{1.25}$$

It has been shown that the mild solution to (1.24) is a weak solution.

Lemma B.1 Let A be a densely defined, closed linear operator on a Banach space X. If $x,\ y \in X$ satisfy $\langle y,\psi\rangle = \langle x, A^*\psi\rangle$ for all $\psi \in$dom(A^*), then $x \in$ dom(A) and $y = Ax$.

Proof: Let $G(A) \subset X \times X$ denotes the graph of A. Since A is closed $G(A)$ is closed. Suppose $y \neq Ax$. By Hahn-Banach theorem there exist $z,\ z^* \in X^*$ such that $\langle Ax,z\rangle + \langle x,z^*\rangle = 0$ and $\langle y,z\rangle + \langle x,z^*\rangle \neq 0$. Thus $z \in$dom(A^*) and $z^* = A^*z$. By the condition $\langle y,z\rangle + \langle x,z^*\rangle = 0$, which is a contradiction. □

Then we have the following theorem.

Theorem (Ball) There exists for each $x \in X$ and $f \in L^1(0,\tau;X)$ a unique weak solution of (1.24)satisfying $u(0) = x$ if and only if A is the generator of a strongly continuous semigroup $\{T(t)\}$ of bounded linear operator on X, and in this case $u(t)$ is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds. \tag{1.26}$$

Proof: Let A generate the strongly continuous semigroup $\{T(t)\}$ on X. Then, for some M, $|T(t)| \leq M$ on $t \in [0,\tau]$. Suppose $x \in$dom(A) and $f \in W^{1,1}(0,\tau;X)$. Then we have

$$\frac{d}{dt}\langle u(t),\psi\rangle = \langle Au(t)+f(t),\psi\rangle = \langle u(t),A^*\psi\rangle + \langle f(t),\psi\rangle.$$

For $(x,f) \in X \times L^1(0,\tau;X)$ there exists a sequence (x_n,f_n) in dom$(A) \times W^{1,1}(0,\tau;X)$ such that $|x_n - x|_X + |f_n - f|_{L^1(0,\tau;X)} \to 0$ as $n \to \infty$ If we define

$$u_n(t) = T(t)x_n + \int_0^t T(t-s)f_n(s)\,ds,$$

then we have

$$\langle u_n(t),\psi\rangle = \langle x,\psi\rangle + \int_0^t (\langle u_n(s),A^*\psi\rangle + \langle f_n(s),\psi\rangle)\,ds$$

and

$$|u_n(t)-u(t)|_X \leq M\,(|x_n - x|_X + \int_0^t |f_n(s)-f(s)|_X\,ds).$$

Passing limit $n \to]\infty$, we see that $u(t)$ is a weak solution of (1.24).

Next we prove that $u(t)$ is the only weak solution to (1.24) satisfying $u(0)=x$. Let $\tilde{u}(t)$ be another such weak solution and set $v = u - \tilde{u}$. Then we have

$$\langle v(t),\psi\rangle = \langle \int_0^t v(s)\,dt, A^*\psi\rangle$$

for all $\psi \in$dom(A^*) and $t \in [0,\tau]$. By Lemma B.1 this implies $z(t) = \int_0^t v(s)\,ds \in$dom$(A)$ and $\frac{d}{dt}z(t) = Az(t)$ with $z(0)=0$. Thus $z=0$ and hence $u(t) = \tilde{u}(t)$ on $[0,\tau]$.

Suppose that A such that (1.24) has a unique weak solution $u(t)$ satisfying $u(0)=x$. For $t \in [0,\tau]$ we define the linear operator $T(t)$ on X by $T(t)x = u(t) - u_0(t)$, where u_0 is the weak solution of (1.24) satisfying $u_(0)=0$. If for $t = nT + s$, where n is a nonnegative integer and $s \in [0,\tau)$ we define $T(t)x = T(s)T(\tau)^n x$, then $T(t)$ is a semigroup. The map $\theta : x \to C(0,\tau;X)$ defined by $\theta(x) = T(\cdot)x$ has a closed graph by the uniform bounded principle and thus $T(t)$ is a strongly continuous semigroup. Let B be the generator of $\{T(t)\}$ and $x \in$dom(B). For $\psi \in$dom(A^*)

$$\frac{d}{dt}\langle T(t)x,\psi\rangle|_{t=0} = \langle Bx,\psi\rangle = \langle x,A^*\psi\rangle.$$

It follows from Lemma that $x \in$dom(A) and $Ax = Bx$. Thus dom$(B) \subset$dom(A). The proof of Theorem is completed by showing dom$(A) \subset$dom(B). Let $x \in$dom(A). Since for $z(t) = T(t)x$

$$\langle z(t),\psi\rangle = \langle \int_0^t z(s)\,dt, A^*\psi\rangle$$

it follows from Lemma that $\int_0^t T(s)x\,ds$ and $\int_0^t T(s)Ax\,ds$ belong to dom(A) and

$$\begin{aligned} T(t)x &= x + A\int_0^t T(s)x\,ds \\ T(t)Ax &= Ax + A\int_0^t T(s)Ax\,ds \end{aligned} \qquad (1.27)$$

Consider the function

$$w(t) = \int_0^t T(s)Ax\,ds - A\int_0^t T(s)x\,ds.$$

It then follows from (1.27) that $z \in C(0,\tau;\ X)$. Clearly $w(0) = 0$ and it also follows from (1.27) that

$$\frac{d}{dt}\langle w(t),\psi\rangle = \langle w(t), A^*\psi\rangle. \tag{1.28}$$

for $\psi \in$dom(A^*). But it follows from our assumptions that (1.28) has the unique solution $w = 0$. Hence from (1.27)

$$T(t)x - x = A\int_0^t T(s)x\,ds$$

and thus

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = Ax$$

which implies $x \in$dom(B). □

1.3 Lumer-Phillips Theorem

The condition (1.19) is very difficult to check for a given A in general. For the case $M = 1$ we have a very complete characterization.

Lumer-Phillips Theorem The followings are equivalent:
(a) A is the infinitesimal generator of a C_0 semigroup of $G(1,\omega)$ class.
(b) $A - \omega\, I$ is a densely defined linear m-dissipative operator,i.e.

$$|(\lambda\, I - A)x| \geq (\lambda - \omega)|x| \quad \text{for all } x \in don(A), \quad \lambda > \omega \tag{1.29}$$

and for some $\lambda_0 > \omega$

$$R(\lambda_0\, I - A) = X. \tag{1.30}$$

Proof: It follows from the m-dissipativity

$$|(\lambda_0\, I - A)^{-1}| \leq \frac{1}{\lambda_0 - \omega}$$

Suppose $x_n \in dom(A) \to x$ and $Ax_n \to y$ in X, the

$$x = \lim_{n\to\infty} x_n = (\lambda_0\, I - A)^{-1} \lim_{n\to\infty} (\lambda_0\, x_n - Ax_n) = (\lambda_0\, I - A)^{-1}(\lambda_0\, x - y).$$

Thus, $x \in dom(A)$ and $y = Ay$ and hence A is closed. Since for $\lambda > \omega$

$$\lambda\, I - A = (I + (\lambda - \lambda_0)(\lambda_0\, I - A)^{-1})(\lambda_0\, I - A),$$

if $\frac{|\lambda-\lambda_0|}{\lambda_0-\omega} < 1$, then $(\lambda\, I - A)^{-1} \in \mathcal{L}(X)$. Thus by the continuation method we have $(\lambda\, I - A)^{-1}$ exists and

$$|(\lambda\, I - A)^{-1}| \leq \frac{1}{\lambda - \omega}, \quad \lambda > \omega.$$

It follows from the Hile-Yosida theorem that (b) $\to$ (a).
(b) $\to$ (a) Since for $x^* \in F(x)$, the dual element of x, i.e. $x^* \in X^*$ satisfying $\langle x, x^* \rangle_{X \times X^*} = |x|^2$ and $|x^*| = |x|$

$$\langle e^{-\omega t} S(t)x, x^* \rangle \leq |x||x^*| = \langle x, x^* \rangle$$

we have for all $x \in dom(A)$

$$0 \geq \lim_{t \to 0^+} \langle \frac{e^{-\omega t} S(t)x - x}{t}, x^* \rangle = \langle (A - \omega\, I)x, x^* \rangle \text{ for all } x^* \in F(x).$$

which implies $A - \omega\, I$ is dissipative. $\square$

Theorem (Dissipative I) (1) A is a ω-dissipative

$$|\lambda\, x - Ax| \geq (\lambda - \omega)|x| \text{ for all } x \in dom\,(A).$$

if and only if (2) for all $x \in dom(A)$ there exists an $x^* \in F(x)$ such that

$$\langle Ax, x^* \rangle \leq \omega\, |x|^2. \tag{1.31}$$

(2) $\to$ (1). Let $x \in dom(A)$ and choose an $x^* \in F(0)$ such that $\langle A, x^* \rangle \leq 0$. Then, for any $\lambda > 0$,

$$\lambda\, |x|^2 = \lambda\, \langle x, x^* \rangle = \langle \lambda\, x - Ax + Ax, x^* \rangle \leq \langle \lambda\, x - Ax, x^* \rangle + \omega\; |x|^2 \leq |\lambda\, x - Ax||x| + \omega\, |x|^2,$$

which implies (1).
(1) $\to$ (2). Without loss of the generality one can assume $\omega = 0$. From (1) we obtain the estimate

$$\frac{1}{\lambda}(|x| - |x - \lambda\, Ax|) \leq 0$$

and

$$\langle Ax, x \rangle_- = -\lim_{\lambda \to 0^+} \frac{1}{\lambda}(|x| - |x - \lambda\, Ax|) \leq 0$$

which implies there exists $x^* \in F(x)$ such that (1.31) holds since $\langle Ax, x \rangle_- = \langle Ax, x^* \rangle$ for some $x^* \in F(x)$. $\square$

Thus, Lumer-Phillips theorem says that if m-diisipative, then (1.31) hold for all $x^* \in F(x)$.

Theorem (Dissipative II) Let A be a closed densely defined operator on X. If A and A^* are dissipative, then A is m-dissipative and thus the infinitesimal generator of a C_0-semigroup of contractions.

Proof: Let $y \in \overline{R(I - A)}$ be given. Then there exists a sequence $x_n \in dom(A)$ such that $y_= x_n - Ax_n \to y$ as $n \to \infty$. By the dissipativity of A we obtain

$$|x_n - x_m| \leq |x_n - x_m - A(x_n - x_m)| \leq |y_- y_m|$$

Hence x_n is a Cauchy sequence in X. We set $x = lim_{n \to \infty} x_n$. Since A is closed, we see that $x \in dom\,(A)$ and $x - Ax = y$, i.e., $y \in R(I - A)$. Thus $R(I - A)$ is closed. Assume that $R(I - A) \neq X$. Then there exists an $x^* \in X^*$ such that

$$\langle (I - A)x, x^*) = 0 \text{ for all } x \in dom\,(A).$$

By definition of the dual operator this implies $x^* \in dom\,(A^*)$ and $(I-A)^*x^* = 0$. The dissipativity of A* implies $|x^*| < |x^* - A^*x^*| = 0$, which is a contradiction. $\square$

Example (revisited example 1)

$$A\phi = -\phi' \text{ in } X = L^2(0,1)$$

and for $\phi \in H^1(0,1)$

$$(A\phi,\phi)_X = -\int_0^1 \phi'(x)\phi\,dx = \frac{1}{2}(|\phi(0)|^2 - |\phi(1)|^2)$$

Thus, A is dissipative if and only if $\phi(0)=0$, the in flow condition. Define the domain of A by

$$dom(A) = \{\phi \in H^1(0,1) : \phi(0) = 0\}$$

The resolvent equation is equivalent to

$$\lambda\,u + \frac{d}{dx}u = f$$

and

$$u(x) = \int_0^x e^{-\lambda\,(x-s)} f(s)\,ds,$$

and $R(\lambda\,I - A) = X$. By the Lumer-Philips theorem A generates the C_0 semigroup on $X = L^2(0,1)$.

Example (Conduction equation) Consider the heat conduction equation:

$$\frac{d}{dt}u = Au = \sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x)\cdot\nabla u + c(x)u, \quad \text{in } \Omega.$$

Let $X = C(\Omega)$ and $dom(A) \subset C^2(\Omega)$. Assume that $a \in R^{n\times n} \in C(\Omega)$ $b \in R^{n,1}$ and $c \in R$ are continuous on $\bar{\Omega}$ and a is symmetric and

$$m\,I \le a(x) \le M\,I \text{ for } 0 < m \le M < \infty.$$

Then, if x_0 is an interior point of Ω at which the maximum of $\phi \in C^2(\Omega)$ is attained. Then,

$$\nabla\phi(x_0) = 0, \quad \sum_{ij} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \le 0.$$

and thus

$$(\lambda\,\phi - A\phi)(x_0) \le \omega\,\phi(x_0)$$

where

$$\omega \le \max_{x\in\Omega} c(x).$$

Similarly, if x_0 is an interior point of Ω at which the minimum of $\phi \in C^2(\Omega)$ is attained, then

$$(\lambda\,\phi - A\phi)(x_0) \ge 0$$

If $x_0 \in \partial\Omega$ attains the maximum, then

$$\frac{\partial}{\partial\nu}\phi(x_0) \le 0.$$

Consider the domain with the Robin boundary condition:

$$dom(A) = \{u \in \alpha(x)\, u(x) + \beta(x)\frac{\partial}{\partial\nu}u = 0 \text{ at } \partial\Omega\}$$

with $\alpha,\ \beta \ge 0$ and $\inf_{x\in\partial\Omega}(\alpha(x)+\beta(x)) > 0$. Then,

$$|\lambda\phi - A\phi|_X \ge (\lambda-\omega)|\phi|_X. \tag{1.32}$$

for all $\phi \in C^2(\Omega)$. It follows from the he Lax Milgram theory that

$$(\lambda_0\, I - A)^{-1} \in \mathcal{L}(L^2(\Omega), H^2(\Omega)),$$

assuming that coefficients (a, b, c) are sufficiently smooth. Let

$$dom(A) = \{(\lambda_0\, I - A)^{-1}C(\Omega)\}.$$

Since $C^2(\Omega)$ is dense in $dom(A)$, (1.32) holds for all $\phi \in dom(A)$, which shows A is dissipative.

Example (Advection equation and Mass transport equation) Consider the advection equation

$$u_t + \nabla\cdot(\vec{b}(x)u) = \nu\,\Delta u.$$

Let $X = L^1(\Omega)$. Assume

$$\vec{b} \in L^\infty(\Omega)$$

Let $\rho \in C^1(R)$ be a monotonically increasing function satisfying $\rho(0) = 0$ and $\rho(x) = sign(x),\ |x| \ge 1$ and $\rho_\epsilon(s) = \rho(\frac{s}{\epsilon})$ for $\epsilon > 0$. For $u \in C^1(\Omega)$

$$(Au, u) = \int_\Gamma (\nu\,\frac{\partial}{\partial n}u - n\cdot\vec{b}\,u, \rho_\epsilon(u))\,ds + (\vec{b}\,u - \nu\,\nabla u, \frac{1}{\epsilon}\rho'_\epsilon(u)\,\nabla u) + (c\,u, \rho_\epsilon(u)).$$

where

$$(\vec{b}\,u, \frac{1}{\epsilon}\rho'_\epsilon(u)\,\nabla u) \le \nu\,(\nabla u, \frac{1}{\epsilon}\rho'_\epsilon(u)\,\nabla u) + \frac{\epsilon}{4\nu}\,meas(\{|u| \le \epsilon\}).$$

Assume the inflow condition $\nu\,\frac{\partial}{\partial n}u - n\cdot\vec{b}\,u = 0$ on $\{s \in \partial\Omega : n\cdot b < 0\}$ and otherwise $\nu\,\frac{\partial}{\partial n}u = 0$. Note that for $u \in L^1(R^d)$

$$(u, \rho_\epsilon(u)) \to |u|_1 \quad \text{and} \quad (\psi, \rho_\epsilon(u)) \to (\psi, sign_0(u)) \text{ for } \psi \in L^1(\Omega)$$

as $\epsilon \to 0$. If $c(x) \le \omega$, then it follows that

$$(\lambda-\omega)\,|u| \le |\lambda\, u - \lambda\, Au|. \tag{1.33}$$

Since $H^1(\Omega)$ is dense in $L^1(\Omega)$, (1.33) holds for $u \in dom(A)$. For $\nu = 0$ case letting $\nu \to 0^+$ (1.33) holds for $dom(A) = \{u \in L^1(\Omega) : (\rho\, u)_x \in L^1(\Omega)\}$.

Example ($X = L^p(\Omega)$) Let $Au = \nu\,\Delta u + b\cdot\nabla u$ with homogeneous boundary condition $u = 0$ on $X = L^p(\Omega)$. Since

$$\langle \Delta u, u^*\rangle = \int_\Omega (\Delta u, |u|^{p-2}u) = -(p-1)\int_\Omega (\nabla u, |u|^{p-2}\nabla u)$$

and

$$(b\cdot\nabla u, |u|^{p-2}u)_{L^2} \le \frac{(p-1)\nu}{2}|(\nabla u, |u|^{p-2}\nabla u)_{L^2} + \frac{|b|_\infty^2}{2\nu(p-1)}\,(|u|^p, 1)_{L^2}$$

we have

$$\langle Au, u^*\rangle \le \omega |u|^2$$

for some $\omega > 0$.

Example (Fractional PDEs I)

In this section we consider the nonlocal diffusion equation of the form

$$u_t = Au = \int_{R^d} J(z)(u(x+z) - u(x))\,dz.$$

Or, equivalently

$$Au = \int_{(R^d)^+} J(z)(u(x+z) - 2u(x) + u(x-z))\,dz$$

for the symmetric kernel J in R^d. It will be shown that

$$(Au, \phi)_{L^2} = -\int_{R^d}\int_{(R^d)^+} J(z)(u(x+z) - u(x))(\phi(x+z) - \phi(x))\,dz\,dx$$

and thus A has a maximum extension.

Also, the nonlocal Fourier law is given by

$$Au = \nabla\cdot(\int_{R^d} J(z)\nabla u(x+z)\,dz).$$

Thus,

$$(Au, \phi)_{L^2} = \int_{R^d\times R^d} J(z)\nabla u(x+z)\cdot\nabla\phi(x)\,dz\,dx$$

Under the kernel J is completely monotone, one can prove that A has a maximal monotone extension.

1.4 Jump diffusion Model for American option

In this section we discuss the American option for the jump diffusion model

$$u_t + (x - \frac{\sigma^2}{2})u_x + \frac{\sigma^2 x^2}{2}u_{xx} + Bu + \lambda = 0, \quad u(T, x) = \psi,$$

$$(\lambda, u - \psi) = 0, \quad \lambda \le 0, \quad u \ge \psi$$

where the generator B for the jump process is given by

$$Bu = \int_{-\infty}^{\infty} k(s)(u(x+s) - u(x) + (e^s - 1)u_x)\, ds.$$

The CMGY model for the jump kernel k is given by

$$k(s) = \begin{cases} Ce^{-M|s|}|s|^{1+Y} = k^+(s) & s \geq 0 \\ \\ Ce^{-G|s|}|s|^{1+Y} = k^-(s) & s \leq 0 \end{cases}$$

Since

$$\int_{-\infty}^{\infty} k(s)(u(x+s) - u(x))\, ds = \int_0^{\infty} k^+(s)(u(x+s) - u(x))\, ds + \int_0^{\infty} k^-(s)(u(x-s) - u(x))\, ds$$

$$= \int_0^{\infty} \frac{k^+(s) + k^-(s)}{2}(u(x+s) - 2u(x) + u(x-s))\, ds + \int_0^{\infty} \frac{k^+(s) - k^-(s)}{2}(u(x+s) - u(x-s))\, ds.$$

Thus,

$$\int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} k(s)(u(x+s) - u(x)\, ds)\phi(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_0^{\infty} k_s(s)(u(x+s) - u(x))(\phi(x+s) - \phi(x))\, ds\, dx$$

$$+ \int_{-\infty}^{\infty} (\int_0^{\infty} k_u(s)(u(x+s) - u(s)))\phi(x)\, dx$$

where

$$k_s(s) = \frac{k^+(s) + k^-(s)}{2}, \quad k_u(s) = \frac{k^+(s) - k^-(s)}{2}$$

and hence

$$(Bu, \phi) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_s(s)(u(x+s) - u(x))(\phi(x+s) - \phi(s))\, ds\, dx$$

$$+ \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} k_u(s)(u(x+s) - u(s)))\phi(x)\, dx + \omega \int_{-\infty}^{\infty} u_x \phi\, dx.$$

where

$$\omega = \int_{-\infty}^{\infty} (e^s - 1)k(s)\, ds.$$

If we equip $V = H^1(R)$ by

$$|u|_V^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_s(s)|u(x+s) - u(x)|^2\, ds\, dx + \frac{\sigma^2}{2} \int_{-\infty}^{\infty} |u_x|^2\, dx,$$

then $A + B \in \mathcal{L}(V, V^*)$ and $A + B$ generates the analytic semigroup on $X = L^2(R)$.

1.5 Numerical approximation of nonlocal operator

In this section we describe our higher order integration method for the convolution;

$$\int_0^\infty \frac{k^+(s)+k^-(s)}{2}(u(x+s)-2u(x)+u(x-s))\,ds+\int_0^\infty \frac{k^+(s)-k^-(s)}{2}(u(x+s)-u(x-s))\,ds.$$

For the symmetric part,

$$\int_{-\infty}^{\infty} s^2 k_s(s)\,\frac{u(x+s)-2u(x)+u(x-s)}{s^2}\,ds,$$

where we have

$$\frac{u(x+s)-2u(x)+u(x-s)}{s^2} \sim u_{xx}(x)+\frac{s^2}{12}u_{xxxx}(x)+O(s^4)$$

We apply the fourth order approximation of u_{xx} by

$$u_{xx}(x) \sim \frac{u(x+h)-2u(x)+u(x-h)}{h^2}-\frac{1}{12}\frac{u(x+2h)-4u(x)+6u(x)-4u(x-h)+u(x-2h)}{h^2}$$

and we apply the second order approximation of $u_{xxxx}(x)$ by

$$u_{xxxx}(x) \sim \frac{u(x+2h)-4u(x)+6u(x)-4u(x-h)+u(x-2h)}{h^4}.$$

Thus, one can approximate

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s)\,\frac{u(x+s)-2u(x)+u(x-s)}{s^2}\,ds$$

by

$$\rho_0\,(\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}-\frac{1}{12}\frac{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}}{h^2})$$
$$+\frac{\rho_1}{12}\,\frac{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}}{h^2},$$

where

$$\rho_0=\int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s)\,ds \quad \text{and} \quad \rho_1=\frac{1}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} s^4 k_s(s)\,ds.$$

The remaining part of the convolution

$$\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j}+s)k_s(s)\,ds$$

can be approximated by three point quadrature rule based on

$$u(x_{k+j}+s) \sim u(x_{k+j})+u'(x_{k+j})s+\frac{s^2}{2}u''(x_{k+j})$$

with

$$u'(x_{k+j}) \sim \frac{u_{k+j+1} - u_{k+j-1}}{2h}$$

$$u''(x_{k+j}) \sim \frac{u_{k+j+1} - 2u_{k+j} + u_{k+j-1}}{h^2}.$$

That is,

$$\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j} + s) k_s(s)\, ds$$

$$\sim \rho_0^k u_{k+j} + \rho_1^k \frac{u_{k+j-1} - u_{k+j+1}}{2} + \rho_2^k \frac{u_{j+k+1} - 2u_{k+j} + u_{j+k-1}}{2}$$

where

$$\rho_0^k = \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} k_s(s)\, ds$$

$$\rho_1^k = \frac{1}{h} \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s - x_k) k_s(s)\, ds$$

$$\rho_2^k = \frac{1}{h^2} \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s - x_k)^2 k_s(s)\, ds.$$

For the skew-symmetric integral

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} k_u(s)(u(x+s) - u(x-s))\, ds \sim \rho_2\, u_x(x) + \frac{\rho_3}{6} h^2\, u_{xxx}(x)$$

where

$$\rho_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} 2s k_u(s)\, ds, \quad \rho_3 = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} 2s^3 k_u(s)\, ds.$$

We may use the forth order difference approximation

$$u_x(x) \sim \frac{u(x+h) - u(x-h)}{2h} - \frac{u(x+2h) - 2u(x+h) + 2u(x-h) - u(x-2h)}{6h}$$

and the second order difference approximation

$$u_{xxx}(x) \sim \frac{u(x+2h) - 2u(x+h) + 2u(x-h) - u(x-2h)}{h^3}$$

and obtain

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} k_u(s)(u(x+s) - u(x-s))\, ds$$

$$\sim \rho_2 \left(\frac{u_{k+1} - u_{k-1}}{2h} - \frac{u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-1}}{6h}\right) + \frac{\rho_3}{6} \frac{u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-1}}{h}.$$

Example (Fractional PDEs II) Consider the fractional equation of the form

$$\int_{-t}^{0} g(\theta) u'(t+\theta)\, d\theta = Au, \quad u(0) = u_0,$$

where the kernel g satisfies

$$g > 0, \quad g \in L^1(-\infty,0) \text{ and non-decreasing.}$$

For example, the case of the Caputo (fractional) derivative has

$$g(\theta) = \frac{1}{\Gamma(1-\alpha)}|\theta|^{-\alpha}.$$

Define $z(t,\theta) = u(t+\theta), \theta \in (-\infty,0]$. Then, $\frac{d}{dt}z = \frac{\partial}{\partial\theta}z$. Thus, we define the linear operator $\mathcal{A}$ on $Z = C((-\infty,0];X)$ by

$$\mathcal{A}z = z'(\theta) \text{ with } dom(\mathcal{A}) = \{z' \in X : \int_{-\infty}^0 g(\theta)z'(\theta)\,d\theta = Az(0)\}$$

Theorem 1.1 *Assume A is m-dissipative in a Banach space X. Then, $\mathcal{A}$ is dissipative and $R(\lambda I - \mathcal{A}) = Z$ for $\lambda > 0$. Thus, $\mathcal{A}$ generates the C_0-semigroup $T(t)$ on $Z = C((-\infty,0];X)$.*

Proof: First we show that $\mathcal{A}$ is dissipative. For $\phi \in dom\,(\mathcal{A})$ suppose $|\phi(0)| > |\phi(\theta)|$ for all $\theta < 0$. Define

$$g_\epsilon(\theta) = \frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} g(\theta)\,d\theta.$$

For all $x^* \in F(\phi(0))$

$$\langle \int_{-\infty}^0 g_\epsilon(\theta)(\phi')d\theta, x^*\rangle$$

$$= -\langle \int_{-\infty}^0 \frac{g(\theta)-g(\theta-\epsilon)}{\epsilon}\,\langle \phi(\theta)-\phi(0), x^*\rangle\,d\theta > 0$$

since

$$\langle \phi(\theta)-\phi(0), x^*\rangle \le (|\phi(\theta)|-|\phi(0)|)|\phi(0)| < 0, \quad \theta < 0.$$

Thus,

$$\lim_{\epsilon\to 0^+}\langle \int_{-\infty}^0 g_\epsilon(\theta)(\phi')d\theta, x^*\rangle = \langle \int_{-\infty}^0 g(\theta)\phi'd\theta, x^*\rangle > 0. \tag{1.34}$$

But, since there exists a $x^* \in F(\phi(0))$ such that

$$\langle Ax, x^*\rangle \le 0$$

which contradicts to (1.34). Thus, there exists θ_0 such that $|\phi(\theta_0)| = |\phi|_Z$. Since $\langle \phi(\theta), x^*\rangle \le |\phi(\theta)|$ for $x^* \in F(\phi(\theta_0))$, $\theta \to \langle \phi(\theta), x^*\rangle$ attains the maximum at θ_0 and thus $\langle \phi'(\theta_0), x^*\rangle = 0$ Hence,

$$|\lambda\,\phi - \phi'|_Z \ge \langle \lambda\,\phi(\theta_0) - \phi'(\theta_0), x^*\rangle = \lambda\,|\phi(\theta_0)| = \lambda\,|\phi|_Z. \tag{1.35}$$

For the range condition $\lambda\phi - \mathcal{A}\phi = f$ we note that

$$\phi(\theta) = e^{\lambda\theta}\phi(0) + \psi(\theta)$$

where

$$\psi(\theta)=\int_\theta^0 e^{\lambda(\theta-\xi)}f(\xi)\,d\xi.$$

Thus,

$$(\Delta(\lambda)\,I-A)\phi(0)=\int_{-\infty}^0 g(\theta)\psi'(\theta)\,d\theta)$$

where

$$\Delta(\lambda)=\lambda\int_{-\infty}^0 g(\theta)e^{\lambda\theta}\,d\theta>0$$

Thus,

$$\phi(0)=(\Delta(\lambda)\,I-A)^{-1}\int_{-\infty}^0 g'(\theta)\psi(\theta)\,d\theta.$$

Since $\mathcal{A}$ is dissipative and

$$\lambda\psi-\psi'=f\in Z,\quad \psi(0)=0,$$

thus $|\psi|_Z\leq\dfrac{1}{\lambda}|f|_Z$. Thus $\phi=(\lambda\,I-\mathcal{A})^{-1}f\in Z$. $\square$

Example (Renewable system) We discuss the renewable system of the form

$$\tfrac{dp_0}{dt}=-\textstyle\sum_i\lambda_i\,p_0(t)+\sum_i\int_0^L\mu_i(x)p_i(x,t)\,dx$$

$$(p_i)_t+(p_i)_x=-\mu_i(x)p,\quad p(0,t)=\lambda_i\,p_0(t)$$

for $(p_0,p_i,\ 1\leq i\leq d)\in R\times L^1(0,T)^d$. Here, $p_0(t)\geq 0$ is the total utility and $\lambda_i\geq 0$ is the rate for the energy conversion to the i-th process p_i. The first equation is the energy balance law and s is the source = generation $-$consumption. The second equation is for the transport (via pipeline and storage) for the process p_i and $\mu_i\geq 0$ is the renewal rate and $\bar\mu\geq 0$ is the natural loos rate. $\{\lambda_i\geq 0\}$ represent the distribution of the utility to the i-th process.

Assume at the time $t=0$ we have the available utility $p_0(0)=1$ and $p_i=0$. Then we have the following conservation

$$p_0(t)+\int_0^t p_i(s)\,ds=1$$

if $t\leq L$. Let $X=R\times L^1(0,L)^d$. Let $A(\mu)$ defined by

$$Ax=(-\sum_i\lambda_i\,p_0+\sum_i\int_0^L\mu_i(x)\,dx,-(p_i)_x-\mu_i(x)p_i)$$

with domain

$$dom(A)=\{(p_0,p_i)\in R\times W^{11}(0,L)^d:p_i(0)=\lambda_i\,p_0\}$$

Let

$$\mathrm{sign}_\epsilon(s)=\begin{cases}\frac{s}{|s|} & |s|>\epsilon\\ \frac{s}{\epsilon} & |s|\leq\epsilon\end{cases}$$

Then,

$$(A(p_0,p),(\text{sign}_0(p_0),\text{sign}_\epsilon(p)) \leq -(\sum_i \lambda_i)|p_0| + |\int_0^L \mu_i p_i\,dx|$$

$$\sum_i(\Psi_\epsilon(p_i(0)) - \Psi_\epsilon(p_i(L)) - \int_0^L \mu_i p_i \text{sign}_\epsilon\,dx)$$

where

$$\Psi_\epsilon(s) = \begin{cases} |s| & |s| > \epsilon \\ \\ \frac{s^2}{2\epsilon} + \frac{\epsilon}{2} & |s| \leq \epsilon \end{cases}$$

Since

$$\text{sign}_\epsilon \to \text{sign}_\epsilon, \quad \Psi_\epsilon \to |s|$$

by the Lebesgue dominated convergence theorem, we have

$$(A(p_0,p),(\text{sign}_0(p_0),\text{sign}_0(p)) \leq 0.$$

The resolvent equation

$$A(p_0,p) = (s,f), \tag{1.36}$$

has a solution

$$p_i(x) = \lambda_i p_0 e^{-\int_0^x \mu_i} + \int_0^x e^{-\int_s^x \mu_i} f(s)\,ds$$

$$(\sum_i \lambda_i)(1 - e^{\int_0^L \mu_i})\,p_0 = s + \int_0^L \mu_i p_i(x)\,dx$$

Thus, A generates the contractive C_0 semigroup $S(t)$ on X. Moreover, it is cone preserving $S(t)\mathcal{C}^+ \subset \mathcal{C}^+$ since the resolvent is positive cone preserving.

Example (Bi-domain equation)

The electrical behavior of the cardiac tissue is described by a system consisting of PDEs coupled with ordinary differential equations which model the ionic currents associated with the reaction terms. The bi-domain model is a mathematical model for the electrical properties of cardiac muscle that takes into account the anisotropy of both the intracellular and extracellular spaces. It is formed of the bi-domain equations. The bi-domain model is now widely used to model defibrillation of the heart. In this paper we consider the feedback control for bi-domain model.

The weak form of the the bi-domain equation is given by

$$\begin{aligned} &(\frac{d}{dt}u,\phi) - (\mathcal{B}(\nabla u + \nabla u_e),\nabla\phi)_\Omega + (F(u,v),\phi) = 0 \\ &(\mathcal{B}\nabla u + (\mathcal{A}+\mathcal{B})\nabla u_e, \nabla\psi)_\Omega = \langle s,\psi\rangle, \end{aligned} \tag{1.37}$$

for all $(\phi,\psi) \in H^1(\Omega) \times H^1(\Omega)/R$, where $(u,u_e) \in H^1(\Omega) \times H^1(\Omega)/R$ is the solution pair and s is the control current. We consider the boundary current control:

$$\langle s,\psi\rangle = \int_\Gamma s(t,x)\psi(x)\,dx.$$

Here, $\mathcal{A}$ and $\mathcal{B}$ are elliptic operators of the form

$$\mathcal{B}\phi = \nabla\cdot(\bar{\sigma}_i\nabla\phi), \quad \mathcal{A}\phi = \nabla\cdot(\bar{\sigma}_e\nabla\phi),$$

where $\bar{\sigma}_i,\ \bar{\sigma}_e$ are respectively the intracellular and extracellular conductivity tensors. Note that one can write (1.37) as

$$\frac{d}{dt}u(t) - \mathcal{L}u(t) + F(u(t), v(t))) + \mathcal{C}s(t) = 0, \tag{1.38}$$

where

$$\mathcal{L} = (\mathcal{A}^{-1} + \mathcal{B}^{-1})^{-1} = \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}.$$

and

$$\mathcal{C}s = \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}s.$$

That is, $v = u + u_e$ satisfies

$$(\mathcal{A} + \mathcal{B})v = \mathcal{A}u + s$$

where

$$\langle s, \phi\rangle = \int_\Gamma s(t,x)\phi(x)\,dx$$

with $\langle s, 1\rangle = 0$. Thus, $\mathcal{L}$ is an self adjoint elliptic operator on $L^2(\Omega)$. The boundary current control becomes the distributed control of the form $\mathcal{C}s(t)$.

Example (Second order equation) Let $V \subset H = H^* \subset V^*$ be the Gelfand triple. Let ρ be a bounded bilinear form on $H \times H$, μ and σ be bounded bilinear forms on $V \times V$. Assume ρ and σ are symmetric and coercive and $\mu(\phi, \phi) \geq 0$ for all $\phi \in V$. Consider the second order equation

$$\rho(u_{tt}, \phi) + \mu(u_t, \phi) + \sigma(u, \phi) = \langle f(t), \phi\rangle \text{ for all } \phi \in V. \tag{1.39}$$

Define linear operators M (mass), D (dampping), K and (stiffness) by

$$(M\phi, \psi)_H = \rho(\phi, \psi), \quad \phi,\ \psi \in H$$

$$\langle D\phi, \psi\rangle = \mu(\phi, \psi) \quad \phi,\ \psi \in V$$

$$\langle K\phi, \psi\rangle_{V^*\times V} = \sigma(\phi, \psi), \quad \phi,\ \psi \in V$$

We assume ρ is symmetric and H-coercive, σ is symmetric and V-coercive and $\mu(\phi, \phi) \geq 0$ for $\phi \in V$. Let $v = u_t$ and define A on $X = V \times H$ by

$$A(u, v) = (v, -M^{-1}(Ku + Dv))$$

with domain

$$dom\,(A) = \{(u, v) \in X : v \in V \text{ and } Ku + Dv \in H\}$$

The state space X is a Hilbert space with inner product

$$((u_1, v_1), (u_, v_2)) = \sigma(u_1, u_2) + \rho(v_1, v_2)$$

and

$$E(t) = |(u(t), v(t))|_X^2 = \sigma(u(t), u(t)) + \rho(v(t), v(t))$$

defines the energy of the state $x(t)=(u(t),v(t))\in X$. First, we show that A is dissipative:

$$(A(u,v),(u,v))_X=\sigma(u,v)+\rho(-M^{-1}(Ku+Dv),v)=\sigma(u,v)-\sigma(u,v)-\mu(v,v)=-\mu(v,v)\leq 0$$

Next, we show that $R(\lambda\,I-A)=X$. That is, for $(f,g)\in X$ the exists a solution $(u,v)\in$ $dom\,(A)$ satisfying

$$\lambda\,u-v=f,\quad \lambda Mv+Dv+Ku=Mg,$$

or equivalently $v=\lambda\,u-f$ and

$$\lambda^2 Mu+\lambda Du+Ku=Mg+\lambda\,Mf+Df \tag{1.40}$$

Define the bilinear form a on $V\times V$

$$a(\phi,\psi)=\lambda^2\,\rho(\phi,\psi)+\lambda\,\mu(\phi,\psi)+\sigma(\phi,\psi)$$

Then, a is bounded and V-coercive and if we let

$$F(\phi)=(M(g+\lambda\,f)\phi)_H+\mu(f,\phi)$$

then $F\in V^*$. It thus follows from the Lax-Milgram theory there exists a unique solution $u\in V$ to (1.40) and $Dv+Ku\in H$.

For example, consider the wave equation

$$\tfrac{1}{c^2(x)}u_{tt}+\kappa(x)u_t=\Delta u$$

$$[\tfrac{\partial u}{\partial n}]+\alpha u=\gamma\,u_t \text{ at } \Gamma$$

In this example we let $V=H^1(\Omega)/R$ and $H=L^2(\Omega)$ and define

$$\sigma(\phi,\psi)=\int_\Omega(\nabla\phi,\nabla\psi)\,dx+\int_\Gamma\alpha\phi\psi\,ds$$

$$\mu(\phi,\psi)=\int_{\partial\Omega}\kappa(x)\phi,\psi\,dx+\int_\Gamma\gamma)\phi,\psi\,ds$$

$$\rho(\phi,\psi)=\int_\Omega\frac{1}{c^2(x)}\,\phi\psi\,dx.$$

Example (Maxwell system for electro-magnetic equations)

$$\epsilon\,E_t=\nabla\times H,\quad \nabla\cdot E=\rho$$

$$\mu\,H_t=-\nabla\times E,\quad \nabla\cdot B=0$$

with boundary condition

$$E\times n=0$$

where E is Electric field, $B\mu\, H$ is Magnetic field and $D = \epsilon\, E$ is dielectric with $\epsilon,\ \mu$ is electric and magnetic permittivity, respectively. Let $X = L^2(\Omega)^d \times L^2(\Omega)^d$ with the norm defined by

$$|(E,H)|_X^2 = \int_\Omega (\epsilon\,|E|^2 + \mu\,|H|^2)\,dx.$$

The dissipativity follows from

$$\int_\Omega (E\cdot(\nabla\times H) - H\cdot(\nabla\times E))\,dx = \int_\Omega \nabla\cdot(E\times H)\,dx = \int_{\partial\Omega} n\cdot(E\times H)\,ds = 0$$

Let $\rho = 0$ and thus $\nabla\cdot E = 0$. The range condition is equivalent to

$$\epsilon\, E + \nabla\times\frac{1}{\mu}(\nabla\times E - g) = f$$

The weak form is given by

$$(\epsilon\, E,\psi) + (\frac{1}{\mu}\nabla\times E, \nabla\times\psi) = (f,\psi) + (g, \frac{1}{\mu}\nabla\times\psi). \tag{1.41}$$

for $\psi \in V = \{H^1(\Omega) : \nabla\cdot\psi = 0,\ \ n\times\psi = 0 \text{ at } \partial\Omega\}$. Since $|\nabla\times\psi|^2 = |\nabla\psi|^2$ for $\nabla\cdot\psi = 0$. the right hand side of (1.41) defines the bounded coercive quadratic form on $V\times V$, it follows from the Lax-Milgram equation that (1.41) has a unique solution in V.

1.6 Dual semigroup

Theorem (Dual semigroup) Let X be a reflexive Banach space. The adjoint $S^*(t)$ of the C_0 semigroup $S(t)$ on X forms the C_0 semigroup and the infinitesimal generator of $S^*(t)$ is A^*. Let X be a Hilbert space and $dom(A^*)$ be the Hilbert space with graph norm and X_{-1} be the strong dual space of $dom(A^*)$, then the extension $S(t)$ to X_{-1} defines the C_0 semigroup on X_{-1}.

Proof: (1) Since for $t,\ s \ge 0$

$$S^*(t+s) = (S(s)S(t))^* = S^*(t)S^*(s)$$

and

$$\langle x, S^*(t)y - y\rangle_{X\times X^*} = \langle S(t)x - x, y\rangle_{X\times X^*} \to 0.$$

for $x \in X$ and $y \in X^*$ Thus, $S^*(t)$ is weakly star continuous at $t = 0$ and let B is the generator of $S^*(t)$ as

$$Bx = w^* - \lim\,\frac{S^*(t)x - x}{t}.$$

Since

$$(\frac{S(t)x - x}{t}, y) = (x, \frac{S^*(t)y - y}{t}),$$

for all $x \in dom(A)$ and $y \in dom(B)$ we have

$$\langle Ax, y\rangle_{X\times X^*} = \langle x, By\rangle_{X\times X^*}$$

and thus $B = A^*$. Thus, A^* is the generator of a w^*- continuous semigroup on X^*.
(2) Since

$$S^*(t)y - y = A^* \int_0^t S^*(s)y\,ds$$

for all $y \in Y = \overline{dom\,(A^*)}$. Thus, $S^*(t)$ is strongly continuous at $t = 0$ on Y.
(3) If X is reflexive, $\overline{dom\,(A^*)} = X^*$. If not, there exists a nonzero $y_0 \in X$ such that $\langle y_0, x^*\rangle_{X\times X^*} = 0$ for all $x^* \in dom(A^*)$. Thus, for $x_0 = (\lambda\, I - A)^{-1}y_0$ $\langle \lambda\, x_0 - Ax_0, x^*\rangle = \langle x_0, \lambda\, x^* - A^*x^*\rangle = 0$. Letting $x^* = (\lambda\, I - A^*)^{-1}x_0^*$ for $x_0^* \in F(x_0)$, we have $x_0 = 0$ and thus $y_0 = 0$, which yields a contradiction.
(4) $X_1 = dom\,(A^*)$ is a closed subspace of X^* and is a invariant set of $S^*(t)$. Since A^* is closed, $S^*(t)$ is the C_0 semigroup on X_1 equipped with its graph norm. Thus,

$$(S^*(t))^* \text{ is the } C_0 \text{ semigroup on } X_{-1} = X_1^*$$

and defines the extension of $S(t)$ to X_{-1}. Since for $x \in X \subset X_{-1}$ and $x^* \in X^*$

$$\langle S(t)x, x^*\rangle = \langle x, S^*(t)x^*\rangle,$$

$S(t)$ is the restriction of $(S^*(t))^*$ onto X. $\square$

1.7 Stability

Theorem (Datko 1970, Pazy 1972). A strongly continuous semigroup $S(t)$, $t \geq 0$ on a Banach space X is uniformly exponentially stable if and only if for $p \in [1, \infty)$ one has

$$\int_0^\infty |S(t)x|^p\,dt < \infty \text{ for all } x \in X.$$

Theorem. (Gearhart 1978, Pruss 1984, Greiner 1985) A strongly continuous semigroup on $S(t)$, $t \geq 0$ on a Hilbert space X is uniformly exponentially stable if and only if the half-plane $\{\lambda \in C : Re\lambda > 0\}$ is contained in the resolvent set $\rho(A)$ of the generator A with the resolvent satisfying

$$|(\lambda\, I - A)^{-1}|_\infty < \infty$$

1.8 Sectorial operator and Analytic semigroup

In this section we have the representation of the semigroup $S(t)$ in terms of the inverse Laplace transform. Taking the Laplace transform of

$$\frac{d}{dt}x(t) = Ax(t) + f(t)$$

we have

$$\hat{x} = (\lambda\, I - A)^{-1}(x(0) + \hat{f})$$

where for $\lambda > \omega$

$$\hat{x} = \int_0^\infty e^{-\lambda t}x(t)\,dt$$

is the Laplace transform of $x(t)$. We have the following the representation theory (inverse formula).

Theorem (Resolvent Calculus) For $x \in dom(A^2)$ and $\gamma > \omega$

$$S(t)x = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda\, I - A)^{-1}x\, d\lambda. \tag{1.42}$$

Proof: Let A_μ be the Yosida approximation of A. Since $\mathrm{Re}\,\sigma(A_\mu) \le \dfrac{\omega_0}{1-\mu\omega_0} < \gamma$, we have

$$u_\mu(t) = S_\mu(t)x = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda\, I - A_\mu)^{-1}x\, d\lambda.$$

Note that

$$\lambda(\lambda\, I - A)^{-1} = I + (\lambda\, I - A)^{-1}A. \tag{1.43}$$

Since

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda}\, d\lambda = 1$$

and

$$\int_{\gamma-i\infty}^{\gamma+i\infty} |\lambda-\omega|^{-2}\, d\lambda < \infty,$$

we have

$$|S_\mu(t)x| \le M\, |A^2 x|,$$

uniformly in $\mu > 0$. Since

$$(\lambda\, I - A_\mu)^{-1}x - (\lambda\, I - A)^{-1}x = \frac{\mu}{1+\lambda\mu}(\nu\, I - A)^{-1}(\lambda\, I - A)^{-1}A^2 x,$$

where $\nu = \dfrac{\lambda}{1+\lambda\mu}$, $\{u_\mu(t)\}$ is Cauchy in $C(0,T;X)$ if $x \in dom(A^2)$. Letting $\mu \to 0^+$, we obtain (1.42). $\square$

Next we consider the sectorial operator. For $\delta > 0$ let

$$\Sigma_\omega^\delta = \{\lambda \in C : arg(\lambda-\omega) < \frac{\pi}{2} + \delta\}$$

be the sector in the complex plane C. A closed, densely defined, linear operator A on a Banach space X is a sectorial operator if

$$|(\lambda\, I - A)^{-1}| \le \frac{M}{|\lambda-\omega|} \text{ for all } \lambda \in \Sigma_\omega^\delta.$$

For $0 < \theta < \delta$ let $\Gamma = \Gamma_{\omega,\theta}$ be the integration path defined by

$$\Gamma^\pm = \{z \in C : |z| \ge \delta,\ arg(z-\omega) = \pm(\tfrac{\pi}{2}+\theta)\},$$

$$\Gamma_0 = \{z \in C : |z| = \delta,\ |arg(z-\omega)| \le \tfrac{\pi}{2}+\theta\}.$$

For $0 < \theta < \delta$ define a family $\{S(t),\ t \geq 0\}$ of bounded linear operators on X by

$$S(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda\, I - A)^{-1}x\, d\lambda. \tag{1.44}$$

Theorem (Analytic semigroup) If A is a sectorial operator on a Banach space X, then A generates an analytic (C_0) semigroup on X, i.e., for $x \in X$ $t \to S(t)x$ is an analytic function on $(0, \infty)$. We have the representation (1.44) for $x \in X$ and

$$|AS(t)x|_X \leq \frac{M_\theta}{t}|x|_X \ \ (\omega = 0).$$

Proof: Since

$$AS(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda(\lambda\, I - A)^{-1}x - x)\, d\lambda.$$

we have

$$|AS(t)x| \leq M\int_0^\infty e^{-sin\theta\, tz}\, dz|x| = \frac{M}{\sin\theta\, t}|x|.\square$$

The elliptic operator A defined by the Lax-Milgram theorem defines a sectorial operator on Hilbert space X.

Theorem (Sectorial operator) Let $V,\ H$ are Hilbert spaces and assume $H \subset V^*$. Let $\rho(u, v)$ is bounded bilinear form on $H \times H$ and

$$\rho(u, u) \geq |u|_H^2 \text{ for all } u \in H$$

Let $a(u, v)$ to be a bounded bilinear form on $V \times V$ with

$$\sigma(u, u) \geq \delta\, |u|_V^2 \text{ for all } u \in V.$$

Define the linear operator A by

$$\rho(Au, \phi) = a(u, \phi) \text{ for all } \phi \in V.$$

Then, for $Re\,\lambda > 0$we have

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V,V^*)} \leq \tfrac{1}{\delta}$$

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H)} \leq \tfrac{M}{|\lambda|}$$

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V^*,H)} \leq \tfrac{M}{\sqrt{|\lambda|}}$$

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H,V)} \leq \tfrac{M}{\sqrt{|\lambda|}}$$

Proof: Let $a(u, v)$ to be a bounded bilinear form on $V \times V$. Define $M \in \mathcal{L}(H, H)$ by

$$(Mu, v) = \rho(u, v) \text{ for all } u,\ v \in H$$

and $A_0 \in \mathcal{L}(V, V^*)$ by

$$\langle A_0 u, v\rangle = \sigma(u, v) \text{ for } v \in V$$

Then, $A = M^{-1}A_0$ and for $f \in V^*$ and $Re\,\lambda > 0$, $(\lambda\, I - A)u = M^{-1}f$ is equivalent to

$$\lambda\rho(u,\phi) + a(u,\phi) = \langle f,\phi\rangle, \text{ for all } \phi \in V. \tag{1.45}$$

It follows from the Lax-Milgram theorem that (1.45) has a unique solution, given $f \in V^*$ and

$$Re\,\lambda\rho(u,u) + a(u,u) \leq |f|_{V^*}|u|_V.$$

Thus,

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V^*,V)} \leq \frac{1}{\delta}.$$

Also,

$$|\lambda|\,|u|_H^2 \leq |f|_{V^*}|u|_V + M\,|u|_V^2 = M_1\,|f|_{V^*}^2$$

for $M_1 = 1 + \frac{M}{\delta^2}$ and thus

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V^*,H)} \leq \frac{\sqrt{M_1}}{|\lambda|^{1/2}}.$$

For $f \in H \subset V^*$

$$\delta\,|u|_V^2 \leq Re\,\lambda\,\rho(u,u) + a(u,u) \leq |f|_H|u|_H, \tag{1.46}$$

and

$$|\lambda|\rho(u,u) \leq |f|_H|u|_H + M|u|_V^2 \leq M_1|f|_H|u|_H$$

Thus,

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H)} \leq \frac{M_1}{|\lambda|}.$$

Also, from (1.46)

$$\delta\,|u|_V^2 \leq |f|_H|u|_H \leq M_1\,|f|^2.$$

which implies

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H,V)} \leq \frac{M_2}{|\lambda|^{1/2}}.$$

1.9 Approximation Theory

In this section we discuss the approximation theory for the linear C_0-semigroup. Equivalence Theorem (Lax-Richtmyer) states that for consistent numerical approximations, stability and convergence are equivalent. In terms of the linear semigroup theory we have

Theorem (Trotter-Kato theorem) Let X and X_n be Banach spaces and A and A_n be the infinitesimal generator of C_0 semigroups $S(t)$ on X and $S_n(t)$ on X_n of $G(M,\omega)$ class. Assume a family of uniformly bounded linear operators $P_n \in \mathcal{L}(X,X_n)$ and $E_n \in \mathcal{L}(X_n,X)$ satisfy

$$P_nE_n = I \quad |E_nP_nx - x|_X \to 0 \text{ for all } x \in X \tag{1.47}$$

Then, the followings are equivalent.
(1) there exist a $\lambda_0 > \omega$ such that for all $x \in X$

$$|E_n(\lambda_0\, I - A_n)^{-1}P_nx - (\lambda_0\, I - A)^{-1}x|_X \to 0 \quad \text{as } n \to \infty, \tag{1.48}$$

(2) For every $x \in X$ and $T \geq 0$

$$|E_n S_n(t) P_n x - S(t)x|_X \to \quad \text{as } n \to \infty.$$

uniformly on $t \in [0, T]$.

Proof: Since for $\lambda > \omega$

$$E_n(\lambda\, I - A)^{-1} P_n x - (\lambda\, I - A)^{-1} x = \int_0^\infty E_n S_n(t) P_n x - S(t) x\, dt$$

(1) follows from (2). Conversely, from the representation theory

$$E_n S_n(t) P_n x - S(t) x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (E_n(\lambda\, I - A)^{-1} P_n x - (\lambda\, I - A)^{-1} x)\, d\lambda$$

where

$$(\lambda\, I - A)^{-1} - (\lambda_0\, I - A)^{-1} = (\lambda - \lambda_0)(\lambda\, I - A)^{-1}(\lambda_0\, I - A)^{-1}.$$

Thus, from the proof of Theorem (Resolvent Calculus) (1) holds for $x \in dom(A^2)$. But since $dom(A^2)$ is dense in X, (2) implies (1). □

Remark (Stability) If A_n is uniformly dissipative:

$$|\lambda u_n - A_n u_n| \geq (\lambda - \omega)\, |u_n|$$

for all $u_n \in dom(A_n)$ and some $\omega \geq 0$, then A_n generates ω contractive semigroup $S_n(t)$ on X_n.

Remark (Consistency)

$$(\lambda I - A_n) u_n = P_n f$$

$$P_n(\lambda I - A) u = P_n f$$

we have

$$(\lambda\, I - A_n)(P_n u - u_n) + P_n A u - A_n P_n u = 0$$

Thus

$$|P_n u - u_n| \leq M\, |P_n A u - A_n P_n u|$$

The consistency(1.48) follows from

$$|P_n A u - A_n P_n u| \to 0$$

for all u in a dense subset of $dom(A)$.

Corollary Let the assumptions of Theorem hold. The statement (1) of Theorem is equivalent to (1.47) and the followings:

(C.1) there exists a subset D of $dom(A)$ such that $\overline{D} = X$ and $\overline{(\lambda_0\, I - A) D} = X$.

(C.2) for all $u \in D$ there exists a sequence $\bar{u}_n \in dom(A_n)$ such that $\lim E_n \bar{u}_n = u$ and $\lim E_n A_n \bar{u}_n = Au$.

Proof: Without loss of generality we can assume $\lambda_0 = 0$. First we assume that condition (1) hold. We set $D = dom(A)$ and thus $AD = X$. For $u \in dom(A)$ we set $\bar{u}_n = A_n^{-1} P_n A u$ and $u = A^{-1}x$. Then,

$$E_n \bar{u}_n - u = E_n A_n^{-1} P_n x - A^{-1} x \to 0$$

and

$$E_n A_n \bar{u}_n - Au = E_n A_n A_n^{-1} P_n x - AA^{-1} x = E_n P_n x - x \to 0$$

as $n \to \infty$. Hence conditions (C.1)–(C.2) hold.

Conversely, we assume conditions (C.1)–(C.2) hold. For $x \in AD$ we choose $u \in D$ such that $u = A^{-1}x$ and set $u_n = A_n^{-1} P_n x = A_n^{-1} P_n A u$. We then for u we choose $\bar{u}_n$ according to (C.2). Thus, we obtain

$$|\bar{u}_n - P_n u| = |P_n(E_n \bar{u}_n - u)| \le M \, |E_n \bar{u}_n - u| \to 0$$

as $n \to \infty$ and

$$|\bar{u}_n - u_n| \le |A_n^{-1}(A_n \bar{u}_n - P_n A u)| \le |A_n^{-1} P_n| |E_n A_n \bar{u}_n - Au| \to 0$$

as $n\infty$. It thus follows that $|u_n - P_n u| \to 0$ as $n \to \infty$. Since

$$E_n A_n^{-1} P_n - A^{-1} = E_n(A_n^{-1} P_n A - P_n) A^{-1} + (E_n P_n - I) A^{-1},$$

we have

$$|E_n A_n^{-1} P_n x - A^{-1} x| \le |E_n(u_n - P_n u)| + |E_n P_n u - u| \le M \, |u_n - P_n u| + |E_n P_n u - u| \to 0$$

as $n \to \infty$ for all $x \in AD$. □

Example 1 (Trotter-Kato theoarem) Consider the heat equation on $\Omega = (0,1) \times (0,1)$:

$$\frac{d}{dt} u(t) = \Delta u, \quad u(0,x) = u_0(x)$$

with boundary condition $u = 0$ at the boundary $\partial\Omega$. We use the central difference approximation on uniform grid points: $(i\,h, j\,h) \in \Omega$ with mesh size $h = \frac{1}{n}$:

$$\frac{d}{dt} u_{i,j}(t) = \Delta_h u = \frac{1}{h}\left(\frac{u_{i+1,j} - u_{i,j}}{h} - \frac{u_{i,j} - u_{i-1,j}}{h}\right) + \frac{1}{h}\left(\frac{u_{i,j+1} - u_{i,j}}{h} - \frac{u_{i,j} - u_{i,j-1}}{h}\right)$$

for $1 \le i,\ j \le n_1$, where $u_{i,0} = u_{i,n} = u1,j = u_{n,j} = 0$ at the boundary node. First, let $X = C(\Omega)$ and $X_n = R^{(n-1)^2}$ with sup norm. Let $E_n u_{i,j} =$ the piecewise linear interpolation and $(P_n u)_{i,j} = u(i\,h, j\,h)$ is the point-wise evaluation. We will prove that Δ_h is dissipative on X_n. Suppose $u_{ij} = |u_n|_\infty$. Then, since

$$\lambda\, u_{i,j} - (\Delta_h u)_{i,j} = f_{ij}$$

and

$$-(\Delta_h u)_{i,j} = \frac{1}{h^2}(4u_{i,j} - u_{i+1,j} - u_{i,j+1} - u_{i-1,j} - u_{i,j-1}) \ge 0$$

we have

$$0 \leq u_{i,j} \leq \frac{f_{i,j}}{\lambda}.$$

Thus, Δ_h is dissipative on X_n with sup norm. Next $X = L^2(\Omega)$ and X_n with ℓ^2 norm. Then,

$$(-\Delta_h u_n, u_n) = \sum_{i,j} |\frac{u_{i,j} - u_{i-1,j}}{h}|^2 + |\frac{u_{i,j} - u_{i,j-1}}{h}|^2 \geq 0$$

and thus Δ_h is dissipative on X_n with ℓ^2 norm.

Example 2 (Galerkin method) Let $V \subset H = H^* \subset V^*$ is the Gelfand triple. Consider the parabolic equation

$$\rho(\frac{d}{dt} u_n, \phi) = a(u_n, \phi) \tag{1.49}$$

for all $\phi \in V$, where the ρ is a symmetric mass form

$$\rho(\phi, \phi) \geq c\, |\phi|_H^2$$

and a is a bounded coercive bilinear form on $V \times V$ such that

$$a(\phi, \phi) \geq \delta\, |\phi|_V^2.$$

Define A by

$$\rho(Au, \phi) = a(u, \phi) \text{ for all } \phi \in V.$$

By the Lax-Milgram theorem

$$(\lambda I - A)u = f \in H$$

has a unique solution satisfying

$$\lambda \rho(u, \phi) - a(u, \phi) = (f, \phi)_H$$

for all $\phi \in V$. Let $dom(A) = (I - A)^{-1} H$. Assume

$$V_n = \{u = \sum a_k \phi_k^n, \;\; \phi_k^n \in V\} \text{ is dense in } V$$

Consider the Galerkin method, i.e. $u_n(t) \in V_n$ satisfies

$$\rho(\frac{d}{dt} u_n(t), \phi) = a(u_n, \phi).$$

Since for $u = (\lambda\, I - A)^{-1} f$ and $\bar{u}_n \in V_n$

$$\lambda \rho(u_n, \phi) + a(u_n, \phi) = (f, \phi) \text{ for } \phi \in V_n$$

$$\lambda \rho(\bar{u}_n, \phi) + a(\bar{u}_n, \phi) = \lambda \rho(\bar{u}_n - u, \phi) + a(\bar{u}_n - u, \phi) + (f, \phi) \text{ for } \phi \in V_n$$

$$\lambda \rho(u_n - \bar{u}_n, \phi) + a(u_n - \bar{u_n}, \phi) = \lambda \rho(\bar{u}_n - u, \phi) + a(\bar{u}_n - u, \phi).$$

Thus,

$$|u_n - \bar{u}_n| \leq \frac{M}{\delta} |\bar{u_n} - u|_V.$$

Example 2 (Discontinuous Galerkin method) Consider the parabolic system for $u = \vec{u} \in L^2(\Omega)^d$

$$\frac{\partial}{\partial t} u = \nabla \cdot (a(x) \nabla u) + c(x) u$$

where $a \in R^d \times d$ is symmetric and

$$\underline{a} |\xi|^2 \leq (\xi, a(x), \xi)_{R^d} \leq \overline{a} |\xi|^2, \quad \xi \in R^d$$

for $0 < \underline{a} \leq \overline{a} < \infty$. The region Ω is dived into n non-overlapping sub-domains Ω_i with boundaries $\partial \Omega_i$ such that $\Omega = \cup \Omega_i$. At the interface $\Gamma_{ij} = \partial \Omega_i \cap \partial \Omega_j$ define

$$[[u]] = u|_{\partial \Omega_i} - u|_{\partial \Omega_j}$$

$$<< u >>= \tfrac{1}{2} (u|_{\partial \Omega_i} + u|_{\partial \Omega_j}).$$

The approximate solution $u_h(t)$ in

$$V_h = \{ u_h \in L^2(\Omega) : u_h \text{ is linear on } \Omega_i \}.$$

Define the bilinear for on $V_h \times V_h$

$$a_h(u, v) = \sum_i \int_{\Omega_i} (a(x) \nabla u, \nabla v) \, dx - \sum_{i > j} \int_{\Gamma_{ij}} (<< n \cdot (a \nabla u) >> [[v]] \pm << n \cdot (a \nabla v) >> [[u]] + \frac{\beta}{h} [[u]][[v]] \, ds),$$

whee h is the meshsize and $\beta > 0$ is sufficiently large. If $+$ on the third term a_h is symmetric and for the case $-$ then a_h enjoys the coercivity

$$a_h(u, u) \geq \sum_i \int_{\Omega_i} (a(x) \nabla u, \nabla u) \, dx, \in u \in V_h,$$

regardless of $\beta > 0$.

Example 3 (Population dynaimcs) The transport equation

$$\tfrac{\partial p}{\partial t} + \tfrac{\partial p}{\partial x} + m(x) p(x, t) = 0$$

$$p(0, t) = \textstyle\int \beta(x) p(x, t) \, dx$$

Define the difference approximation

$$A_n p = (-\frac{p_i - p_{i-1}}{h} - m(x_i) p_i, \ 1 \leq i \leq n), \quad p_0 = \sum_i \beta_i \, p_i$$

Then,

$$(A_n p, sign_0(p)) \leq (\sum m_i - \beta_i) |p_i| \leq 0$$

Thus, A_n on $L^1(0, 1)$ is dissipative.

$$(A, E_n \phi) - (P_n A_n, \phi).$$

Example 4 (Yee's scheme)

Consider the two dimensional Maxwell's equation. Consider the staggered grid; i.e. $E = (E^1_{i-\frac{1}{2},j}, E^2_{i,j-\frac{1}{2}})$ is defined at the the sides and $H = H_{i-\frac{1}{2},j-\frac{1}{2}}$ is defined at the center of the cell $\Omega_{i,j} = ((i-1)h, ih) \times ((j-1)h, jh)$.

$$\begin{aligned}
&\epsilon_{i-\frac{1}{2},j}\,\tfrac{d}{dt}E^1_{i-\frac{1}{2},j} = -\tfrac{H_{i-\frac{1}{2},j+\frac{1}{2}}-H_{i-\frac{1}{2},j-\frac{1}{2}}}{h}\\
&\epsilon_{i,j+\frac{1}{2}}\,\tfrac{d}{dt}E^2_{i,j+\frac{1}{2}} = \tfrac{H_{i+\frac{1}{2},j+\frac{1}{2}}-H_{i-\frac{1}{2},j-+\frac{1}{2}}}{h}\\
&\mu_{i-\frac{1}{2},j-\frac{1}{2}}\,\tfrac{d}{dt}H_{i-\frac{1}{2},j-\frac{1}{2}} = \tfrac{E^2_{i,j-\frac{1}{2}}-E^2_{i-1,j-\frac{1}{2}}}{h} - \tfrac{E^1_{i-\frac{1}{2},j}-E^1_{i-\frac{1}{2},j-1}}{h},
\end{aligned} \tag{1.50}$$

where $E^1_{i-\frac{1}{2},j} = 0,\ j = 0,\ j = N$ and $E^2_{i,j-\frac{1}{2},j} = 0,\ i = 0,\ j = N$.

Since

$$\sum_{i=1}^N \sum_{j=1}^N -\frac{H_{i-\frac{1}{2},j+\frac{1}{2}} - H_{i-\frac{1}{2},j-\frac{1}{2}}}{h} E^1_{i-\frac{1}{2},j} + \frac{H_{i+\frac{1}{2},j+\frac{1}{2}} - H_{i-\frac{1}{2},j-+\frac{1}{2}}h^2}{E}_{i,j+\frac{1}{2}}$$

$$+(\frac{E^2_{i,j-\frac{1}{2}} - E^2_{i-1,j-\frac{1}{2}}}{h} - \frac{E^1_{i-\frac{1}{2},j} - E^1_{i-\frac{1}{2},j-1}}{h})H_{i-\frac{1}{2},j-\frac{1}{2}} = 0$$

(1.50) is uniformly dissipative. The range condition $\lambda I - A_h = (f,g) \in X_h$ is equivalent to the minimization for E

$$\min \quad \frac{1}{2}(\epsilon_{i-\frac{1}{2},j}E^1_{i-\frac{1}{2},j} + \epsilon_{i,j+\frac{1}{2}}E^2_{i,j+\frac{1}{2}}) + \frac{1}{2}\frac{1}{\mu_{i,j}}(|\frac{E^2_{i,j-\frac{1}{2}} - E^2_{i-1,j-\frac{1}{2}}}{h}|^2 + |\frac{E^1_{i-\frac{1}{2},j} - E^1_{i-\frac{1}{2},j-1}}{h}|^2)$$

$$-(f^1_{i-\frac{1}{2},j} - \tfrac{1}{\mu_{i,j}}\tfrac{g_{i-\frac{1}{2},j+\frac{1}{2}}-g_{i-\frac{1}{2},j-\frac{1}{2}}}{h}, E^1_{i-\frac{1}{2},j} - (f^2_{i,j+\frac{1}{2}} + \tfrac{1}{\mu_{i,j}}\tfrac{H_{i+\frac{1}{2},j+\frac{1}{2}}-H_{i-\frac{1}{2},j-+\frac{1}{2}}}{h})E^2_{i,j+\frac{1}{2}}.$$

Example 5 Legende-Tau method

2 Dissipative Operators and Semigroup of Nonlinear Contractions

In this section we consider

$$\frac{du}{dt} \in Au(t), \quad u(0) = u_0 \in X$$

for the dissipative mapping A on a Banach space X.

Definition (Dissipative) A mapping A on a Banach space X is dissipative if

$$|x_1 - x_2 - \lambda\,(y_1 - y_2)| \ge |x_1 - x_2| \text{ for all } \lambda > 0 \text{ and } [x_1, y_1],\ [x_2, y_2] \in A,$$

or equivalently

$$\langle y_1 - y_2, x_1 - x_2\rangle_- \le 0 \text{ for all } [x_1, y_1],\ [x_2, y_2] \in A.$$

and if in addition $R(I - \lambda A) = X$, then A is m-dissipative.

In particular, it follows that if A is dissipative, then for $\lambda > 0$ the operator $(I - \lambda\, A)^{-1}$ is a single-valued and nonexpansive on $R(I - \lambda\, A)$, i.e.,

$$|(I - \lambda\, A)^{-1} x - (I - \lambda\, A)^{-1} y| \le |x - y| \quad \text{for all } x,\ y \in R(I - \lambda\, A).$$

Define the resolvent and Yosida approximation A by

$$\begin{aligned} &J_\lambda x = (I - \lambda\, A)^{-1} x \in dom\,(A), \quad x \in dom\,(J_\lambda) = R(I - \lambda\, A) \\ &A_\lambda = \lambda^{-1}(J_\lambda x - x), \quad x \in dom\,(J_\lambda). \end{aligned} \tag{2.1}$$

We summarize some fundamental properties of J_λ and A_λ in the following theorem.

Theorem 1.4 Let A be an $\omega-$ dissipative subset of $X \times X$, i.e.,

$$|x_1 - x_2 - \lambda\,(y_1 - y_2)| \ge (1 - \lambda\omega)\,|x_1 - x_2| \tag{2.2}$$

for all $0 < \lambda < \omega^{-1}$ and $[x_1, y_1],\ [x_2, y_2] \in A$ and define $\|Ax\|$ by

$$\|Ax\| = \inf\{|y| : y \in Ax\}.$$

Then for $0 < \lambda < \omega^{-1}$,

(i) $|J_\lambda x - J_\lambda y| \le (1 - \lambda\omega)^{-1}\,|x - y|$ for $x,\ y \in dom\,(J_\lambda)$.
(ii) $A_\lambda x \in A J_\lambda x$ for $x \in R(I - \lambda\, A)$.
(iii) For $x \in dom(J_\lambda) \cap dom\,(A)$ $|A_\lambda x| \le (1 - \lambda\omega)^{-1}\,\|Ax\|$ and thus

$$|J_\lambda x - x| \le \lambda(1 - \lambda\omega)^{-1}\,\|Ax\|.$$

(iv) If $x \in dom\,(J_\lambda),\ \lambda,\ \mu > 0$, then

$$\frac{\mu}{\lambda}\,x + \frac{\lambda - \mu}{\lambda}\,J_\lambda x \in dom\,(J_\mu)$$

and

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}\,x + \frac{\lambda - \mu}{\lambda}\,J_\lambda x\right).$$

(v) If $x \in dom\,(J_\lambda) \cap dom\,(A)$ and $0 < \mu \le \lambda < \omega^{-1}$, then

$$(1 - \lambda\omega)|A_\lambda x| \le (1 - \mu\omega)\,|A_\mu y|.$$

(vi) A_λ is $\omega^{-1}(1 - \lambda\omega)^{-1}$-dissipative and for $x,\ y \in dom(J_\lambda)$

$$|A_\lambda x - A_\lambda y| \le \lambda^{-1}(1 + (1 - \lambda\omega)^{-1})\,|x - y|.$$

Proof: $(i) - (ii)$ If $x,\ y \in dom\,(J_\lambda)$ and we set $u = J_\lambda x$ and $v = J_\lambda y$, then there exist $\hat{u}$ and $\hat{v}$ such that $x = u - \lambda\,\hat{u}$ and $y = v - \lambda\,\hat{v}$. Thus, from (2.2)

$$|J_\lambda x - J_\lambda y| = |u - v| \le (1 - \lambda\omega)^{-1}|u - v - \lambda\,(\hat{u} - \hat{v})| = (1 - \lambda\omega)^{-1}\,|x - y|.$$

Next, by the definition $A_\lambda x = \lambda^{-1}(u - x) = \hat{u} \in Au = AJ_\lambda x$.
(iii) Let $x \in dom\,(J_\lambda) \cap dom\,(A)$ and $\hat{x} \in Ax$ be arbitrary. Then we have $J_\lambda(x - \lambda\,\hat{x}) = x$ since $x - \lambda\,\hat{x} \in (I - \lambda\,A)x$. Thus,

$$|A_\lambda x| = \lambda^{-1}|J_\lambda x - x| = \lambda^{-1}|J_\lambda x - J_\lambda(x - \lambda\,\hat{x})| \le (1-\lambda\omega)^{-1}\lambda^{-1}\,|x - (x - \lambda\,\hat{x})| = (1-\lambda\omega)^{-1}\,|\hat{x}|.$$

which implies (iii).
(iv) If $x \in dom\,(J_\lambda) = R(I - \lambda\,A)$ then we have $x = u - \lambda\,\hat{u}$ for $[u, \hat{u}] \in A$ and thus $u = J_\lambda x$. For $\mu > 0$

$$\frac{\mu}{\lambda}\,x + \frac{\lambda - \mu}{\lambda}\,J_\lambda x = \frac{\mu}{\lambda}\,(u - \lambda\,\hat{u}) + \frac{\lambda - \mu}{\lambda}\,u = u - \mu\,\hat{u} \in R(I - \mu\,A) = dom\,(J_\mu).$$

and

$$J_\mu\left(\frac{\mu}{\lambda}\,x + \frac{\lambda - \mu}{\lambda}\,J_\lambda x\right) = J_\mu(u - \mu\,\hat{u}) = u = J_\lambda x.$$

(v) From (i) and (iv) we have

$$\begin{aligned}
\lambda\,|A_\lambda x| &= |J_\lambda x - x| \le |J_\lambda x - J_\mu x| + |J_\mu x - x| \\
&\le \left|J_\mu\left(\frac{\mu}{\lambda}\,x + \frac{\lambda - \mu}{\lambda}\,J_\lambda x\right) - J_\mu x\right| + |J_\mu x - x| \\
&\le (1 - \mu\omega)^{-1}\left|\frac{\mu}{\lambda}\,x + \frac{\lambda - \mu}{\lambda}\,J_\lambda x - x\right| + |J_\mu x - x| \\
&= (1 - \mu\omega)^{-1}(\lambda - \mu)\,|A_\lambda x| + \mu\,|A_\mu x|,
\end{aligned}$$

which implies (v) by rearranging.
(vi) It follows from (i) that for $\rho > 0$

$$\begin{aligned}
|x - y - \rho\,(A_\lambda x - A_\lambda y)| &= |(1 + \frac{\rho}{\lambda})\,(x - y) - \frac{\rho}{\lambda}\,(J_\lambda x - J_\lambda y)| \\
&\ge (1 + \frac{\rho}{\lambda})\,|x - y| - \frac{\rho}{\lambda}\,|J_\lambda x - J_\lambda y| \\
&\ge ((1 + \frac{\rho}{\lambda}) - \frac{\rho}{\lambda}(1 - \lambda\omega)^{-1})\,|x - y| = (1 - \rho\,\omega(1 - \lambda\omega)^{-1})\,|x - y|.
\end{aligned}$$

The last assertion follows from the definition of A_λ and (i). □

Theorem 1.5
(1) A dissipative set $A \subset X \times X$ is m-dissipative, if and only if

$$R(I - \lambda_0\,A) = X \quad \text{for some } \lambda_0 > 0.$$

(2) An m-dissipative mapping is maximal dissipative, i.e., all dissipative set containing A in $X \times X$ coincide with A.

(3) If $X = X^* = H$ is a Hilbert space, then the notions of the maximal dissipative set and m-dissipative set are equivalent.

Proof: (1) Suppose $R(I - \lambda_0 A) = X$. Then it follows from Theorem 1.4 (i) that J_{λ_0} is contraction on X. We note that

$$I - \lambda A = \frac{\lambda_0}{\lambda}\left(I - (1 - \frac{\lambda_0}{\lambda})\, J_{\lambda_0}\right)(I - \lambda_0 A) \tag{2.3}$$

for $0 < \lambda < \omega^{-1}$. For given $x \in X$ define the operator $T : X \to X$ by

$$Ty = x + (1 - \frac{\lambda_0}{\lambda})\, J_{\lambda_0} y, \quad y \in X$$

Then

$$|Ty - Tz| \le |1 - \frac{\lambda_0}{\lambda}|\, |y - z|$$

where $|1 - \frac{\lambda}{\lambda_0}| < 1$ if $2\lambda > \lambda_0$. By Banach fixed-point theorem the operator T has a unique fixed point $z \in X$, i.e., $x = (I - (1 - \frac{\lambda_0}{\lambda}\, J_{\lambda_0})z$. Thus,

$$x \in (I - (1 - \frac{\lambda_0}{\lambda})\, J_{\lambda_0})(I - \lambda_0 A) dom\,(A).$$

and it thus follows from (2.3) that $R(I - \lambda A) = X$ if $\lambda > \frac{\lambda_0}{2}$. Hence, (1) follows from applying the above argument repeatedly.

(2) Assume A is m-dissipative. Suppose $\tilde{A}$ is a dissipative set containing A. We need to show that $\tilde{A} \subset A$. Let $[x, \hat{x}] \in \tilde{A}$ Since $x - \lambda\, \hat{x} \in X = R(I - \lambda A)$, for $\lambda > 0$, there exists a $[y, \hat{y}] \in A$ such that $x - \lambda\, \hat{x} = y - \lambda\, \hat{y}$. Since $A \subset \tilde{A}$ it follows that $[y, \hat{y}] \in A$ and thus

$$|x - y| \le |x - y - \lambda\, (\hat{x} - \hat{y})| = 0.$$

Hence, $[x, \hat{x}] = [y, \hat{y}] \in A$.

(3) It suffices to show that if A is maximal dissipative, then A is m-dissipative. We use the following extension lemma (Lemma 1.6). Let y be any element of H. By Lemma 1.6, taking $C = H$, we have that there exists $x \in H$ such that

$$(\xi - x, \eta - x + y) \le 0 \quad \text{for all } \; [\xi, \eta] \in A.$$

and thus

$$(\xi - x, \eta - (x - y)) \le 0 \quad \text{for all } \; [\xi, \eta] \in A.$$

Since A is maximal dissipative, this implies that $[x, x - y] \in A$, that is $x - y \in Ax$, and therefore $y \in R(I - H)$. $\square$

Lemma 1.6 Let A be dissipative and C be a closed, convex, non-empty subset of the Hilbert space H such that $dom\,(A) \in C$. Then for every $y \in H$ there exists $x \in C$ such that

$$(\xi - x, \eta - x + y) \le 0 \quad \text{for all } \; [\xi, \eta] \in A.$$

Proof: Without loss of generality we can assume that $y = 0$, for otherwise we define $A_y = \{[\xi, \eta + y] : [\xi, \eta] \in A\}$ with $dom\,(A_y) = dom\,(A)$. Since A is dissipative if and only if A_y is dissipative , we can prove the lemma for A_y. For $[\xi, \eta] \in A$, define the set

$$C([\xi, \eta]) = \{x \in C : (\xi - x, \eta - x) \le 0\}.$$

Thus, the lemma is proved if we can show that $\bigcap_{[\xi,\eta]\in A}\; C([\xi, \eta])$ is non-empty.

2.0.1 Properties of m-dissipative operators

In this section we discuss some properties of m-dissipative sets.

Lemma 1.7 Let X^* be a strictly convex Banach space. If A is maximal dissipative, then Ax is a closed convex set of X for each $x \in dom\,(A)$.

Proof: It follows from Lemma that the duality mapping F is single-valued. First, we show that Ax is convex. Let $\hat{x}_1,\ \hat{x_2} \in Ax$ and set $\hat{x} = \alpha \hat{x}_1 + (1-\alpha)\,\hat{x}_2$ for $0 \leq \alpha \leq 1$. Then, Since A is dissipative, for all $[y, \hat{y}] \in A$

$$Re\,\langle \hat{x} - \hat{y}, F(x-y)\rangle = \alpha\, Re\,\langle \hat{x}_1 - \hat{y}, F(x-y)\rangle + (1-\alpha)\, Re\,\langle \hat{x}_2 - \hat{y}, F(x-y)\rangle \leq 0.$$

Thus, if we define a subset $\tilde{A}$ by

$$\tilde{A}z = \begin{cases} Az & \text{if } \ z \in dom\,(A) \setminus \{x\} \\ \\ Ax \cup \{\hat{x}\} & \text{if } \ z = x, \end{cases}$$

then $\tilde{A}$ is a dissipative extension of A and $dom\,(\tilde{A}) = dom\,(A)$. Since A is maximal dissipative, it follows that $\tilde{A}x = Ax$ and thus $\hat{x} \in Ax$ as desired.

Next, we show that Ax is closed. Let $\hat{x}_n \in Ax$ and $\lim_{n\to\infty} \hat{x}_n = \hat{x}$. Since A is dissipative, $Re\,\langle \hat{x}_n - \hat{y}, x - y\rangle \leq 0$ for all $[y, \hat{y}] \in A$. Letting $n \to \infty$, we obtain $Re\,\langle \hat{x} - \hat{y}, x-y\rangle \leq 0$. Hence, as shown above $\hat{x} \in Ax$ as desired. □

Definition 1.4 A subset A of $X \times X$ is said to be demiclosed if $x_n \to x$ and $y_n \rightharpoonup y$ and $[x_n, y_n] \in A$ imply that $[x, y] \in A$. A subset A is closed if $[x_n, y_n]$, $x_n \to x$ and $y_n \to y$ imply that $[x, y] \in A$.

Theorem 1.8 Let A be m-dissipative. Then the followings hold.
(i) A is closed.
(ii) If $\{x_\lambda\} \subset X$ such that $x_\lambda \to x$ and $A_\lambda x_\lambda \to y$ as $\lambda \to 0^+$, then $[x, y] \in A$.

Proof: (i) Let $[x_n, \hat{x}_n] \in A$ and $(x_n, \hat{x}_n) \to (x, \hat{x})$ in $X \times X$. Since A is dissipative $Re\,\langle \hat{x}_n - \hat{y}, x_n - y\rangle_i \leq 0$ for all $[y, \hat{y}] \in A$. Since $\langle \cdot, \cdot \rangle_i$ is lower semicontinuous, letting $n \to \infty$, we obtain $Re\,\langle \hat{x} - \hat{y}, x - y\rangle_i \leq$ for all $[y, \hat{y}] \in A$. Then $A_1 = [x, \hat{x}] \cup A$ is a dissipative extension of A. Since A is maximal dissipative, $A_1 = A$ and thus $[x, \hat{x}] \in A$. Hence, A is closed.
(ii) Since $\{A_\lambda x\}$ is a bounded set in X, by the definition of A_λ, $\lim\, |J_\lambda x_\lambda - x_\lambda| \to 0$ and thus $J_\lambda x_\lambda \to x$ as $\lambda \to 0^+$. But, since $A_\lambda x_\lambda \in A J_\lambda x_\lambda$, it follows from (i) that $[x, y] \in A$. □

Theorem 1.9 Let A be m-dissipative and let X^* be uniformly convex. Then the followings hold.
(i) A is demiclosed.
(ii) If $\{x_\lambda\} \subset X$ such that $x_\lambda \to x$ and $\{|A_\lambda x|\}$ is bounded as $\lambda \to 0^+$, then $x \in dom\,(A)$. Moreover, if for some subsequence $A_{\lambda_n} x_n \rightharpoonup y$, then $y \in Ax$.
(iii) $lim_{\lambda\to 0^+} |A_\lambda x| = \|Ax\|$.

Proof: (i) Let $[x_n, \hat{x}_n] \in A$ be such that $\lim\, x_n = x$ and $w - \lim \hat{x}_n = \hat{x}$ as $n \to \infty$. Since X^* is uniformly convex, from Lemma the duality mapping is single-valued and uniformly continuous on the bounded subsets of X. Since A is dissipative $Re\,\langle \hat{x}_n - \hat{y}, F(x_n - y)\rangle \leq 0$

for all $[y, \hat{y}] \in A$. Thus, letting $n \to \infty$, we obtain $Re\,\langle \hat{x} - \hat{y}, F(x-y)\rangle \le 0$ for all $[y, \hat{y}] \in A$. Thus, $[x, \hat{x}] \in A$, by the maximality of A.

Definition 1.5 The minimal section A^0 of A is defined by

$$A^0 x = \{y \in Ax : |y| = \|Ax\|\} \quad \text{with } \ dom\,(A^0) = \{x \in dom\,(A) : A^0 x \text{ is non-empty}\}.$$

Lemma 1.10 Let X^* be a strictly convex Banach space and let A be maximal dissipative. Then, the followings hold.
(i) If X is strictly convex, then A^0 is single-valued.
(ii) If X reflexible, then $dom\,(A^0) = dom\,(A)$.
(iii) If X strictly convex and reflexible, then A^0 is single-valued and $dom\,(A^0) = dom\,(A)$.

Theorem 1.11 Let X^* is a uniformly convex Banach space and let A be m-dissipative. Then the followings hold.
(i) $lim_{\lambda \to 0^+} F(A_\lambda x) = F(A^0 x)$ for each $x \in dom\,(A)$.
Moreover, if X is also uniformly convex, then
(ii) $\lim_{\lambda \to 0^+} A_\lambda x = A^0 x$ for each $x \in dom\,(A)$.

Proof: (1) Let $x \in dom\,(A)$. By (ii) of Theorem 1.4

$$|A_\lambda x| \le \|Ax\|$$

Since $\{A_\lambda x\}$ is a bounded sequence in a reflexive Banach space (i.e., since X^* is uniformly convex, X^* is reflexive and so is X), there exists a weak convergent subsequence $\{A_{\lambda_n} x\}$. Now we set $y = w - \lim_{n\to\infty} A_{\lambda_n} x$. Since from Theorem 1.4 $A_{\lambda_n} x \in AJ_{\lambda_n} x$ and $\lim_{n\to\infty} J_{\lambda_n} x = x$ and from Theorem 1.10 A is demiclosed, it follows that $[x, y] \in A$. Since by the lower-semicontinuity of norm this implies

$$\|Ax\| \le |y| \liminf_{n\to\infty} |A_{\lambda_n} x| \le \limsup_{n\to\infty} |A_{\lambda_n} x| \le \|Ax\|,$$

we have $|y| = \|Ax\| = \lim_{n\to\infty} |A_{\lambda_n} x|$ and thus $y \in A^0 x$. Next, since $|F(A_{\lambda_n} x)| = |A_{\lambda_n} x| \le \|Ax\|$, $F(A_{\lambda_n} x)$ is a bounded sequence in the reflexive Banach space X^* and has a weakly convergent subsequence $F(A_{\lambda_k} x)$ of $F(A_{\lambda_n} x)$. If we set $y^* = w - \lim_{k\to\infty} F(A_{\lambda_k} x)$, then it follows from the dissipativity of A that

$$Re\,\langle A_{\lambda_k} x - y, F(A_{\lambda_k} x)\rangle = \lambda_n^{-1} Re\,\langle A_{\lambda_k} x - y, F(J_{\lambda_k} xx)\rangle \le 0,$$

or equivalently $|A_{\lambda_k} x|^2 \le Re\,\langle y, F(A_{\lambda_n} x)\rangle$. Letting $k \to \infty$, we obtain $|y|^2 \le Re\,\langle y, y^*\rangle$. Combining this with

$$|y^*| \le \lim_{k\to\infty} |F(A_{\lambda_k} x)| = \lim_{k\to\infty} |A_{\lambda_k} x| = |y|,$$

we have

$$|y^*| \le Re\,\langle y, y^*\rangle \le |\langle y, y^*\rangle| \le |y||y^*| \le |y|^2.$$

Hence,

$$\langle y, y^*\rangle = |y|^2 = |y^*|^2$$

and we have $y^* = F(y)$. Also, $\lim_{k\to\infty} |F(A_{\lambda_k}x)| = |y| = |F(y)|$. It thus follows from the uniform convexity of X^* that

$$\lim_{k\to\infty} F(A_{\lambda_k}x) = F(y).$$

Since Ax is a closed convex set of X from Theorem , we can show that $x \to F(A^0x)$ is single-valued. In fact, if C is a closed convex subset of X, the y is an element of minimal norm in C, if and only if

$$|y| \le |(1-\alpha)y + \alpha\, z| \quad \text{for all } z \in C \text{ and } 0 \le \alpha \le 1.$$

Hence,

$$\langle z - y, y\rangle_+ \ge 0.$$

and from Theorem 1.10

$$0 \le Re\,\langle z - y, f\rangle = Re\,\langle z, f\rangle - |y|^2 \tag{2.4}$$

for all $z \in C$ and $f \in F(y)$. Now, let $y_1,\ y_2$ be arbitrary in A^0x. Then, from (2.4)

$$|y_1|^2 \le Re\,\langle y_2, F(y_1)\rangle \le |y_1||y_2|$$

which implies that $\langle y_2, F(y_1)\rangle = |y_2|^2$ and $|F(y_1)| = |y_2|$. Therefore, $F(y_1) = F(y_2)$ as desired. Thus, we have shown that for every sequence $\{\lambda\}$ of positive numbers that converge to zero, the sequence $\{F(A_\lambda x)\}$ has a subsequence that converges to the same limit $F(A^0x)$. Therefore, $\lim_{\lambda\to 0} F(A_\lambda x) \to F(A^0x)$.

Furthermore, we assume that X is uniformly convex. We have shown above that for $x \in dom\,(A)$ the sequence $\{A_\lambda\}$ contains a weak convergent subsequence $\{A_{\lambda_n}x\}$ and if $y = w - \lim_{n\to\infty} A_{\lambda_n}x$ then $[x, y] \in A^0$ and $|y| = \lim_{n\to\infty} |A_{\lambda_n}x|$. But since X is uniformly convex, it follows from Theorem 1.10 that A^0 is single-valued and thus $y = A^0x$. Hence, $w - \lim_{n\to\infty} A_{\lambda_n}x = A^0x$ and $\lim_{n\to\infty} |A_{\lambda_n}x| = |A^0x|$. Since X is uniformly convex, this implies that $\lim_{n\to\infty} A_{\lambda_n}x = A^0x$. □

Theorem 1.12 Let X is a uniformly convex Banach space and let A be m-dissipative. Then $\overline{dom\,(A)}$ is a convex subset of X.

Proof: It follows from Theorem 1.4 that

$$|J_\lambda x - x| \le \lambda\,\|Ax\| \quad \text{for } \ x \in dom\,(A)$$

Hence $|J_\lambda x - x| \to 0$ as $\lambda \to 0^+$. Since $J_\lambda x \in dom\,(A)$ for $X \in X$, it follows that

$$\overline{dom\,(A)} = \{x \in X : |J_\lambda x - x| \to 0 \text{ as } \lambda \to 0^+\}.$$

Let $x_1,\ x_2 \in \overline{dom\,(A)}$ and $0 \le \alpha \le 1$ and set

$$x = \alpha\, x_1 + (1-\alpha)\, x_2.$$

Then, we have

$$\begin{aligned} &|J_\lambda x - x_1| \le |x - x_1| + |J_\lambda x_1 - x_1| \\ &|J_\lambda x - x_2| \le |x - x_2| + |J_\lambda x_2 - x_2| \end{aligned} \tag{2.5}$$

where $x - x_1 = (1-\alpha)(x_2 - x_1)$ and $x - x_2 = \alpha(x_1 - x_2)$. Since $\{J_\lambda x\}$ is a bounded set and a uniformly convex Banach space is reflexive, it follows that there exists a subsequence $\{J_{\lambda_n} x\}$ that converges weakly to z. Since the norm is weakly lower semicontinuous, letting $n \to \infty$ in (2.5) with $\lambda = \lambda_n$, we obtain

$$|z - x_1| \leq (1-\alpha)\,|x_1 - x_2|$$

$$|z - x_2| \leq \alpha\,|x_1 - x_2|$$

Thus,

$$|x_1 - x_2| = |(x_1 - z) + (z - x_2)| \leq |x_1 - z| + |z - x_2| \leq |x_1 - x_2|$$

and therefore $|x_1 - z| = (1-\alpha)\,|x_1 - x_2|$, $|z - x_2| = \alpha\,|x_1 - x_2|$ and $|(x_1 - z) + (z - x_2)| = |x_1 - x_2|$. But, since X is uniformly convex we have $z = x$ and $w - \lim\, J_{\lambda_n} x = x$ as $n \to \infty$. Since we also have

$$|x - x_1| \leq \liminf_{n\to\infty}\, |J_{\lambda_n} x - J_{\lambda_n} x_1| \leq |x - x_1|$$

$|J_{\lambda_n} x - J_{\lambda_n} x_1| \to |x - x_1|$ and $w - \lim\, J_{\lambda_n} x - J_{\lambda_n} x_1 = x - x_1$ as $n \to \infty$. Since X is uniformly convex, this implies that $\lim_{\lambda \to 0^+}\, J_\lambda x = x$ and $x \in \overline{dom\,(A)}$. $\square$

2.1 Generation of Nonlinear Semigroups

In this section, we consider the generation of nonlinear semigroup by Crandall-Liggett on a Banach space X.

Definition 2.1 Let X_0 be a subset of X. A semigroup $S(t),\ t \geq$ of nonlinear contractions on X_0 is a function with domain $[0,\infty) \times X_0$ and range in X_0 satisfying the following conditions:

$$S(t+s)x = S(t)S(s)x \quad \text{and} \quad S(0)x = x \quad \text{for } \ x \in X_0,\ t,s \geq 0$$

$$t \to S(t)x \in X \ \text{ is continuous}$$

$$|S(t)x - S(t)y| \leq |x - y| \quad \text{for } \ t \geq 0,\ x, y \in X_0$$

Let A be a ω-dissipative operator and $J_\lambda = (I - \lambda\, A)^{-1}$ is the resolvent. The following estimate plays an essential role in the Crandall-Liggett generation theory.

Lemma 2.1 Assume a sequence $\{a_{n,m}\}$ of positive numbers satisfies

$$a_{n,m} \leq \alpha\, a_{n-1,m-1} + (1-\alpha)\, a_{n-1,m} \tag{2.6}$$

and $a_{0,m} \leq m\,\lambda$ and $a_{n,0} \leq n\,\mu$ for $\lambda \geq \mu > 0$ and $\alpha = \frac{\mu}{\lambda}$. Then we have the estimate

$$a_{n,m} \leq [(m\lambda - n\mu)^2 + m\lambda^2]^{\frac{1}{2}} + [(m\lambda - n\mu)^2 + n\lambda\mu]^{\frac{1}{2}}. \tag{2.7}$$

Proof: From the assumption, (2.7) holds for either $m = 0$ or $n = 0$. We will use the induction in $n,\ m$, that is if (2.7) holds for $(n+1, m)$ when (2.7) is true for (n, m) and $(n, m-1)$, then (2.7) holds for all (n, m). Let $\beta = 1 - \alpha$. We assume that (2.7) holds for (n, m) and $(n, m-1)$. Then, by (2.6) and Cauchy-Schwarz inequality $\alpha x + \beta y \leq (\alpha+\beta)^{\frac{1}{2}}(\alpha x^2 + \beta y^2)^{\frac{1}{2}}$

$$a_{n+1,m} \leq \alpha\, a_{n,m-1} + \beta\, a_{n,m}$$

$$\leq \alpha\,([((m-1)\lambda - n\mu)^2 + (m-1)\lambda^2]^{\frac{1}{2}} + [((m-1)\lambda - n\mu)^2 + n\lambda\mu]^{\frac{1}{2}})$$
$$+\beta\,([(m\lambda - n\mu)^2 + m\lambda^2]^{\frac{1}{2}} + [(m\lambda - n\mu)^2 + n\lambda\mu]^{\frac{1}{2}})$$

$$= (\alpha+\beta)^{\frac{1}{2}}(\alpha\,[((m-1)\lambda - n\mu)^2 + (m-1)\lambda^2] + \beta\,[(m\lambda - n\mu)^2 + m\lambda^2])^{\frac{1}{2}}$$
$$+(\alpha+\beta)^{\frac{1}{2}}(\alpha\,[((m-1)\lambda - n\mu)^2 + n\lambda\mu] + \beta\,[(m\lambda - n\mu)^2 + n\lambda\mu])^{\frac{1}{2}}$$

$$\leq [(m\lambda - (n+1)\mu)^2 + m\lambda^2]^{\frac{1}{2}} + [(m\lambda - (n+1)\mu)^2 + (n+1)\lambda\mu]^{\frac{1}{2}}.$$

Here, we used $\alpha + \beta = 1$, $\alpha\lambda = \mu$ and

$$\alpha\,[((m-1)\lambda - n\mu)^2 + (m-1)\lambda^2] + \beta\,[(m\lambda - n\mu)^2 + m\lambda^2])$$

$$\leq (m\lambda - n\mu)^2 + m\lambda^2 - \alpha\lambda(m\lambda - n\mu) = (m\lambda - (n+1)\mu)^2 + m\lambda^2 - \mu^2$$

$$\alpha\,[((m-1)\lambda - n\mu)^2 + n\lambda\mu] + \beta\,[(m\lambda - n\mu)^2 + n\lambda\mu]$$

$$\leq (m\lambda - n\mu)^2 + (n+1)\lambda\mu - 2\alpha\lambda(m\lambda - n\mu) \leq (m\lambda - (n+1)\mu)^2 + (n+1)\lambda\mu - \mu^2. \square$$

Theorem 2.2 Assume A be a dissipative subset of $X \times X$ and satisfies the range condition

$$\overline{dom\,(A)} \subset R(I - \lambda\, A) \quad \text{for all sufficiently small } \lambda > 0. \tag{2.8}$$

Then, there exists a semigroup of type ω on $S(t)$ on $\overline{dom\,(A)}$ that satisfies for $x \in \overline{dom\,(A)}$

$$S(t)x = \lim_{\lambda \to 0^+} (I - \lambda\, A)^{-[\frac{t}{\lambda}]}x, \quad t \geq 0 \tag{2.9}$$

and

$$|S(t)x - S(t)x| \leq |t - s|\, \|Ax\| \quad \text{for } x \in dom\,(A), \text{ and } t,\ s \geq 0.$$

Proof: First, note that from (2.10) $\overline{dom\,(A)} \subset dom\,(J_\lambda)$. Let $x \in dom\,(A)$ and set $a_{n,m} = |J_\mu^n x - J_\lambda^m x|$ for $n,\ m \geq 0$. Then, from Theorem 2.4

$$a_{0,m} = |x - J_\lambda^m x| \leq |x - J_\lambda x| + |J_\lambda x - J_\lambda^2 x| + \cdots + |J_\lambda^{m-1}x - J_\lambda^m x|$$

$$\leq m\,|x - J_\lambda x| \leq m\lambda\, \|Ax\|.$$

Similarly, $a_{n,0} = |J_\mu^n x - x| \leq n\mu\, \|Ax\|$. Moreover,

$$a_{n,m} = |J_\mu^n x - J_\lambda^m x| \leq |J_\mu^n x - J_\mu\left(\frac{\mu}{\lambda}\, J_\lambda^{m-1}x + \frac{\lambda - \mu}{\lambda}\, J_\lambda^m x\right)|$$

$$\leq \frac{\mu}{\lambda}\, |J_\mu^{n-1}x - J_\lambda^{m-1}x| + \frac{\lambda - \mu}{\lambda}\, |J_\mu^{n-1}x - J_\lambda^m x|$$

$$= \alpha\, a_{n-1,m-1} + (1 - \alpha)\, a_{n-1,m}.$$

It thus follows from Lemma 2.1 that

$$|J_\mu^{[\frac{t}{\mu}]}x - J_\lambda^{[\frac{t}{\lambda}]}x| \leq 2(\lambda^2 + \lambda t)^{\frac{1}{2}}\, \|Ax\|. \tag{2.10}$$

Thus, $J_\lambda^{[\frac{t}{\lambda}]}x$ converges to $S(t)x$ uniformly on any bounded intervals, as $\lambda \to 0^+$. Since $J_\lambda^{[\frac{t}{\lambda}]}$ is non-expansive, so is $S(t)$. Hence (2.9) holds. Next, we show that $S(t)$ satisfies the semigroup property $S(t+s)x = S(t)S(s)x$ for $x \in \overline{dom\,(A)}$ and $t,\ s \geq 0$. Letting $\mu \to 0^+$ in (2.10), we obtain

$$|S(t)x - J_\lambda^{[\frac{t}{\lambda}]}x| \leq 2(\lambda^2 + \lambda t)^{\frac{1}{2}}. \tag{2.11}$$

for $x \in dom\,(A)$. If we let $x = J_\lambda^{[\frac{s}{\lambda}]}z$, then $x \in dom\,(A)$ and

$$|S(t)J_\lambda^{[\frac{s}{\lambda}]}z - J_\lambda^{[\frac{t}{\lambda}]}J_\lambda^{[\frac{s}{\lambda}]}z| \leq 2(\lambda^2 + \lambda t)^{\frac{1}{2}}\, \|Az\| \tag{2.12}$$

where we used that $\|AJ_\lambda^{[\frac{s}{\lambda}]}z\| \leq \|Az\|$ for $z \in dom\,(A)$. Since $[\frac{t+s}{\lambda}] - ([\frac{t}{\lambda}] + [\frac{s}{\lambda}])$ equals 0 or 1, we have

$$|J_\lambda^{[\frac{t+s}{\lambda}]}z - J_\lambda^{[\frac{t}{\lambda}]}J_\lambda^{[\frac{s}{\lambda}]}z| \leq |J_\lambda z - z| \leq \lambda\, \|Az\|. \tag{2.13}$$

It thus follows from (2.11)–(2.13) that

$$|S(t+s)z - S(t)S(s)z| \leq |S(t+s)x - J_\lambda^{[\frac{t+s}{\lambda}]}z| + |J_\lambda^{[\frac{t+s}{\lambda}]}z - J_\lambda^{[\frac{t}{\lambda}]}J_\lambda^{[\frac{s}{\lambda}]}z|$$

$$+|J_\lambda^{[\frac{t}{\lambda}]}J_\lambda^{[\frac{s}{\lambda}]}z - S(t)J_\lambda^{[\frac{s}{\lambda}]}z| + |S(t)J_\lambda^{[\frac{s}{\lambda}]}z - S(t)S(s)z| \to 0$$

as $\lambda \to 0^+$. Hence, $S(t+s)z = S(t)S(s)z$.

Finally, since

$$|J_\lambda^{[\frac{t}{\lambda}]}x - x| \leq [\frac{t}{\lambda}]\, |J_\lambda x - x| \leq t\, \|Ax\|$$

we have $|S(t)x - x| \leq t\, \|Ax\|$ for $x \in dom\,(A)$. From this we obtain $|S(t)x - x| \to 0$ as $t \to 0^+$ for $x \in \overline{dom\,(A)}$ and also

$$|S(t)x - S(s)x| \leq |S(t-s)x - x| \leq (t-s)\, \|Ax\|$$

for $x \in dom\,(A)$ and $t \geq s \geq 0$. □

2.2 Cauchy Problem

Definition 3.1 Let $x_0 \in X$ and $\omega \in R$. Consider the Cauchy problem

$$\frac{d}{dt}u(t) \in Au(t), \quad u(0) = x_0. \tag{2.14}$$

(1) A continuous function $u(t) : [0, T] \to X$ is called a strong solution of (2.14) if $u(t)$ is Lipschitz continuous with $u(0) = x_0$, strongly differentiable a.e. $t \in [0, T]$, and (2.14) holds a.e. $t \in [0, T]$.

(2) A continuous function $u(t):[0,T]\to X$ is called an integral solution of type ω of (2.14) if $u(t)$ satisfies

$$|u(t)-x|-|u(\hat{t})-x|\le\int_{\hat{t}}^t(\omega\,|u(s)-x|+\langle y,u(s)-x\rangle_+)\,ds \tag{2.15}$$

for all $[x,y]\in A$ and $t\ge\hat{t}\in[0,T]$.

<u>Theorem 3.1</u> Let A be a dissipative subset of $X\times X$. Then, the strong solution to (2.14) is unique. Moreover if the range condition (2.10) holds, then then the strong solution $u(t):[0,\infty)\to X$ to (2.14) is given by

$$u(t)=\lim_{\lambda\to 0^+}(I-\lambda\,A)^{-[\frac{t}{\lambda}]}x\quad\text{for } x\in\overline{dom\,(A)}\text{ and } t\ge 0.$$

Proof: Let $u_i(t),\ i=1,2$ be the strong solutions to (2.14). Then, $t\to|u_1(t)-u_2(t)|$ is Lipschitz continuous and thus a.e. differentiable $t\ge 0$. Thus

$$\frac{d}{dt}|u_1(t)-u_2(t)|^2=2\,\langle u_1'(t)-u_2'(t),u_1(t)-u_2(t)\rangle_i$$

a.e. $t\ge 0$. Since $u_i'(t)\in Au_i(t),\ i=1,2$ from the dissipativeness of A, we have $\frac{d}{dt}|u_1(t)-u_2(t)|^2\le 0$ and therefore

$$|u_1(t)-u_2(t)|^2\le\int_0^t\frac{d}{dt}|u_1(t)-u_2(t)|^2\,dt\le 0,$$

which implies $u_1=u_2$.

For $0<2\lambda<s$ let $u_\lambda(t)=(I-\lambda\,A)^{-[\frac{t}{\lambda}]}x$ and define $g_\lambda(t)=\lambda^{-1}(u(t)-u(t-\lambda))-u'(t)$ a.e. $t\ge\lambda$. Since $\lim_{\lambda\to 0^+}|g_\lambda|=0$ a.e. $t>0$ and $|g_\lambda(t)|\le 2M$ for a.e. $t\in[\lambda,s]$, where M is a Lipschitz constant of $u(t)$ on $[0,s]$, it follows that $\lim_{\lambda\to 0^+}\int_\lambda^s|g_\lambda(t)|\,dt=0$ by Lebesgue dominated convergence theorem. Next, since

$$u(t-\lambda)+\lambda\,g_\lambda(t)=u(t)-\lambda\,u'(t)\in(I-\lambda\,A)u(t),$$

we have $u(t)=(I-\lambda\,A)^{-1}(u(t-\lambda)+\lambda\,g_\lambda(t))$. Hence,

$$|u_\lambda(t)-u(t)|\le|(I-\lambda\,A)^{-[\frac{t-\lambda}{\lambda}]}x-u(t-\lambda)-\lambda\,g_\lambda(t)|$$

$$\le|u_\lambda(t-\lambda)-u(t-\lambda)|+\lambda\,|g_\lambda(t)|$$

a.e. $t\in[\lambda,s]$. Integrating this on $[\lambda,s]$, we obtain

$$\lambda^{-1}\int_{s-\lambda}^s|u_\lambda(t)-u(t)|\,dt\le\lambda^{-1}\int_0^\lambda|u_\lambda(t)-u(t)|\,dt+\int_\lambda^s|g_\lambda(t)|\,dt.$$

Letting $\lambda\to 0^+$, it follows from Theorem 2.2 that $|S(s)x-u(s)|=0$ since u is Lipschitz continuous, which shows the desired result. □

In general, the semigroup $S(t)$ generated on $\overline{dom\,(A)}$ in Theorem 2.2 in not necessary strongly differentiable. In fact, an example of an m-dissipative A satisfying (2.10) is given, for which the semigroup constructed in Theorem 2.2 is not even weakly differentiable for all $t \geq 0$. Hence, from Theorem 3.1 the corresponding Cauchy problem (2.14) does not have a strong solution. However, we have the following.

Theorem 3.2 Let A be an ω-dissipative subset satisfying (2.10) and $S(t),\ t \geq 0$ be the semigroup on $\overline{dom\,(A)}$, as constructed in Theorem 2.2. Then, the followings hold.
(1) $u(t) = S(t)x$ on $\overline{dom\,(A)}$ defined in Theorem 2.2 is an integral solution of type ω to the Cauchy problem (2.14).
(2) If $v(t) \in C(0,T;X)$ be an integral of type ω to (2.14), then $|v(t)-u(t)| \leq e^{\omega t}\,|v(0)-u(0)|$.
(3) The Cauchy problem (2.14) has a unique solution in $\overline{dom\,(A)}$ in the sense of Definition 2.1.

Proof: A simple modification of the proof of Theorem 2.2 shows that for $x_0 \in \overline{dom\,(A)}$

$$S(t)x_0 = \lim_{\lambda \to 0^+} (I - \lambda\, A)^{-[\frac{t}{\lambda}]} x_0$$

exists and defines the semigroup $S(t)$ of nonlinear ω-contractions on $\overline{dom\,(A)}$, i.e.,

$$|S(t)x - S(t)y| \leq e^{\omega t}\,|x-y| \quad \text{for } \ t \geq 0 \text{ and } x, y \in \overline{dom\,(A)}.$$

For $x_0 \in dom\,(A)$ we define for $\lambda > 0$ and $k \geq 1$

$$y_\lambda^k = \lambda^{-1}(J_\lambda^k x_0 - J_\lambda^{k-1} x_0) = A_\lambda J_\lambda^{k-1} x_0 \in A J_\lambda^k. \tag{2.16}$$

Since A is ω-dissipative, $\langle y_{\lambda,k} - y, J_\lambda^k x_0 - x\rangle_- \leq \omega\,|J_\lambda^k x_0 - x|$ for $[x,y] \in A$. Since from Lemma 1.1 (4) $\langle y, x\rangle_- - \langle z, x\rangle_+ \leq \langle y - z, x\rangle_-$, it follows that

$$\langle y_\lambda^k, J_\lambda^k x_0 - x\rangle_- \leq \omega\,|J_\lambda^k x_0 - x| + \langle y, J_\lambda^k x_0 - x\rangle_+ \tag{2.17}$$

Since from Lemma 1.1 (3) $\langle x+y, x\rangle_- = |x| + \langle y, x\rangle_-$, we have

$$\langle \lambda y_\lambda^k, J_\lambda^k x_0 - x\rangle_- = |J_\lambda^k x_0 - x| + \langle -(J_\lambda^{k-1} x_0 - x), J_\lambda^k x_0 - x\rangle_- \geq |J_\lambda^k x_0 - x| - |J_\lambda^{k-1} x_0 - x|.$$

It thus follows from (2.17) that

$$|J_\lambda^k x_0 - x| - |J_\lambda^{k-1} x_0 - x| \leq \lambda\,(\omega\,|J_\lambda^k x_0 - x| + \langle y, J_\lambda^k x_0 - x\rangle_+).$$

Since $J^{[\frac{t}{\lambda}]} = J_\lambda^k$ on $t \in [k\lambda, (k+1)\lambda)$, this inequality can be written as

$$|J_\lambda^k x_0 - x| - |J_\lambda^{k-1} x_0 - x| \leq \int_{k\lambda}^{(k+1)\lambda} (\omega\,|J_\lambda^{[\frac{t}{\lambda}} x_0 - x| + \langle y, J_\lambda^{[\frac{t}{\lambda}]} x_0 - x\rangle_+\, dt.$$

Hence, summing up this in k from $k = [\frac{\hat{t}}{\lambda}] + 1$ to $[\frac{t}{\lambda}]$ we obtain

$$|J_\lambda^{[\frac{t}{\lambda}]} x_0 - x| - |J_\lambda^{[\frac{\hat{t}}{\lambda}]} x_0 - x| \leq \int_{[\frac{\hat{t}}{\lambda}]\lambda}^{[\frac{t}{\lambda}]\lambda} (\omega\,|J_\lambda^{[\frac{s}{\lambda}]} x_0 - x| + \langle y, J_\lambda^{[\frac{s}{\lambda}]} x_0 - x\rangle_+\, ds.$$

Since $|J_\lambda^{[\frac{s}{\lambda}]} x_0| \le (1-\lambda\omega)^{-k}\,|x_0| \le e^{k\lambda\omega}\,|x_0|$, by Lebesgue dominated convergence theorem and the upper semicontinuity of $\langle\cdot,\cdot\rangle_+$, letting $\lambda \to 0^+$ we obtain

$$|S(t)x_0 - x| - |S(\hat{t})x_0 - x| \le \int_{\hat{t}}^t (\omega\,|S(s)x_0 - x| + \langle y, S(t)x_0 - x\rangle_+)\,ds \tag{2.18}$$

for $x_0 \in dom\,(A)$. Similarly, since $S(t)$ is Lipschitz continuous on $\overline{dom\,(A)}$, again by Lebesgue dominated convergence theorem and the upper semicontinuity of $\langle\cdot,\cdot\rangle_+$, (2.18) holds for all $x_0 \in \overline{dom\,(A)}$.

(2) Let $v(t) \in C(0,T;X)$ be an integral of type ω to (2.14). Since $[J_\lambda^k x_0, y_\lambda^k] \in A$, it follows from (2.15) that

$$|v(t) - J_\lambda^k x_0| - |v(\hat{t}) - J_\lambda^k x_0| \le \int_{\hat{t}}^t (\omega\,|v(s) - J_\lambda^k x_0| + \langle y_\lambda^k, v(s) - J_\lambda^k x_0\rangle_+)\,ds. \tag{2.19}$$

Since $\lambda\, y_\lambda^k = -(v(s) - J_\lambda^k x_0) + (v(s) - J_\lambda^{k-1} x_0)$ and from Lemma 1.1 (3) $\langle -x+y, x\rangle_+ = -|x| + \langle y, x\rangle_+$, we have

$$\langle \lambda y_\lambda^k, v(s) - J_\lambda^k x_0\rangle_+ = -|v(s) - J_\lambda^k x_0| + \langle v(s) - J_\lambda^{k-1} x_0, v(s) - J_\lambda^k x_0\rangle_+ \le -|v(s) - J_\lambda^k x_0| + |v(s) - J_\lambda^{k-1} x_0|.$$

Thus, from (2.19)

$$(|v(t) - J_\lambda^k x_0| - |v(\hat{t}) - J_\lambda^k x_0|)\lambda \le \int_{\hat{t}}^t (\omega\lambda\,|v(s) - J_\lambda^k x_0| - |v(s) - J_\lambda^k x_0| + |v(s) - J_\lambda^{k-1} x_0|)\,ds$$

Summing up the both sides of this in k from $[\frac{\hat{\tau}}{\lambda}]+1$ to $[\frac{\tau}{\lambda}]$, we obtain

$$\int_{[\frac{\hat{\tau}}{\lambda}]\lambda}^{[\frac{\tau}{\lambda}]\lambda} (|v(t) - J_\lambda^{[\frac{\sigma}{\lambda}]} x_0| - |v(\hat{t}) - J_\lambda^{[\frac{\sigma}{\lambda}]} x_0|\,d\sigma$$

$$\le \int_{\hat{t}}^t (-|v(s) - J_\lambda^{[\frac{\tau}{\lambda}]} x_0| + |v(s) - J_\lambda^{[\frac{\hat{\tau}}{\lambda}]} x_0| + \int_{[\frac{\hat{\tau}}{\lambda}]\lambda}^{[\frac{\tau}{\lambda}]\lambda} \omega\,|v(s) - J_\lambda^{[\frac{\sigma}{\lambda}]} x_0|\,d\sigma)\,ds.$$

Now, by Lebesgue dominated convergence theorem, letting $\lambda \to 0^+$

$$\begin{aligned} &\int_{\hat{\tau}}^{\tau} (|v(t) - u(\sigma)| - |v(\hat{t}) - u(\sigma)|)\,d\sigma + \int_{\hat{t}}^t (|v(s) - u(\tau)| - |v(s) - u(\hat{\tau})|)\,ds \\ &\qquad \le \int_{\hat{t}}^t \int_{\hat{\tau}}^{\tau} \omega\,|v(s) - u(\sigma)|\,d\sigma\ ds. \end{aligned} \tag{2.20}$$

For $h > 0$ we define F_h by

$$F_h(t) = h^{-2} \int_t^{t+h} \int_t^{t+h} |v(s) - u(\sigma)|\,d\sigma\,ds.$$

Then from (2.20) we have $\dfrac{d}{dt} F_h(t) \le \omega\, F_h(t)$ and thus $F_h(t) \le e^{\omega t} F_h(0)$. Since u, v are continuous we obtain the desired estimate by letting $h \to 0^+$. □

Lemma 3.3 Let A be an ω-dissipative subset satisfying (2.10) and $S(t),\ t \geq 0$ be the semigroup on $\overline{dom\,(A)}$, as constructed in Theorem 2.2. Then, for $x_0 \in \overline{dom\,(A)}$ and $[x,y] \in A$

$$|S(t)x_0 - x|^2 - |S(\hat{t})x_0 - x|^2 \leq 2 \int_{\hat{t}}^{t} (\omega\,|S(s)x_0 - x|^2 + \langle y, S(s)x_0 - x\rangle_s\, ds \tag{2.21}$$

and for every $f \in F(x_0 - x)$

$$\limsup_{t \to 0^+} \, Re\, \langle \frac{S(t)x_0 - x_0}{t}, f\rangle \leq \omega\,|x_0 - x|^2 + \langle y, x_0 - x\rangle_s. \tag{2.22}$$

Proof: Let y_k^λ be defined by (2.16). Since A is ω-dissipative, there exists $f \in F(J_\lambda^k x_0 - x)$ such that $Re\,(y_\lambda^k - y, f\rangle \leq \omega\,|J_\lambda^k x_0 - x|^2$. Since

$$Re\,\langle y_\lambda^k, f\rangle = \lambda^{-1}\, Re\,\langle J_\lambda^k x_0 - x - (J_\lambda^{k-1}x_0 - x), f\rangle$$

$$\geq \lambda^{-1}(|J_\lambda^k x_0 - x|^2 - |J_\lambda^{k-1}x_0 - x||J_\lambda^k x_0 - x|) \geq (2\lambda)^{-1}(|J_\lambda^k x_0 - x|^2 - |J_\lambda^{k-1}x_0 - x|^2),$$

we have from Theorem 1.4

$$|J_\lambda^k x_0 - x|^2 - |J_\lambda^{k-1}x_0 - x|^2 \leq 2\lambda\, Re\,\langle y_\lambda^k, f\rangle \leq 2\lambda\,(\omega\,|J_\lambda^k x_0 - x|^2 + \langle y, J_\lambda^k x_0 - x\rangle_s.$$

Since $J_\lambda^{[\frac{t}{\lambda}]}x_0 = J_\lambda^k x_0$ on $[k\lambda, (k+1)\lambda)$, this can be written as

$$|J_\lambda^k x_0 - x|^2 - |J_\lambda^{k-1}x_0 - x|^2 \leq \int_{k\lambda}^{(k+1)\lambda} (\omega\,|J_\lambda^{[\frac{t}{\lambda}}x_0 - x|^2 + \langle y, J_\lambda^{[\frac{t}{\lambda}]}x_0 - x\rangle_s)\, dt.$$

Hence,

$$|J_\lambda^{[\frac{t}{\lambda}]}x_0 - x|^2 - |J_\lambda^{[\frac{\hat{t}}{\lambda}]}x_0 - x|^2 \leq 2\lambda \int_{[\frac{\hat{t}}{\lambda}]\lambda}^{[\frac{t}{\lambda}]\lambda} (\omega\,|J_\lambda^{[\frac{s}{\lambda}]}x_0 - x|^2 + \langle y, J_\lambda^{[\frac{s}{\lambda}]}x_0 - x\rangle_s)\, ds.$$

Since $|J_\lambda^{[\frac{s}{\lambda}]}x_0| \leq (1-\lambda\omega)^{-k}\,|x_0| \leq e^{k\lambda\omega}\,|x_0|$, by Lebesgue dominated convergence theorem and the upper semicontinuity of $\langle\cdot,\cdot\rangle_s$, letting $\lambda \to 0^+$ we obtain (2.21).

Next, we show (2.22). For any given $f \in F(x_0 - x)$ as shown above

$$2Re\,\langle S(t)x_0 - x_0, f\rangle \leq |S(t)x_0 - x|^2 - |x_0 - x|^2.$$

Thus, from (2.21)

$$Re\,\langle S(t)x_0 - x_0, f\rangle \leq \int_0^t (\omega\,|S(s)x_0 - x|^2 + \langle y, S(s)x_0 - x\rangle_s)\, ds$$

Since $s \to S(s)x_0$ is continuous, by the upper semicontinuity of $\langle\cdot,\cdot\rangle_s$, we have (2.22). □

Theorem 3.4 Assume that A is a close dissipative subset of $X \times X$ and satisfies the range condition (2.10) and let $S(t),\ t \geq 0$ be the semigroup on $\overline{dom\,(A)}$, defined in Theorem 2.2. Then, if $S(t)x$ is strongly differentiable at $t_0 > 0$ then

$$S(t_0)x \in dom\,(A) \quad \text{and} \quad \frac{d}{dt}S(t)x\,|_{t=t_0} \in AS(t)x,$$

and moreover

$$S(t_0)x \in dom\,(A^0) \quad \text{and} \quad \frac{d}{dt}S(t)x\,\big|_{t=t_0} = A^0 S(t_0)x.$$

Proof: Let $\frac{d}{dt}S(t)x\,|_{t=t_0} = y$. Then $S(t_0-\lambda)x - (S(t_0)x - \lambda\, y) = o(\lambda)$, where $\frac{|o(\lambda)|}{\lambda} \to 0$ as $\lambda \to 0^+$. Since $S(t_0-\lambda)x \in \overline{dom\,(A)}$, there exists a $[x_\lambda, y_\lambda] \in A$ such that $S(t_0-\lambda)x = x_\lambda - \lambda\, y_\lambda$ and

$$\lambda\,(y - y_\lambda) = S(t_0)x - x_\lambda + o(\lambda). \tag{2.23}$$

If we let $x = x_\lambda$, $y = y_\lambda$ and $x_0 = S(t_0)x$ in (2.22), then we obtain

$$Re\,\langle y, f\rangle \le \omega\,|S(t_0)x - x_\lambda|^2 + \langle y_\lambda, S(t_0)x - x_\lambda\rangle_s.$$

for all $f \in (S(t_0)x - x_\lambda)$. It follows from Lemma 1.3 that there exists a $g \in F(S(t_0)x - x_\lambda)$ such that $\langle y_\lambda, S(t)x - x_\lambda\rangle_s = Re\,\langle y_\lambda, g\rangle$ and thus

$$Re\,\langle y - y_\lambda, g\rangle \le \omega\,|S(t_0)x - x_\lambda|^2.$$

From (2.23)

$$\lambda^{-1}|S(t_0)x - x_\lambda|^2 \le \frac{|o(\lambda)|}{\lambda}\,|S(t_0)x - x_\lambda| + \omega\,|S(t_0)x - x_\lambda|^2$$

and thus

$$(1-\lambda\omega)\left|\frac{S(t_0)x - x_\lambda}{\lambda}\right| \to 0 \quad \text{as} \;\; \lambda \to 0^+.$$

Combining this with (2.23), we obtain

$$x_\lambda \to S(t_0)x \quad \text{and} \quad y_\lambda \to y$$

as $\lambda \to 0^+$. Since A is closed, it follows that $[S(t_0)x, y] \in A$, which shows the first assertion.

Next, from Theorem 2.2

$$|S(t_0+\lambda)x - S(t_0)x| \le \lambda\,\|AS(t_0)x\| \quad \text{for} \;\; \lambda > 0$$

This implies that $|y| \le \|AS(t_0)x\|$. Since $y \in AS(t_0)x$, it follows that $S(t_0)x \in dom\,(A^0)$ and $y \in A^0S(t_0)x$. $\square$

<u>**Theorem 3.5**</u> Let A be a dissipative subset of $X \times X$ satisfying the range condition (2.10) and $S(t),\ t \ge 0$ be the semigroup on $\overline{dom\,(A)}$, defined in Theorem 2.2. Then the followings hold.

(1) For ever $x \in \overline{dom\,(A)}$

$$\lim_{\lambda\to 0^+} |A_\lambda x| = \liminf_{t\to 0^+} |S(t)x - x|.$$

(2) Let $x \in \overline{dom\,(A)}$. Then, $\lim_{\lambda\to 0^+} |A_\lambda x| < \infty$ if and only if there exists a sequence $\{x_n\}$ in $dom\,(A)$ such that $\lim_{n\to\infty} x_n = x$ and $\sup_n \|Ax_n\| < \infty$.

Proof: Let $x \in \overline{dom\,(A)}$. From Theorem 1.4 $|A_\lambda x|$ is monotone decreasing and $\lim |A_\lambda x|$ exists (including ∞). Since $|J_\lambda^{[\frac{t}{\lambda}]} - x| \le t\,|A_\lambda x|$, it follows from Theorem 2.2 that

$$|S(t)x - x| \le t \lim_{\lambda\to 0^+} |A_\lambda x|$$

thus

$$\liminf_{t\to 0^+} \frac{1}{t}\,|S(t)x-x| \le \lim_{\lambda\to 0^+} |A_\lambda x|.$$

Conversely, from Lemma 3.3

$$-\liminf_{t\to 0^+} \frac{1}{t}\,|S(t)x-x||x-u| \le \langle v, x-u\rangle_s$$

for $[u,v]\in A$. We set $u=J_\lambda x$ and $v=A_\lambda x$. Since $x-u=-\lambda\,A_\lambda x$,

$$-\liminf_{t\to 0^+} \frac{1}{t}\,|S(t)x-x|\lambda|A_\lambda x| \le -\lambda\,|A_\lambda x|^2$$

which implies

$$\liminf_{t\to 0^+} \frac{1}{t}\,|S(t)x-x| \ge |A_\lambda x|.$$

Theorem 3.7 Let A be a dissipative set of $X\times X$ satisfying the range condition (2.10) and let $S(t),\ t\ge 0$ be the semigroup defined on $\overline{dom\,(A)}$ in Theorem 2.2.

(1) If $x\in dom\,(A)$ and $S(t)x$ is differentiable a.e., $t>0$, then $u(t)=S(t)x,\ t\ge 0$ is a unique strong solution of the Cauchy problem (2.14).

(2) If X is reflexive. Then, if $x\in dom\,(A)$ then $u(t)=S(t)x,\ t\ge 0$ is a unique strong solution of the Cauchy problem (2.14).

Proof: The assertion (1) follows from Theorems 3.1 and 3.5. If X is reflexive, then since an X-valued absolute continuous function is a.e. strongly differentiable, (2) follows from (1). □

2.2.1 Infinitesimal generator

Definition 4.1 Let X_0 be a subset of a Banach space X and $S(t),\ t\ge 0$ be a semigroup of nonlinear contractions on X_0. Set $A_h=h^{-1}(T(h)-I)$ for $h>0$ and define the strong and weak infinitesimal generators A_0 and A_w by

$$\begin{aligned} &A_0x=\lim_{h\to 0^+} A_hx \quad \text{with } dom\,(A_0)=\{x\in X_0 : \lim_{h\to 0^+} A_hx \text{ exists}\}\\ &A_0x=w-\lim_{h\to 0^+} A_hx \quad \text{with } dom\,(A_0)=\{x\in X_0 : w-\lim_{h\to 0^+} A_hx \text{ exists}\}, \end{aligned} \tag{2.24}$$

respectively. We define the set $\hat{D}$ by

$$\hat{D}=\{x\in X_0 : \liminf_{h\to 0} |A_hx|<\infty\} \tag{2.25}$$

Theorem 4.1 Let $S(t),\ t\ge 0$ be a semigroup of nonlinear contractions defined on a closed subset X_0 of X. Then the followings hold.
(1) $\langle A_wx_1-A_wx_2, x^*\rangle \le 0$ for all $x_1,x_2\in dom\,(A_w)$ and $x^*\in F(x_1-x_2)$. In particular, A_0 and A_w are dissipative.
(2) If X is reflexive, then $\overline{dom\,(A_0)}=\overline{dom\,(A_w)}=\overline{\hat{D}}$.
(3) If X is reflexive and strictly convex, then $dom(A_w)=\hat{D}$. In addition, if X is uniformly convex, then $dom\,(A_w)=dom\,(A_0)=\hat{D}$ and $A_w=A_0$.

Proof: (1) For $x_1, x_2 \in X_0$ and $x^* \in F(x_1 - x_2)$ we have

$$\langle A_h x_1 - A_h x_2, x^* \rangle = h^{-1}(\langle S(h)x_1 - S(h)x_2, x^* \rangle - |x_1 - x_2|^2)$$

$$\leq h^{-1}(|S(h)x_1 - S(h)x_2||x_1 - x_2| - |x_1 - x_2|^2) \leq 0.$$

Letting $h \to 0^+$, we obtain the desired inequality.
(2) Obviously, $dom\,(A_0) \subset dom\,(A_w) \subset \hat{D}$. Let $x \in \hat{D}$. It suffices to show that $x \in \overline{dom\,(A_0)}$. We can show that $t \to S(t)x$ is Lipschitz continuous. In fact, there exists a monotonically decreasing sequence $\{t_k\}$ of positive numbers and $L > 0$ such that $t_k \to 0$ as $k \to \infty$ and $|S(t_k)x - x| \leq L\, t_k$. Let $h > 0$ and n_k be a nonnegative integer such that $0 \leq h - n_k t_k < t_k$. Then we have

$$|S(t+h)x - S(t)x| \leq |S(t)x - x| = |S(h - n_k t_k + n_k t_k)x - x|$$

$$\leq |S(h - n_k t_k)x - x| + L\, n_k t_k \leq |S(h - n_k t_k)x - x| + L\, h$$

By the strong continuity of $S(t)x$ at $t = 0$, letting $k \to \infty$, we obtain $|S(t+h)x - S(t)x| \leq L\, h$. Now, since X is reflexive this implies that $S(t)x$ is a.e. differentiable on $(0, \infty)$. But since $S(t)x \in dom\,(A_0)$ whenever $\frac{d}{dt}S(t)x$ exists, $S(t)x \in dom\,(A_0)$ a.e. $t > 0$. Thus, since $|S(t)x - x| \to 0$ as $t \to 0^+$, it follows that $x \in \overline{dom\,(A_0)}$.
(3) Assume that X is reflexive and strictly convex. Let $x_0 \in \hat{D}$ and Y be the set of all weak cluster points of $t^{-1}(S(t)x_0 - x_0)$ as $t \to 0^+$. Let $\tilde{A}$ be a subset of $X \times X$ defined by

$$\tilde{A} = A_0 \cup [x_0, \overline{co\, Y}] \quad \text{and} \quad dom\,(\tilde{A}) = dom\,(A_0) \cup \{x_0\}$$

where $\overline{co\, Y}$ denotes the closure of the convex hull of Y. Note that from (1)

$$\langle A_0 x_1 - A_0 x_2, x^* \rangle \leq 0 \quad \text{for all } \ x_1, x_2 \in dom\,(A_0) \text{ and } x^* \in F(x_1 - x_2)$$

and for every $y \in Y$

$$\langle A_0 x_1 - y, x^* \rangle \leq 0 \quad \text{for all } \ x_1 \in dom\,(A_0) \text{ and } x^* \in F(x_1 - x_0).$$

This implies that $\tilde{A}$ is a dissipative subset of $X \times X$. But, since X is reflexive, $t \to S(t)x_0$ is a.e. differentiable and

$$\frac{d}{dt}S(t)x_0 = A_0 S(t)x_0 \in \tilde{A}S(t)x_0, \quad a.e.,\ t > 0.$$

It follows from the dissipativity of $\tilde{A}$ that

$$\langle \frac{d}{dt}(S(t)x_0 - x_0), x^* \rangle \leq \langle y, x^* \rangle, \quad a.e,\ t > 0 \text{ and } y \in \tilde{A}x_0 \tag{2.26}$$

for $x^* \in F(S(t)x_0 - x_0)$. Note that for $h > 0$

$$\langle h^{-1}(S(t+h)x_0 - S(t)x_0), x^* \rangle \leq h^{-1}(|S(t+h)x_0 - x_0| - |S(t)x_0 - x_0|)|x^*| \quad \text{for } x^* \in F(S(t)x_0 - x_0).$$

Letting $h \to 0^+$, we have

$$\langle \frac{d}{dt}(S(t)x_0 - x_0), x^* \rangle \leq |S(t)x_0 - x_0| \, \frac{d}{dt}|S(t)x_0 - x_0|, \quad a.e.\ t > 0$$

The converse inequality follows much similarly. Thus, we have

$$|S(t)x_0 - x_0|\frac{d}{dt}|S(t)x_0 - x_0| = \langle \frac{d}{dt}(S(t)x_0 - x_0), x^* \rangle \quad \text{for } \ x^* \in F(S(t)x_0 - x_0). \tag{2.27}$$

It follows from (2.26)–(2.27) that $\dfrac{d}{dt}|S(t)x_0 - x_0| \leq |y|$ for $y \in Y$ and a.e. $t > 0$ and thus

$$|S(t)x_0 - x_0| \leq t\,\|\tilde{A}x_0\| \quad \text{for all } t > 0. \tag{2.28}$$

Note that $\tilde{A}x_0 = \overline{co\,Y}$ is a closed convex subset of X. Since X is reflexive and strictly convex, there exists a unique element $y_0 \in \tilde{A}x_0$ such that $|y_0| = \|\tilde{A}x_0\|$. Hence, (2.28) implies that $\overline{co\,Y} = y_0 = A_w x_0$ and therefore $x_0 \in dom\,(A_w)$.

Next, we assume that X is uniformly convex and let $x_0 \in dom(A_w) = \hat{D}$. Then

$$w - \lim \ \frac{S(t)x_0 - x_0}{t} = y_0 \quad \text{as } \ t \to 0^+.$$

From (2.28)

$$|t^{-1}(S(t)x_0 - x_0)| \leq |y_0|, \quad a.e.\ t > 0.$$

Since X is uniformly convex, these imply that

$$\lim \ \frac{S(t)x_0 - x_0}{t} = y_0 \quad \text{as } \ t \to 0^+.$$

which completes the proof. □

Theorem 4.2 Let X and X^* be uniformly convex Banach spaces. Let $S(t),\ t \geq 0$ be the semigroup of nonlinear contractions on a closed subset X_0 and A_0 be the infinitesimal generator of $S(t)$. If $x \in dom\,(A_0)$, then
(*i*) $S(t)x \in dom\,(A_0)$ for all $t \geq 0$ and the function $t \to A_0 S(t)x$ is right continuous on $[0, \infty)$.
(*ii*) $S(t)x$ has a right derivative $\frac{d^+}{dt}S(t)x$ for $t \geq 0$ and $\frac{d^+}{dt}S(t)x = A_0 S(t)x,\ t \geq 0$.
(*iii*) $\frac{d}{dt}S(t)x$ exists and is continuous except a countable number of values $t \geq 0$.

Proof: (*i*) – (*ii*) Let $x \in dom\,(A_0)$. By Theorem 4.1, $dom\,(A_0) = \hat{D}$ and thus $S(t)x \in dom\,(A_0)$ and

$$\frac{d^+}{dt}S(t)x = A_0 S(t)x \quad \text{for } \ t \geq 0. \tag{2.29}$$

Moreover, $t \to S(t)x$ a.e. differentiable and $\frac{d}{dt}S(t)x = A_0 S(t)x$ a.e. $t > 0$. We next prove that $A_0 S(t)x$ is right continuous. For $h > 0$

$$\frac{d}{dt}(S(t+h)x - S(t)x) = A_0 S(t+h)x - A_0 S(t)x, \quad a.e.\ t > 0.$$

From (2.27)

$$|S(t+h)x - S(t)x|\,\frac{d}{dt}|S(t+h)x - S(t)x| = \langle A_0S(t+h)x - A_0S(t)x, x^*\rangle \leq 0$$

for all $x^* \in F(S(t+h)x - S(t)x)$, since A_0 is dissipative. Integrating this over $[s,t]$, we obtain

$$|S(t+h)x - S(t)x| \leq |S(s+h)x - S(s)x| \quad \text{for } \ 0 \leq s \leq t$$

and therefore

$$|\frac{d^+}{dt}S(t)x| \leq |\frac{d^+}{ds}S(s)x|.$$

Hence $t \to |A_0S(t)x|$ is monotonically non-increasing function and thus it is right continuous. Let $t_0 \geq 0$ and let $\{t_k\}$ be a decreasing sequence of positive numbers such that $t_k \to t_0$. Without loss of generality, we may assume that $w - \lim_{k\to\infty} A_0S(t_k) = y_0$. The right continuity of $|A_0S(t)x|$ at $t = t_0$, thus implies that

$$|y_0| \leq |A_0S(t_0)x| \tag{2.30}$$

since norm is weakly lower semicontinuous. Let $\tilde{A}_0$ be the maximal dissipative extension of A_0. It then follows from Theorem 1.9 that $\tilde{A}$ is demiclosed and thus $y_0 \in \tilde{A}S(t)x$. On the other hand, for $x \in dom\,(A_0)$ and $y \in \tilde{A}x$, we have

$$\langle\frac{d}{dt}(S(t)x - x), x^*\rangle \leq \langle y, x^*\rangle \quad \text{for all } \ x^* \in F(S(t)x - x)$$

a.e. $t > 0$, since $\tilde{A}$ is dissipative and $\frac{d}{dt}S(t)x = A_0S(t)x \in \tilde{A}S(t)x$ a.e. $t > 0$. From (2.27) we have

$$t^{-1}|S(t)x - x| \leq |\tilde{A}x| = \|\tilde{A}^0x\| \quad \text{for } \ t \geq 0$$

where $\tilde{A}^0$ is the minimal section of $\tilde{A}$. Hence $A_0x = \tilde{A}^0x$. It thus follows from (2.30) that $y_0 = A_0S(t_0)x$ and $\lim_{k\to\infty} A_0S(t_k)x = y_0$ since X is uniformly convex. Thus, we have proved the right continuity of $A_0S(t)x$ for $t \geq 0$.
(*iii*) Integrating (2.29) over $[t, t+h]$, we have

$$S(t+h)x - S(t)x = \int_t^{t+h} A_0S(s)x\,ds$$

for $t,\ h \geq 0$. Hence it suffices to prove that the function $t \to A_0S(t)x$ is continuous except a countable number of $t > 0$. Using the same arguments as above, we can show that if $|A_0S(t)x|$ is continuous at $t = t_0$, then $A_0S(t)x$ is continuous at $t = t_0$. But since $|A_0S(t)x|$ is monotone non-increasing, it follows that it has at most countably many discontinuities, which completes the proof. □

Theorem 4.3 Let X and X^* be uniformly convex Banach spaces. If A be m-dissipative, then A is demiclosed, $\overline{dom\,(A)}$ is a closed convex set and A^0 is single-valued operator with $\overline{dom\,(A^0)} = \overline{dom\,(A)}$. Moreover, A^0 is the infinitesimal generator of a semigroup of contractions on $\overline{dom\,(A)}$.

Proof: It follows from Theorem 1.9 that A is demiclosed. The second assertion follows from Theorem 1.12. Also, from Theorem 3.4

$$\frac{d}{dt}S(t)x = A^0 S(t)x, \quad a.e.\ t > 0$$

and

$$|S(t)x - x| \le t\,|A^0 x|, \quad t > 0 \tag{2.31}$$

for $x \in dom\,(A)$. Let A_0 be the infinitesimal generator of the semigroup $S(t),\ t \ge 0$ generated by A defined in Theorem 2.2. Then, (2.31) implies that by Theorem 4.1 $x \in dom\,(A_0)$ and by Theorem 4.2 $\frac{d^+}{dt}S(t)x = A_0 S(t)x$ and $A_0 S(t)x$ is right continuous in t. Since A is closed,

$$A_0 x = \lim_{t \to 0^+} A_0 S(t)x \in Ax.$$

Hence, (2.30) implies that $A_0 x = A^0 x$.

When X is a Hilbert space we have the nonlinear version of Hille-Yosida theorem as follows.

Theorem 4.4 Let H be a Hilbert space. Then,
(1) The infinitesimal generator A_0 of a semigroup of contractions $S(t),\ t \ge 0$ on a closed convex set X_0 has a dense domain in X_0 and there exists a unique maximal dissipative operator A such that $A^0 = A_0$.

Conversely,
(2) If A_0 is a maximal dissipative operator, then $\overline{dom\,(A)}$ is a closed convex set and A^0 is the infinitesimal generator of contractions on $\overline{dom\,(A)}$.

Proof: (2) Since from Theorem 1.7 the maximal dissipative operator in a Hilbert space is m-dissipative, (2) follows from Theorem 4.3. □

Example (Nonlinear Diffusion) Consider the nonlinear diffusion equation of the form

$$u_t = Au = \Delta\gamma(u) - \beta(u)$$

on $X = L^1(\Omega)$. Assume $\gamma : R \to R$ is maximal monotone. and $\gamma : R \to R$ is monotone. Let

$$dom(A) = \{\text{there exists a } v \in W^{1,1}(\Omega) \text{ such that } v \in \gamma(u) \text{ and } \Delta v \in X\}$$

Thus, $sign(x-y) = sign(\gamma(x)-\gamma(y))$. Let $\rho \in C^2(R)$ be a monotonically increasing function satisfying $\rho(0) = 0$ and $\rho(x) = sign(x),\ |x| \ge 1$ and $\rho_\epsilon(x) = \rho(\frac{x}{\epsilon})$ for $\epsilon > 0$. Note that for $u \in X$

$$(u, \rho_\epsilon(u)) \to |u| \quad \text{and} \quad (\psi, \rho_\epsilon(u)) \to (\psi, sign_0(u)) \ \text{ for } \ \psi \in X$$

as $\epsilon \to 0^+$.

$$(Au_1 - Au_2, \rho_\epsilon(\gamma(u_1) - \gamma(u_2))) = -(\nabla(\gamma(u_1) - \gamma(u_2)), \rho_\epsilon'\, \nabla(\gamma(u_1) - \gamma(u_2)))$$

$$-(\beta(u_1) - \beta(u_2), \rho_\epsilon(\gamma(u_1) - \gamma(u_2)) \le 0.$$

Letting $\epsilon \to 0^*$ we obtain

$$(Au_1 - Au_2, sign_0(u_1 - u_2)) \leq 0$$

for all $u_1,\ u_2 \in dom(A)$.

For the range condition: we consider $v \in \gamma(u)$ such that

$$\lambda\, \gamma^{-1}(v) - \Delta v + \beta(\gamma^{-1}(v)) = f, \tag{2.32}$$

Let $\partial j = \lambda\, \gamma^{-1}(\cdot) + \beta(\gamma^{-1}(\cdot))$. For $f \in L^2(\Omega)$ consider the minimization

$$\frac{1}{2}\int_\Omega (|\nabla v|^2 + j(v) - f(x)v)\, dx$$

over $v \in H^1_0(\Omega)$. It has a unique solution $v \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$\Delta v + f \in \partial j(v).$$

For $f \in X$ we choose $f_n \in L^2(\Omega)$ such that $|f_n - f|_X \to 0$ as $n \to \infty$. As show above $u_n = \Delta v_n + f_n$,

$$|u_n - u_m|_X \leq M\, |f_n - f_m|_X$$

Thus, there exists $u \in X$ such that $\Delta v_n \to u - f$. Moreover, there exists a $v \in W^{1,q},\ 1 < q < \frac{d}{d-1}$ such that $v_n \to v$ in X and $\Delta v = u - f$. In fact, let $p > d$. For all $h_0 \in L^p(\Omega)$ and $\vec{h} \in L^p(\Omega)^d$

$$-\Delta\phi = h_0 + \nabla \cdot \vec{h}$$

has a unique solution $\phi \in W^{1,p}(\Omega)$ and

$$|\phi|_{W^{1,p}} \leq M\, (|h|_p + |\vec{h}|_p).$$

By the Green's formula

$$|(h_0, v_n) - (h, \nabla v_n)| = |-(\Delta v_n, \phi)| \leq M\, (|h|_p + |\vec{h}|_p|)|\Delta v_n|_1.$$

Since $h_0 \in L^p(\Omega)$ and $h \in L^p(\Omega)^d$ are arbitraly

$$|v_n|_{W^{1,q}} \leq M\, |\Delta v_n|,$$

Since $W^{1,q}(\Omega)$ is compactly embedded into $L^1(\Omega)$,

$$v_n \to v, \quad v \in W^{1,q}_0(\Omega).$$

Since ∂j is maximal monotone $u \in \partial j(v)$ equivalently $u \in \gamma^{-1}(v)$ and $\Delta v + f \in \partial j(v)$.

Example (Conservation law) We consider the scalar conservation law

$$u_t + (f(u))_x + f_0(x,u) = 0,\ \ t > 0 \quad u(x,0) = u_0(x),\ x \in R^d \tag{2.33}$$

where $f : R \to R^d$ is C^1. Let $X = L^1(R^d)$ and define

$$Au = -(f(u))_x,$$

where we assume $f_0 = 0$ for the sake of simplicity of our presentation. Define

$$\mathcal{C} = \{\phi \in Z : \phi \geq 0\}.$$

Since $Ac = 0$ for all constant c, it follows that

$$\phi - c \in \mathcal{C} \Longrightarrow (I - \lambda\,\mathcal{A})^{-1}\phi - c \in \mathcal{C}.$$

Similarly,

$$c - \phi \in \mathcal{C} \Longrightarrow c - (I - \lambda\,\mathcal{A})^{-1}\phi \in \mathcal{C}.$$

Thus, without loss of generality, one can assume f is bounded.
We use the following lemma.

Lemma 2.1 For $\psi \in H^1(R^d)$ and $\phi \in L^2_{div} = \{\phi \in (L^2(R^d))^d : \nabla \cdot \phi \in L^2(R^d)\}$ we have

$$(\phi, \nabla\psi) + (\nabla \cdot \phi, \psi) = 0$$

Proof: Note that for $\zeta \in C_0^\infty(R^d)$

$$(\phi, \nabla(\zeta\,\psi)) + (\nabla \cdot \phi, \zeta\,\psi) = 0$$

Let $g \in C^\infty(R^d)$ satisfying $g = 1$ for $|x| \leq 1$ and $g = 0$ if $|x| \geq 2$ and set $\zeta = g(\frac{x}{r})$. Then we have

$$(\phi, \zeta\nabla\psi + \frac{1}{r}\,\psi\nabla g) + (\nabla \cdot \phi, \zeta\,\psi) = 0.$$

Since $g(\frac{x}{r}) \to 1$ a.e. in R^d as $r \to \infty$ thus the lemma follows from Fatou's lemma. $\Box$.

<u>dissipativity</u> Note that

$$-(f(u_1)_x - f(u_2)_x, \rho(u_1 - u_2)) = (f(u_1) - f(u_2), \rho'(u_1 - u_2)\,(u_1 - u_2)_x),$$

If we define $\Psi(x) = \int_0^x \sigma\rho'(\sigma)\,d\sigma$ and $\rho_\epsilon(x) = \rho(\frac{x}{\epsilon})$ for $\epsilon > 0$. then

$$|(\eta\,(u_1 - u_2), \rho'_\epsilon(u_1 - u_2)\,(u_1 - u_2)_x)| = \epsilon\,(\Psi(\frac{u_1 - u_2}{\epsilon}), \eta_x) \leq M\,\epsilon\,|\eta_x|_1 \to 0$$

as $\epsilon \to 0$. Note that for $u \in L^1(R^d)$

$$(u, \rho_\epsilon(u)) \to |u| \quad \text{and} \quad (\psi, \rho_\epsilon(u)) \to (\psi, sign_0(u)) \;\text{ for }\; \psi \in L^1(R^d)$$

as $\epsilon \to 0^+$. Thus,

$$\langle Au_1 - Au_2, sign_0(u_1 - u_2)\rangle \leq 0$$

and A is dissipative.

It will be show that

$$range(\lambda\,I - A) = X,$$

i.e., for any $g \in X$ there exists an entropy solution satisfying

$$(sign(u - k)(\lambda u - g), \psi) \leq (sing(u - k)(f(u) - f(k)), \psi_x)$$

for all $\psi \in C_0^1(R^d)$ and $k \in R$. Hence A has a maximal monotone extension in $L^1(R^d)$.

In fact, for $\epsilon > 0$ consider the viscous equation

$$\lambda\, u - \epsilon\, \Delta u + f(u)_x = g \tag{2.34}$$

First, assume f is Lipschitz continuos, one can show that

$$\lambda\, u - \epsilon \Delta u - f(u)_x : H^1(R^d) \to (H^1(R^d))^*$$

is monotone, hemi-continuous and coercive. It thus follows from the Minty-Browder theorem that(2.34) has a solution $u^\epsilon \in H^1(R^d)$ for all $g \in L^2(R^d) \cap L^1(R^d)$. Since $f(u)_x \in L^2(R^d)$, $u \in H^2(R^d)$.

<u>$L^\infty(R^d)$ estimate</u> From (2.34)

$$(\lambda\, u - \epsilon\, \Delta u + f(u)_x, |u|^{p-2}u) = (g, |u|^{p-2}u).$$

Since

$$\lambda |u|_p^p + (f'(u)u_x, |u|^{p-2}u) - \epsilon(p-1)(|u|^{\frac{p}{2}-1}u_x, |u|^{\frac{p}{2}-1}u_x)$$

$$\leq (\lambda - \frac{\delta}{2} - \frac{|f'|_\infty}{2\epsilon\delta(p-1)})|u|_p^p,$$

we have

$$|u|_p \leq (\frac{1}{2}\lambda - \frac{|f'|_\infty}{2\epsilon\lambda(p-1)})^{-1}|g|_p.$$

By letting $p \to \infty$ we obtain

$$|u|_\infty \leq \frac{1}{\lambda}|g|_\infty. \tag{2.35}$$

Thus, without loss of generality, f is $C^1(R)^d$.

<u>$W^{1,1}(R^d)$ estimate</u> Assuming $g \in W^{11}(R^d)$, $v = u_x$ satisfies

$$\lambda\, v - \epsilon \Delta v + (f'(u)v)_x = g_x.$$

Using the same arguments as above

$$|v| \leq |g_x|_1.$$

Thus, u^ϵ is of bounded variation uniformly in $\epsilon > 0$. Since $BV(\Omega)$ is compact in $L^1(\Omega)$ for any bounded set in R^d. Thus, for $g \in W^{1,1}$ there exists a strong limit u of u^ϵ in $L^1(R)$ as $\epsilon \to 0^+$ and $u \in BV \cap L^\infty$ such that

$$\int (\lambda\, u - g)\phi\, dx - \int (f(u), \phi_x)\, dx = 0$$

for all $\phi_0(R^d)$, i.e. Since (2.35) holds uniformly $\epsilon > 0$ and W^{11} is dense in L^1, one can show that $R(\lambda\, I - A) = X$.

<u>Entropy solution</u> We show that for every $k \in R$ and nonnegative function $\psi \in C_0^\infty(R^d)$

$$(sign(u-k)(\lambda u - g, , \psi) - \epsilon(\Delta u, \psi) \leq (sign(u-k)(f(x) - f(k)), \psi_x) + \epsilon\, (|u-k|, \Delta\psi) \geq 0 \tag{2.36}$$

It suffices to prove for $u \in C_0^2(R^d)$. Note that

$$(\rho_\epsilon(u-k)f(u)_x,\psi) = ((\int_k^u \rho_\epsilon(s-k)f'(u)\,ds)_x,\psi)$$

$$= -(\int_k^u \rho_\epsilon(s-k)f_u(t,x,u)\,ds,\psi_x)$$

Since $\int_k^{k+\epsilon}\rho_\epsilon(s-k)\,ds = \epsilon\int_0^1 \rho(s)\,ds \to 0$ as $\epsilon \to 0$, letting $\epsilon \to 0$ we obtain

$$(sgn(u-k)f(u)_x,\psi) = -(sgn(u-k)(f(u)-f(k)),\psi_x)$$

Next,

$$-(\rho_\epsilon(u-k)\Delta u,\psi) = (\rho'_\epsilon(u-k)u_x,\psi\, u_x) + (\rho_\epsilon(u-k)u_x,\psi_x)$$

where

$$(\rho_\epsilon(u-k)u_x,\psi_x) = (-\Psi_\epsilon(u-k), De|ta\psi) \to -(|u-k|,\Delta\psi) \quad \text{as} \ \ \epsilon \to 0.$$

Thus, we obtain

$$-(\rho_\epsilon(u-k)\Delta u,\psi) \geq -(|u-k|,\Delta\psi)$$

and u^ϵ satisfies (2.37). Letting $\epsilon \to 0^+$,

$$(sign(u-k)(\lambda u - g,,\psi) - \epsilon(\Delta u,\psi) \leq (sign(u-k)(f(x)-f(k)),\psi_x) \geq 0 \tag{2.37}$$

for all limit of u^ϵ as $\epsilon \to 0^+$, i.e. u is an entropy solution. It can be shown that the entropy solution is unique.

Example (Hamilton Jacobi equation) Consider the Hamilton Jacobi equation for value function $v = v(t,x) \in R$:

$$v_t + f(v_x) = 0. \tag{2.38}$$

Note that u is a solution to a scalar conservation in R^1, then $v = \int^x u\,dx$ satisfies the the Hamilton-Jacobi equation. Let $X = C_0(R^d)$ and

$$Av = -f(v_x) \quad dom(A) = \{f(v_x) \in X\}$$

Then, for $v_1,\ v_2 \in C^1(R^d) \cap X$

$$\langle A(v_1 - v_2), \delta_{x_0}\rangle = -(f((v_1)_x(x_0)) - f((v_2)_x(x_0))) = 0$$

where $x_0 \in R^n$ such that $|v|_X = |v(x_0)|$.

We prove the range condition

$$range(\lambda I - A) = X \quad \text{for } \lambda > 0.$$

That is, there exists a unique viscosity solution to $\lambda v - f(v_x) = g$; for all $\phi \in C^1(\Omega)$ if $v - \phi$ attains a local maximum at $x_0 \in R^d$, then

$$\lambda\, v(x_0) - g(x_0) + f(\phi_x(x_0)) \leq 0$$

and if $v - \phi$ attains a local minimum at $x_0 \in R^d$, then

$$\lambda\, v(x_0) - g(x_0) + f(\phi_x(x_0)) \geq 0.$$

Thus, A is maximal monotone and (2.38) has an intgral solution.

Consider the equation of the form

$$\lambda\, V + \hat{H}(x, V_x) - \nu\, \Delta V = \omega\, f.$$

Assume that $\hat{H}$ is C^1 and there exist $\tilde{c}_1 > 0,\ \tilde{c}_2 > 0$ and $\tilde{c}_3 \in L^2(R^d)$ such that

$$|\hat{H}(x,p) - \hat{H}(x,q)| \leq \tilde{c}_1\, |p - q| \quad \text{and} \quad \hat{H}(t,x,0) \in L^2(R^d)$$

and

$$|\hat{H}_x(x,p)| \leq \tilde{c}_3(x) + \tilde{c}_2\, |p|$$

Define the Hilbert space $H = H^1(R^d)$ by

$$H = \{\phi \in L^2(R^d) : \phi_x \in L^2(R^d)\}$$

with inner product

$$(\phi, \psi)_H = \int_{R^d} \phi(x)\psi(x) + \phi_x(x) \cdot \psi_x(x)\, dx.$$

Define the single valued operator A on H by

$$A\, V = -\hat{H}(x, V_x) + \epsilon\, \Delta V$$

with $dom\,(A) = H^3(R^d)$. We show that $A - \lambda\, I$ is m-dissipative for some λ. First, $A - \hat{\omega}\, I$ with $\lambda = \frac{\tilde{c}_1^2}{2\epsilon}$ is monotone since

$$(A\phi - A\psi, \phi - \psi)_H = -\epsilon\, (|\phi_x - \psi_x|_2^2 + |\Delta(\phi - \psi)|_2^2)$$

$$-(\hat{H}(\cdot, \phi_x) - \hat{H}(\cdot, \psi_x), \phi_x - \psi_x - \Delta(\phi - \psi))$$

$$\leq \lambda\, |\phi - \psi|_H^2 - \frac{\epsilon}{2}\, |\phi_x - \psi_x|_H^2,$$

where we used the following Lemma 2.1.

Let us define the linear operator T on X by $T\phi = \frac{\nu}{2}\, \Delta$. with $dom\,(T) = H^3(R^d)$. Then T is a self-adjoint operator in H. Moreover, if let $\tilde{X} = H^2(R^d)$ then $\tilde{X}^* = L^2(R^d)$ where $H = H^1(R^d)$ is the pivoting space and $H = H^*$ and $T \in \mathcal{L}(\tilde{X}, \tilde{X}^*)$ is hermite and coercive. Thus, T is maximal monotone. Hence equation $\omega\, V - AV = \omega\, f$ in $L^2(R^d)$ has a unique solution $V \in H$. Note that from (2.7) for $\phi \in H^2(R^d)$

$$\hat{H}(x, \phi_x)_x = \hat{H}_x(x, \phi_x) + \hat{H}_p(x, \phi_x)\phi_{xx} \in L^2(R^d).$$

Thus, if $f \in H$ then the solution $V \in dom\,(A)$ and thus A is maximum monotone in H.

Step 3: We establish $W^{1,\infty}(R^d)$ estimate of solutions to

$$\omega\, V + H(x,V_x) - \nu\,\Delta V = \omega\, f,$$

when $f_x,\ H_x(x,0) \in L^\infty(R^d)$ and H satisfies

$$|H_x(x,p) - H_x(x,0)| \le M_1\,|p| \quad \text{and} \quad |H_p(x,p)| \le M_2\,\sqrt{1+|x|^2}. \tag{2.9}$$

Consider the equation of the form

$$\omega\, V + \psi\, H(x,V_x) - \nu\,\Delta V = \omega\, f \tag{2.10}$$

where

$$\psi(x) = \frac{c^2}{c^2+|x|^2} \quad \text{for } \ c \ge 1.$$

Then $\psi(x)\,H(x,p)$ satisfies (2.6)–(2.7) and thus there exists a unique solution $V \in H^3(R^d)$ to (2.10) for sufficiently large $\omega > 0$. Define $U = V_x$. Then, from (2.10) we have

$$\omega\,(U,\phi) + (\psi_x\, H(x,U) + \psi\, H_p(x,U)\cdot U_x, \phi) + \nu\,(U_x,\phi_x) = \omega\,(f_x,\phi) \tag{2.11}$$

for $\phi \in H^1(R^d)^d$. Let

$$|U| = \sqrt{U_1^2 + \cdots + U_n^2} \quad \text{and} \quad |U|_p = \left(\int_{R^d} |U(x)|^p\,dx\right)^{\frac{1}{p}}$$

Define the functions $\Psi,\ \Phi : R^+ \to R^+$ by

$$\Psi(r) = \begin{cases} r^{\frac{p}{2}} & \text{for } \ r \le R^2 \\ R^{p-2} r & \text{for } \ r \ge R^2 \end{cases} \quad \text{and} \quad \Phi(r) = \begin{cases} r^{\frac{p}{2}-1} & \text{for } \ r \le R^2 \\ R^{p-2} & \text{for } \ r \ge R^2. \end{cases}$$

Setting $\phi = \Phi(|U|^2)U \in H^1(R^d)^d$ in (2.11), we obtain

$$\begin{aligned} &\omega\,|\Psi(|U|^2)|_1 - (\frac{2x\cdot U}{c^2+|x|^2}\Phi(|U|^2), \psi\, H(x,U)) \\ &+(\psi\,(H_x(x,U) - \omega\, f_x), \Phi(|U|^2)U) + (\psi\, H_p(x,U), \Phi(|U|^2)(\frac{1}{2}|U|^2)_x) \\ &+\nu\,\{2\,(\Phi'(|U|^2)(\frac{1}{2}|U|^2)_x, (\frac{1}{2}|U|^2)_x) + (\Phi(|U|^2)U_x, U_x))\} = 0. \end{aligned} \tag{2.12}$$

Since from (2.9)

$$|H(x,p)| \le const\,\sqrt{1+|x|^2}\,(1+|p|),$$

there exists constants $k_1,\ k_2$ independent of $c \ge 1$ and $R > 0$ such that

$$(\frac{2x\cdot U}{c^2+|x|^2}\,\Phi(|U|^2), \psi\, H(x,U)))) \le k_1\,(\psi, \Psi(|U|^2)) + k_2\,|\Phi(|U|^2)U|_q$$

where $q = \frac{p}{p-1}$. It thus follows from (2.9) and (2.12) that

$$\omega\,|\Psi(|U|^2)|_1 + \frac{1}{\nu}\,(\Phi(|U|^2)\,U_x, U_x) \le \tilde{\omega}|\Psi(|U|^2)|_1 + (k_2 + |H_x(x,0)|_p + \omega\,|f_x|_p)\,|\Phi(|U|^2)U|_q.$$

for some constant $\tilde{\omega} > 0$ independent of $c \ge 1$ and $R > 0$. Since $|\Psi(|U|^2)|_1 \ge |\Phi(|U|^2)U|_q^q$, it follows that for $\omega \ge \tilde{\omega}$

$$|\Psi(|U|^2)|_1 + \frac{1}{2\nu}\,(\Phi(|U|^2)\,U_x, U_x)$$

is uniformly bounded in $R > 0$. Letting $R \to \infty$, it follows from Fatou's lemma that $U \in L^p(R^d)$ and from (2.12)

$$\omega\,(|U|_p - |f_x|_p) + (\psi\,\frac{2x\cdot U}{c^2+|x|^2}|U|^{p-2}, H(x,U) + (\psi\,H_p(x,U), |U|^{p-2}(\frac{1}{2}|U|^2)_x)$$

$$+(\psi\,H_x(x,U), |U|^{p-2}U) + \nu\,\{(p-2)\,(|U|^{p-4}, (\frac{1}{2}|U|^2)_x, (\frac{1}{2}|U|^2)_x) + (|U|^{p-2}U_x, U_x)\}.$$

Thus we have

$$|U|_p \le |f_x|_p + \frac{1}{\omega}(\sigma_p\,|U|_p + k_2 + |H_x(x,0|_p)$$

where

$$\sigma_p = k_1 + M_1 + \frac{|\psi\,H_p(x,U)|_\infty^2}{(p-2)\nu}.$$

Letting $p \to \infty$, we obtain

$$|U|_\infty \le |f_x|_\infty + \frac{1}{\omega}((k_1+M_1)\,|U|_\infty + k_2 + |H_x(x,0|_\infty). \tag{2.13}$$

For $c \ge 1$ let us denote by V^c, the unique solution of (2.10). Let $\zeta(x) = \chi(x/r) \in C^2(R^d)$, $r \ge 1$ where $\chi = \chi(|x|) \in C^2(R^d)$ is a nonincreasing function such that

$$\chi(s) = 1 \text{ for } 0 \le s \le 1 \quad \text{and} \quad \chi(s) = \exp(-|s|) \text{ for } s \ge 2.$$

and we assume $-\Delta\chi \le k_3\,\chi$. Then

$$\omega\,(V^c, \zeta\,\xi) + (\psi\,H(x,U^c), \zeta\,\xi) - \nu\,(\Delta V^c, \zeta\,\xi) = \omega\,(f, \zeta\,\xi) \tag{2.14}$$

for all $\xi \in L^2(R^d)$, and $U^c = V^c_x$ satisfies

$$\omega\,(U^c, \zeta\,\phi) + (\psi\,H(x,U^c), \nabla\cdot(\zeta\,\phi)) - \nu\,(tr\,U^c_x, \nabla\cdot\zeta\,\phi) = \omega\,(f_x, \zeta\,\phi) \tag{2.15}$$

for all $\phi \in H^1(R^d)^d$. Setting $\phi = U^c$ in (2.15), we obtain

$$\omega\,(\zeta\,U^c, U^c) + (\psi\,H(x,U^c), U^c\cdot\zeta_x + \zeta\nabla\cdot U^c)$$

$$+\nu\,\{(\zeta\,U^c_x, U^c_x) - \frac{1}{2}\,(\Delta\zeta, |U^c|^2)\} = \omega\,(f_x, \zeta\,U).$$

Since $|U^c|_\infty$ is uniformly bounded in $c \geq 1$, it follows that for any compact set Ω in R^d there exists a constant M_Ω independent of $c \geq 1$ such that

$$\omega\,(\zeta\,U^c, U^c) + \nu\,(\zeta\,U^c_x, U^c_x) \leq M_\Omega.$$

Hence for every compact set Ω of R^d Since if $\Omega_r = \{|x| < r\}$, then $H^2(\Omega_r)$ is compactly embedded into $H^1(\Omega_r)$, it follows that there exists a subsequence of $\{V^c\}$ which converges strongly in $H^1(\Omega_r)$. By a standard diagonalization process, we can construct a subsequence $\{V^{\hat{c}}\}$ which converges to a function V strongly in $H^1(\hat{\Omega}))$ and weakly in $H^2(\hat{\Omega})$ for every compact set $\hat{\Omega}$ in R^d. Let $U = V_x$. Then, $U^{\hat{c}}$ converges weakly in $H^1(\hat{\Omega})^d$ and strongly in $L^2(\hat{\Omega})$. Since $L^2(\hat{\Omega})$ convergent sequence has an a.e. pointwise convergent subsequence, without loss of generality we can assume that $U^{\hat{c}}$ converges to U a.e. in R^d. Hence, by Lebesgue dominated convergence theorem

$$H_p(\cdot, U^{\hat{c}}) \to H_p(\cdot, U) \;\text{ and }\; H_x(\cdot, U^{\hat{c}}) \to H_x(\cdot, U)$$

strongly in $L^2(\hat{\Omega})^d$. It follows from (2.14)–(2.15) that the limit V satisfies

$$\omega\,(V, \zeta\,\xi) + (H(x, V_x), \zeta\,\xi) - \nu(\Delta V, \zeta\,\xi) = \omega\,(V, \zeta\,\xi) \tag{2.16}$$

for all $\xi \in L^2_{loc}(R^d)$ and

$$\omega\,(U, \zeta\,\phi) + (H(x, U), \nabla\cdot(\zeta\,\phi)) - \nu\,(tr\,U_x, (\zeta\,\phi)_x) = \omega\,(f_x, \zeta\,\phi) \tag{2.17}$$

for all $\phi \in H^1_{loc}(R^d)$. Setting $\phi = |U|^{p-2}U$ in (2.17), we obtain
(2.18)

$$\omega\,(\zeta, |U|^p) + (\zeta\,H_p(x, U), |U|^{p-2}(\frac{1}{2}|U|^2)_x) + (\zeta\,H_x(x, U), |U|^{p-2}U) - \omega\,(f_x, \zeta\,|U|^{p-2}U)$$

$$+\nu\,\{(p-2)\,(\zeta\,|U|^{p-4}(\frac{1}{2}|U|^2)_x, (\frac{1}{2}|U|^2)_x) + (\zeta\,|U|^{p-2}U_x, U_x)) - \frac{1}{p}(\Delta\zeta, |U|^p)\} = 0$$

for all $\zeta = \chi(x/r)$. Thus,

$$(\zeta, |U|^p)^{\frac{1}{p}} \leq (\zeta, |f_x|^p)^{\frac{1}{p}} + \frac{1}{\omega}\,(c_p\,(\zeta, |U|^p)^{\frac{1}{p}} + (\zeta, |H_x(x, 0)|^p)^{\frac{1}{p}}) \tag{2.19}$$

where $c_p = M_1 + \frac{k_3}{p} + \frac{|\zeta\,H_p(x,U)|^2_\infty}{2\nu\,(p-2)}$ and

$$(H_x(x, 0) - H_x(x, p), p) \leq M_1\,|p|^2 \;\text{ all }\; x \in R^d. \tag{2.20}$$

Now, letting $p \to \infty$ we obtain from (2.19)

$$(\omega - M_1) \sup_{|x|\leq r} |U| \leq \omega\,|f_x|_\infty + |H_x(x, 0)|_\infty.$$

Since $r \geq 1$ is arbitrary, we have

$$(\omega - M_1)\,|U|_\infty \leq \omega\,|f_x|_\infty + |H_x(x, 0)|_\infty \tag{2.21}$$

Setting $\xi = V$ in (2.16), we obtain

$$\omega\,(\zeta, |V|^2) + (\zeta, H_p(x,U)) - \omega\,(f, \zeta\, V) + \nu\,\{\,(\zeta\,|V|^2)_x, -\frac{1}{2}(\Delta\zeta, |U|^2)\} = 0.$$

Also, from (2.18)

$$\omega\,(\zeta, |U|^2) + (\zeta\, H_p(x,U), U\cdot U_x) + (\zeta\, H_x(x,U), U) - \omega\,(f_x, \zeta\, U)$$

$$+\nu\,\{(\zeta\, U_x, U_x)) - \frac{1}{2}(\Delta\zeta, |U|^2)\} = 0$$

Since $|U|_\infty$ is bounded, thus $V \in H^2_{\ell oc}$. In fact

$$\omega\,(\zeta\, V, V) + \nu\,(\zeta\, U_x, U_x) \le a_1\,|U|^2_\infty + a_2\omega\,((\zeta f, f) + (\zeta f_x, f_x)$$

for some constants $a_1,\ a_2$.

Step 4: Next we prove that for $\omega > \max(M_1, \omega_\alpha)$ equation

$$\omega\, V - A_\nu(t)V = \omega\, f$$

has a unique solution satisfying (2.21). We define the sequence $\{V_k\}$ in $H^2_{\ell oc}(R^d)$ by the successive iteration

$$\omega\, V_{k+1} - A_\nu(t)V_{k+1} - (\omega-\omega_0)V_k = \omega_0\, f. \tag{2.22}$$

From *Step 3* (2.22) has a solution V_{k+1} satisfying

$$(\omega - M_1)\,|U_{k+1}|_\infty \le (\omega-\omega_0)\,|U_k|_\infty + \omega_0\,|f_x|_\infty + |H_x(x,0)|_\infty \tag{2.23}$$

Thus,

$$|U_k|_\infty \le (1 - \frac{\omega-\omega_0}{\omega - M_1})^{-1}\frac{(\omega_0\,|f_x|_\infty + |H_x(x,0)|_\infty)}{\omega - M_1} = \frac{\omega_0\,|f_x|_\infty + |H_x(x,0)|_\infty}{\omega_0 - M_1} = \alpha$$

for all $k \ge 1$. $\{V_k\}$ is bounded sequence in $W^{1,\infty}(R^d)$ and thus in $H^2_{\ell oc}(R^d)$. Moreover, we have from (2.4)

$$|V_{k+1} - V_k|_X \le \frac{\omega-\omega_0}{\omega-\omega_\alpha}\,|V_k - V_{k-1}|_X.$$

Thus $\{V_k\}$ is a Cauchy sequence in X and $\{V_k\}$ converges to V in X. Let us define the single-valued operator B on X by $B\phi = -H(x,\phi_x)$. Since $\{BV_k\}$ is bounded in $L^\infty(R^d)$ we may assume that BV_k converges weakly star to w in $L^\infty(R^d)$. If we show that $w = Bu$, then $V \in H^2(R^d)$ solves the desired equation. Since $\{V_k\}$ is bounded in $H^2(\Omega_r)$, $\{V_k\}$ is strongly precompact in $H^1(\Omega_r)$ for each $r > 0$. Hence $BV_k \to BV$ a.e. in Ω_r and thus $w = BV$.

Step 5: Next, we consider the case when $H \in C^1$ is without the Lipschitz bound in p but satisfies (2.2). Consider the cut-off function of $H(x,p)$ by

$$H^M(x,p) = \begin{cases} H_p(x, \frac{p}{|p|})(p - M\,\frac{p}{|p|}) + H(M\frac{p}{|p|}) & \text{if } |p| \ge M \\ H(x,p) & \text{if } |p| \le M. \end{cases}$$

From *Step 3* and (2.20) equation

$$\omega\, V + H^M(x,V_x) - \nu\,\Delta V = \omega\, f$$

has the unique solution V and $U = V_x$ satisfies (2.21) with $M_1 = \beta\, M + a$. Let $M > 0$ be so that $(\omega - (\beta\, M + a))\, M \leq \omega\, |f_x|_\infty + |H_x(x,0)|_\infty$. Then, $|U|_\infty \leq M$. and thus $V \in H^2_{loc}(R^d) \cap W^{1,\infty}(R^d)$ satisfies

$$\omega\, V + H(x,V_x) - \nu\,\Delta V = \omega\, f \tag{2.24}$$

and

$$(\omega - (\beta\, M + a))\, M \leq \omega\, |f_x|_\infty + |H_x(x,0)|_\infty \tag{2.25}$$

for $\omega > M_1$. If $\beta = 0$, then

$$\frac{\varphi(V) - \varphi(f)}{\omega} \leq a\,\varphi(V) + b$$

where $b = |H_x(x,0)|_\infty$.

Step 6: We prove that the solution V^ν to (2.24) converges to a viscosity solution V to

$$\omega\, V + H(x,V_x) = \omega\, f \tag{2.26}$$

as $\nu \to 0^+$. First we show that for all $\phi \in C^2(\Omega)$ if $V^\nu - \phi$ attains a local maximum (minimum, respectively) at $x_0 \in R^n$, then

$$\omega\,(V(x_0) - f(x_0)) + H(x_0,\phi_x(x_0)) - \nu\,(\Delta V)(x_0) \leq 0 \;\; (\geq 0, \text{ respectively}). \tag{2.27}$$

For $\phi \in C^2(\Omega)$ we assume that $V^\nu - \phi$ attains a local maximum at $x_0 \in R^d$. Let $\Omega = \{x \in R^d : |x - x_0| < 1\}$ and Γ denote its boundary. Then without loss of generality we can assume that $V^\nu - \phi$ attains the unique global maximum 1 at x_0 and $V^\nu - \phi \leq 0$ on Γ. In fact we can choose $\zeta \in C^\infty(\Omega)$ such that $V^\nu - (\phi - \zeta)$ attains the unique global maximum 1 at x_0, $V^\nu - (\phi - \zeta) \leq 0$ on Γ and $\zeta_x(x_0) = 0$. Let $\psi = \sup(0, V^\nu - \phi) \in W^{1,\infty}_0(\Omega)$. Multiplying ψ^{p-1} to (2.26) and integrating over Ω, we obtain

$$\omega\, |\psi|_p^p + (\eta \cdot (V^\nu - \phi)_x, \psi^{p-1}) + \nu\,(p-1)\,(\psi_x, \psi^{p-2}\psi_x) = -(\delta\,\psi, \psi^{p-1}),$$

where

$$\eta = \int_0^1 H_p(x, \phi_x + \theta\,(V^\nu - \phi)_x)\, d\theta$$

and

$$\delta = \omega\,(\phi - f) + H(x,\phi_x) - \nu\,\Delta\phi.$$

Since

$$|(\eta \cdot \psi_x, \psi^{p-1})| \leq \frac{1}{4\nu(p-1)} |\eta|^2_{L^\infty(\Omega)}\, |\psi|_p^p + \nu(p-1)\, |\psi^{\frac{p}{2}-1}\psi_x|_2^2$$

it follows that

$$(\omega - \frac{|\eta|^2_\infty}{4\nu(p-1)})\, |\psi|_p^p \leq -(\delta\psi, \psi^{p-1}).$$

Letting $p \to \infty$, we can conclude that $\delta(x_0) \leq 0$ since $\delta \in C(\Omega)$. Setting $\psi = \inf(0, V^\nu - \phi))$ and assuming $V^\nu - \phi$ has the unique global minimum -1 at x_0 and $V^\nu - \phi \geq 0$ on Γ, the same argument as above shows the second assertion.

Next, we show that there exists a subsequence of $\{V^\nu\}$ that converges to a viscosity solution V to

$$\omega\, V + H(x, V_x) = \omega\, f, \tag{2.28}$$

i.e., for all $\phi \in C^1(\Omega)$ if $V - \phi$ attains a local maximum at $x_0 \in R^d$, then

$$\omega\,(V(x_0) - f(x_0)) + H(x_0, \phi_x(x_0)) \leq 0 \tag{2.29a}$$

and if $V - \phi$ attains a local minimum at $x_0 \in R^h$, then

$$\omega\,(V(x_0) - f(x_0)) + H(x_0, \phi_x(x_0)) \geq 0. \tag{2.29b}$$

It follows from *Step 5* that for some $\gamma > 0$ independent of ν

$$|V^\nu|_{W^{1,\infty}(\Omega)} \leq \gamma$$

Thus there exists a subsequence of $\{V^\nu\}$ (denoted by the same) that converges weakly star to V in $W^{1,\infty}(R^d)$, and thus the convergence is uniform in Ω. We prove (2.29a) first for $\phi \in C^2(\Omega)$. Assume that for $\phi \in C^2(\Omega)$ $V^\nu - \phi$ has a local maximum at $x_0 \in \Omega$. We can choose $\zeta \in C^\infty(\Omega)$ such that $\zeta_x(x_0) = 0$ and $V^\nu - (\phi - \zeta)$ has a strict local maximum at x_0. For $\nu > 0$ sufficiently small, $V^\nu - (\phi - \zeta)$ has a local maximum at some $x_\nu \in \Omega$ and $x_\nu \to x_0$ as $\nu \to 0^+$. From (2.27)

$$\omega\,(V^\nu(x_\nu) - f(x_\nu)) + H(x_\nu, \phi_x(x_\nu)) - \nu\,(\Delta\phi)(x_\nu) \leq 0$$

We conclude (2.29a), since $V^\nu(x_\nu) \to V(x_0)$, $\phi_x(x_\nu) - \zeta_x(x_\nu) \to \phi_x(x_0) - \zeta_x(x_0) = \phi_x(x_0)$ and $\nu\,\Delta\phi(x_\nu) \to 0$ as $\nu \to 0^+$. For $\phi \in C^1(\Omega)$ exactly the same argument is applied to the convergent sequence $\phi_n \in C^2(\Omega)$ to ϕ in $C^1(\Omega)$ to prove (2.29a).

Step 7: We show that if $V\ , W \in D_\alpha$ are viscosity solutions to $\omega\,(V - f) + H(x, V_x) = 0$ and $\omega\,(W - g) + H(x, W_x) = 0$, respectively, then

$$(\omega - \omega_\alpha)\,|u - v|_X \leq \omega\,|f - g|_X. \tag{3.30}$$

For $\delta > 0$ let

$$\psi(x) = \frac{1}{\sqrt{1 + |x|^{2+\delta}}}$$

If $u,\ v \in D_\alpha$ then

$$\lim_{|x|\to\infty} \psi(x)V(x) = \lim_{|x|\to\infty} \psi(x)W(x) = 0. \tag{2.31}$$

We choose a function $\beta \in C^\infty(R^d)$ satisfying

$$0 \leq \beta \leq 1,\ \ \beta(0) = 1,\ \ \beta(x) = 0 \ \text{ if } \ |x| > 1.$$

Let $M = \max(|u|_{X_\delta}, |W|_{X_\delta})$. Define the function $\Phi : R^n \times R^n \to R$ by

$$\Phi(x, y) = \psi(x)V(x) - \psi(y)W(y) + 3M\beta_\epsilon(x - y) \tag{2.32}$$

where

$$\beta_\epsilon(x) = \beta(\frac{x}{\epsilon}) \quad \text{for } \ x \in R^d.$$

Off the support of $\beta_\epsilon(x-y)$, $\Phi \leq 2M$, while if $|x|+|y| \to \infty$ on this support, then $|x|,\ |y| \to \infty$ and thus from (2.30) $\lim_{|x|+|y|\to\infty} \Phi \leq 3M$. We may assume that $V(\bar{x}) - W(\bar{x}) > 0$ for some $\bar{x}$. Then,

$$\Phi(\bar{x}, \bar{x}) = \psi(\bar{x})(V(\bar{x}) - W(\bar{x})) + 3M\,\beta_\epsilon(0) > 3M.$$

Hence Φ attains its maximum value at some point $(x_0, y_0) \in R^d \times R^d$. Moreover, $|x_0 - y_0| \leq \epsilon$ since $\beta_\epsilon(x_0 - y_0) > 0$. Now x_0 is a maximum point of

$$\psi(x)\left(V(x) - \frac{\psi(y_0)W(y_0) - 3M\beta_\epsilon(x - y_0) + \Phi(x_0, y_0)}{\psi(x)}\right)$$

and since $\psi > 0$ the function

$$x \to V(x) - \frac{\psi(y_0)W(y_0) - 3M\beta_\epsilon(x - y_0) + \Phi(x_0, y_0)}{\psi(x)}$$

attains a maximum 0 at x_0. Since

$$\psi(y_0)W(y_0) - 3M\beta_\epsilon(x_0 - y_0) + \Phi(x_0, y_0) = \psi(x_0)V(x_0)$$

and V is a viscosity solution

$$\psi(x_0)(\omega\,(V(x_0) - f(x_0)) + H(x_0, p)) \leq 0, \tag{2.33}$$

where

$$p = \frac{2+\delta}{2}\psi(x_0)V(x_0)\psi(x_0)|x_0|^\delta\, x_0 - \frac{3M\beta_\epsilon'(x_0 - y_0)}{\psi(x_0)}.$$

and we used the fact that $(|x|^{2+\delta})' = (2+\delta)|x|^\delta x$. Moreover since $V \in D_\alpha$

$$|p| \leq \alpha \tag{2.34}$$

Similarly, the function

$$y \to W(y) - \frac{\psi(x_0)V(x_0) + 3M\beta_\epsilon(x_0 - y) - \Phi(x_0, y_0)}{\psi(y)}$$

attains a minimum 0 at y_0 and since W is a viscosity solution

$$\psi(y_0)(\omega\,(W(y_0) - g(y_0)) + H(y_0, q)) \geq 0, \tag{2.35}$$

where

$$q = \frac{2+\delta}{2}\psi(y_0)W(y_0)\psi(y_0)|y_0|^\delta\, y_0 - \frac{3M\beta_\epsilon'(x_0 - y_0)}{\psi(y_0)}$$

and $|q| \le \alpha$. Thus by (2.33) and (2.35) we have

$$\begin{aligned} &\omega\,(\psi(x_0)V(x_0) - \psi(y_0)W(y_0)) \\ &\qquad \le \psi(y_0)\,H(y_0,q) - \psi(x_0)\,H(x_0,p) + \omega(\psi(x_0)f(x_0) - \psi(y_0)g(y_0)) \end{aligned} \tag{2.36}$$

Since $\Phi(x_0,y_0) \ge \Phi(\bar{x},\bar{x})$ we have

$$\psi(x_0)V(x_0) - \psi(y_0)W(y_0) \ge \psi(\bar{x})(V(\bar{x}) - W(\bar{x})) + 3M\,(1 - \beta_\epsilon(x_0 - y_0))$$

and thus

$$(\psi(x_0) - \psi(y_0))\,V(x_0) + \psi(y_0)(V(x_0) - W(y_0)) \ge \psi(\bar{x})(V(\bar{x}) - W(\bar{x})) + 3M\,(1 - \beta_\epsilon(x_0 - y_0)).$$

Since

(2.37)

$$|(\psi(x_0) - \psi(y_0))\,V(x_0)| = \psi(y_0)\psi(x_0)|V(x_0)|(\sqrt{1+|y_0|^{2+\delta}} - \sqrt{1+|x_0|^{2+\delta}}) \le const\,|x_0 - y_0|,$$

it follows that $V(x_0) \ge W(y_0)$ for sufficiently small $\epsilon > 0$. Note that

$$\psi(y_0)H(y_0,q) - \psi(x_0)H(x_0,p) = (\psi(y_0) - \psi(x_0))H(x_0,p)$$

$$+\psi(y_0)(H(y_0,p) - H(x_0,p)) + \psi(y_0)(H(y_0,q) - H(y_0,p)).$$

From (2.36)–(2.37) we have that

$$\begin{aligned} &\omega\,(\psi(x_0)V(x_0) - \psi(y_0)W(y_0) - (\psi(x_0)f(x_0) - \psi(y_0)g(y_0))) \\ &\qquad \le O(\epsilon) + \psi(y_0)\,(c_1(y_0), p - q) + c_2\,|p - q|). \end{aligned} \tag{2.38}$$

where $O(\epsilon) \to 0$ as $\epsilon \to 0$. Now we evaluate $p - q$, i.e.,

$$p - q = \frac{2+\delta}{2}(\psi(x_0)V(x_0) - \psi(y_0)W(y_0))\psi(y_0)|y_0|^\delta\, y_0$$

$$+\frac{2+\delta}{2}\psi(x_0)V(x_0)(\psi(x_0)\,|x_0|^\delta x_0 - \psi(y_0)|y_0|^\delta y_0)$$

$$+3M\beta'_\epsilon(x_0 - y_0)\,(\sqrt{1+|x_0|^{2+\delta}} - \sqrt{1+|y_0|^{2+\delta}}).$$

Since

$$|\psi(x_0)V(x_0)\psi(x_0)|x_0|^\delta\, x_0| \le |u|_X \frac{|x_0|^\delta\sqrt{1+|x_0|^2}}{1+|x_0|^{2+\delta}}\,|x_0| \le M_3$$

for some $M_3 > 0$, it follows from (2.34) that

$$3|\beta'_\epsilon(x_0 - y_0)\sqrt{1+|x_0|^{2+\delta}}| \le M_4$$

for some $M_4 > 0$. Thus,

$$|3M\beta'_\epsilon(x_0 - y_0)\,|\sqrt{1+|x_0|^{2+\delta}} - \sqrt{1+|y_0|^{2+\delta}}| \le const\,(2+\delta)MM_4|x_0 - y_0|.$$

and therefore

$$\frac{2+\delta}{2}\psi(x_0)V(x_0)(\psi(x_0)|x_0|^\delta x_0 - \psi(y_0)|y_0|^\delta y_0) + 3M\beta_\epsilon(x_0-y_0)\left(|x_0|^{2+\delta} - |y_0|^{2+\delta}\right) = O(\epsilon).$$

In the right-hand side of (2.38) we have

$$\psi(y_0)((c_1(y_0), p-q) + c_2\,|p-q|)$$

$$\leq O(\epsilon) + \frac{2+\delta}{2}\frac{\beta\,|y_0|^{2+\delta} + c_2\alpha\,|y_0|^{1+\delta}}{1+|y_0|^{2+\delta}}\,(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)).$$

Hence from (2.38) we conclude

$$\omega_{y_0}\,(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)) \leq \psi(x_0)f(x_0) - \psi(y_0)g(y_0) + O(\epsilon) \tag{2.39}$$

where

$$\omega_\delta = \sup_{y_0}\,(\omega - \frac{2+\delta}{2}\,\frac{\beta\,|y_0|^{2+\delta} + c_2\alpha\,|y_0|^{1+\delta}}{1+|y_0|^{2+\delta}}).$$

Assume that $\omega > \lambda_\alpha$. For $x \in R^d$ we have

$$\psi(x)(u(x) - v(x)) + 3M = \Phi(x,x) \leq \Phi(x_0,y_0) \leq \psi(x_0)u(x_0) - \psi(y_0)v(y_0) + 3M$$

and so by (2.39)

$$\begin{aligned}\omega_\delta \sup_{R^d}\,\psi(x)(u(x)-v(x))^+ &\leq \omega\,(\psi(x_0)u(x_0) - \psi(y_0)v(y_0)) \leq \psi(x_0)f(x_0) - \psi(y_0)g(y_0) + O(\epsilon)\\ &\leq \sup_{R^d}\,\psi(f-g)^+ + |\psi(x_0)g(x_0) - \psi(y_0)g(y_0)| + O(\epsilon)\\ &\leq \sup_{R^d}\,\psi(f-g)^+ + \omega_{\psi g}(\epsilon) + O(\epsilon)\end{aligned}$$

where $\omega_{\psi g}(\cdot)$ is the modulus of continuity of ψg. Letting $\epsilon \to 0$, we obtain

$$\omega_\delta \sup_{R^d}\,\psi(x)(u(x)-v(x))^+ \leq \sup_{R^n}\,\psi(f-g)^+ \tag{3.40}$$

Since $|\psi|_{X_\delta} \to |\psi|_X$ as $\delta \to 0^+$ for $\psi \in X$ we obtain (2.30) by taking limit $\delta \to 0^+$ in (2.40).
Example (Plastic equations) Consider the visco-plastic equation of the form

$$v_t + div(\sigma) = 0$$

where the stress $\sigma(\epsilon) = \sigma^t$ minimizes

$$h(\sigma) - \epsilon : \sigma$$

and the strain ϵ is given by

$$\epsilon_{i,j} = \frac{1}{2}(\frac{\partial}{\partial x_i}v_j + \frac{\partial}{\partial x_i}v_j).$$

That is,

$$\sigma \in \partial h^*(\epsilon)$$

where h^* is the conjugate of h

$$h^*(\epsilon) = \sup_{\sigma \in \mathcal{C}} \{\epsilon : \sigma - h(\sigma)\}$$

For the case of the linear elastic system

$$\sigma_{11}$$

3 Evolution equations

In this section we consider the evolution equation of the form

$$\frac{d}{dt} x(t) \in A(t)u(t) \tag{3.1}$$

in a Banach space $X \cap D$, where D is a closed set. We assume the dissipativity: there exist a constant $\omega = \omega_D$ and continuous functions $f : [0, T] \to X$ and $L : R^+ \to R^+$ independent of $t,\ s \in [0, T]$ such that

$$(1-\lambda\omega)\,|x_1-x_2| \leq |x_1-x_2-\lambda\,(y_1-y_2)|+\lambda\,|f(t)-f(s)|L(x_2)|K(|y_2|), \quad K(r) = 1+c\,r \tag{3.2}$$

for all $x_1 \in D \cap dom\,(A(t))$ $x_2 \in D \cap dom\,(A(s))$and $y_1 \in A(t)x_1,\ y_2 \in A(s)x_2$, and

$$A(t),\ t \in [0, T] \text{ is m-dissipative and } J_\lambda(t) = (\lambda\, I - A(t))^{-1} : D \to D \cap dom(A(t)). \tag{3.3}$$

Thus, one constructs the mild solution as

$$u(t) = \lim_{\lambda \to 0^+} \Pi_{k=1}^{[t/\lambda]} J_\lambda(t_k)u_0, \quad t_k = k\lambda.$$

That is,

$$\frac{u_i - u_{i-1}}{\lambda} \in A(t_i)u_i, \quad t_i = i\,\lambda. \tag{3.4}$$

Remark Assume there exists a Liapunov functional φ such that

$$\frac{\varphi(u_i) - \varphi(u_{i-1})}{\lambda} \leq a\,\varphi(u_i) + b. \tag{3.5}$$

for (3.4). Define

$$D_\alpha = \{u \in D : \varphi(u) \leq \alpha\}.$$

That is,

$$J(t_i) : D_{i-1} \to D_i \cap dom(A(t_i))$$

where $D_i = \{u \in D : \varphi(u) \leq \alpha_i\}$ and $\alpha_i = (1-a\lambda)^{-1}(\alpha_{i-1}+b)$. Assume the local dissipativity condition: (3.2) holds with $\omega = \omega_{D_\alpha}$. It is said that $(A(t), X)$ is locally quasi-dissipative operator in the sense of Kobayashi and Oharu.

Theorem 1.1 (Crandall-Pazy Theorem) Let $(A(t), X)$ satisfy (3.2)–(3.3). Then,

$$U(t,s) = \lim_{\lambda \to 0^+} \Pi_{i=1}^{[\frac{t-s}{\lambda}]} J_\lambda(s+i\lambda)x$$

exists for $x \in \overline{dom\,(A(0))}$ and $0 \le s \le t \le T$. The $U(t,s)$ for $0 \le s \le t \le T$ defines an evolution operator on $\overline{dom\,A(0)}$ and moreover satisfies

$$|U(t,s)x - U(t,s)y| \le e^{\omega\,(t-s)}\,|x-y|$$

for $0 \le s \le t \le T$ and $x,\ y \in \overline{dom\,(A(0))}$.

Proof: Let $h_i = \lambda$ and $t_i = s + i\lambda$ in (3.2). Then $x_i = J_\lambda(t_i)x_{i-1}$, where we dropped the superscript λ and $x_m = (\Pi_{i=1}^m J_\lambda(t_i))x$. Thus $|x_i| \le (1-\omega\lambda)^{-m} \le e^{2\omega(T-s)}\,|x| = M_1$ for $0 < m\lambda \le T - s$. Let $\hat{x}_j$ is the approximation solution corresponding to the stepsize $\hat{h} - j = \mu$ and $\hat{t}_j = s + j\mu$. Define $a_{m,n} = |x_m - \hat{x}_n|$. We first evaluate $a_{0,n},\ a_{m,0}$.

$$a_{m,0} = |x_m - x| \le \sum_{k=1}^m\ |(\Pi_{i=k}^m J_\lambda(t_i))x - (\Pi_{i=k+1}^m J_\lambda(t_i))x|$$

$$\le \sum_{k=1}^m (1-\omega\lambda)^{-(m-k+1)}\lambda\,|||A(t_i)x||| \le e^{2\omega(T-s)} m\lambda\, M(x).$$

where $M(x) = \sup_{t\in[0,T]}\ |||A(t)x|||$. Similarly, we have

$$a_{0,n} \le e^{2\omega(T-s)} n\mu\, M(x).$$

Next, we establish the recursive formula for $a_{i,j}$. For $\lambda \ge \mu > 0$

$$a_{i,j} = |x_i - \hat{x}_j| \le |J_\lambda(t_i)x_{i-1} - J_\mu(\hat{t}_j)\hat{x}_{j-1}|$$

$$\le |J_\lambda(t_i)x_{i-1} - J_\mu(t_i)\hat{x}_{j-1}| + |J_\mu(t_i)\hat{x}_{j-1} - J_\mu(\hat{t}_j)\hat{x}_{j-1}|$$

From Theorem 1.4

$$|J_\lambda(t_i)x_{i-1} - J_\mu(t_i)\hat{x}_{j-1}|$$

$$= |J_\mu(t_i)(\frac{\mu}{\lambda}x_{i-1} + \frac{\lambda-\mu}{\lambda}J_\lambda(t_i)x_{i-1}) - J_\mu(t_i)\hat{x}_{j-1}|$$

$$\le (1-\omega\,\mu)^{-1}(\frac{\mu}{\lambda}|x_{i-1} - \hat{x}_{j-1}| + \frac{\lambda-\mu}{\lambda}\,|x_i - \hat{x}_{j-1}|).$$

Hence for $i,\ j \ge 1$ we have

$$a_{i,j} \le (1-\omega\mu)^{-1}(\alpha\, a_{i-1,j-1} + \beta\, a_{i,j-1}) + b_{i,j}$$

where

$$\alpha = \frac{\mu}{\lambda}, \quad \text{and} \quad \beta = \frac{\lambda-\mu}{\lambda}$$

and

$$b_{i,j} = |J_\mu(t_i)\hat{x}_{j-1} - J_\mu(\hat{t}_j)\hat{x}_{j-1}| \le \mu\,|f(t_i) - f(\hat{t}_j)|L(|\hat{x}_j|)K(|A_\mu(\hat{t}_j)\hat{x}_{j-1}|).$$

We show that if $y_i = A_\lambda(t_i)x_{i-1}$, then there exists a constant $M_2 = M_2(x_0, T, ||A(s)x_0||)$ such that $|y_i| \le M_2$ Since $x_i = J_\lambda(t_i)x_{i-1}$ and $A_\lambda(t_i)x_{i-1} \in A(t_i)x_i$, from (H.1) we have

$$|||A(t_i)x_i||| = |A_\lambda(t_i)x_{i-1}| \le |A_\lambda(t_{i-1})x_{i-1}| + |f(t_i) - f(t_{i-1})|L(M_1)(1 + |A_h(t_{i-1})x_{i-1}|))$$

$$\le (1-\lambda)^{-1}(|||A(t_{i-1})x_{i-1}||| + |f(t_i) - f(t_{i-1})|L(M_1)(1 + |||A(t_{i-1})x_{i-1}|||).$$

If we define $a_i = |||A(t_i)x_i|||$, then

$$(1 - \omega\,\lambda)\,a_i \le a_{i-1} + b_i(1 + a_{i-1}).$$

where $b_i = L(M_1)|f(t_i) - f(t_{i-1})|$. Thus, it follows from the proof of Lemma 2.4 that $|||A(t_i)x_i||| \le M_2$, for some constant $M_2 = M(x_0, T)$. Since

$$|y_i| \le (1-\omega\lambda)^{-1}(|||A(t_{i-1})x_{i-1}||| + |f(t_i) - f(t_{i-1})|L(M_1)(1 + |||A(t_{i-1})x_{i-1}|||),$$

thus $|y_i|$ is uniformly bounded.

It follows from [Crandall-Pazy, Ito-Kappel] that

$$a_{m,n} \le e^{2\omega(T-s)}M(x)\,[((n\mu - m\lambda)^2 + n\mu(\lambda-\mu))^{1/2} + ((n\mu - m\lambda)^2 + m\lambda(\lambda-\mu))^{1/2}]$$

$$+(1-\omega\mu)^{-n}\sum_{j=0}^{n-1}\sum_{i=0}^{\min(m-1,j)} \beta^{j-1}\alpha^i \begin{pmatrix} j \\ i \end{pmatrix} b_{m-i,n-j},$$

where $m\lambda,\ n\mu \le T - s$. Let ρ be the modulus of continuity of f on $[0, T]$, i.e.,

$$\rho(r) = \sup\,\{|f(t) - f(\tau) : 0 \le t,\ \tau \le T \text{ and } |t - \tau| \le r\}$$

Then ρ is is nondecreasing and subadditive; i.e., $\rho(r_1 + r_2) \le \rho(r_1) + \rho(r_2)$ for $r_1,\ r_2 \ge 0$. Thus,

$$J = \sum_{j=0}^{n-1}\sum_{i=0}^{\min(m-1,j)} \beta^{j-1}\alpha^i \begin{pmatrix} j \\ i \end{pmatrix} b_{m-i,n-j}$$

$$\le C\mu\left(n\,\rho(|n\mu - m\lambda)|) + \sum_{j=0}^{n-1}\sum_{i=0}(m-1)^j\,\beta^{j-1}\alpha^i \begin{pmatrix} j \\ i \end{pmatrix} \rho(|j\mu - i\lambda)|)\right).$$

where we used () with $C = L(M_1)K(M_2)$, the subadditivity of of ρ and the estimate

$$\sum_{i=0}^{\min(m-1,j)} \beta^{j-1}\alpha^i \begin{pmatrix} j \\ i \end{pmatrix} \le 1.$$

Next, let $\delta > 0$, be given and write

$$\sum_{j=0}^{n-1} \sum_{i=0}^{\min(m-1,j)} \beta^{j-1}\alpha^i \begin{pmatrix} j \\ i \end{pmatrix} \rho(|j\mu - i\lambda|) = I_1 + I_2$$

where I_1 is the sum over indecies such that $|j\mu - i\lambda| < \delta$, while I_2 is the sum over indecies satisfying $|j\mu - i\lambda| \geq \delta$. Clearly $I_1 \leq n\rho(\delta)$, but

$$I_2 \leq \rho(T) \sum_{j=0}^{n-1} \sum_{i=0}^{\min(m-1,j)} \beta^{j-1}\alpha^i \begin{pmatrix} j \\ i \end{pmatrix} \frac{|j\mu - i\lambda|^2}{\delta^2} = \frac{\rho(T)}{\delta^2} n(n-1)(\lambda\mu - \mu^2) \leq \frac{\rho(T)n^2}{\delta^2}\,\mu(\lambda - \mu)$$

Therefore,

$$J \leq Cn\mu \left(\rho(|n\mu - m\lambda| + \rho(\delta) + \frac{\rho(T)}{\delta^2} n\mu(\lambda - \mu) \right).$$

Combining these inealities, we obtain

$$a_{m,n} \leq e^{2\omega(T-s)} M(x) \left[((n\mu - m\lambda)^2 + n\mu(\lambda - \mu))^{1/2} + ((n\mu - m\lambda)^2 + m\lambda(\lambda - \mu))^{1/2}\right]$$

$$+ e^{2\omega(T-s)} C\, n\mu \left(\rho(|n\mu - m\lambda| + \rho(\delta) + \frac{\rho(T)}{\delta^2} n\mu(\lambda - \mu) \right).$$

Now, we can choose, e.g, $\delta^2 = \sqrt{\lambda - \mu}$ and it follows that $a_{n,m}$ as function of $m,\ n$ and $\lambda,\ \mu$, tends to zero as $|n\mu - m\lambda| \to 0$ and $n,\ m \to \infty$, subject to $0 < n\mu,\ m\lambda \leq T - s$, and the convergence is uniform in s.

Theorem 1.1 Let $(A(t), X)$ satisfy (3.2)–(3.3). Then,

$$U(t,s) = \lim_{\lambda \to 0^+} \Pi_{i=1}^{[\frac{t-s}{\lambda}]}\, J_\lambda(s + i\lambda)x$$

exists for $x \in \overline{dom\,(A(0))}$ and $0 \leq s \leq t \leq T$. The $U(t,s)$ for $0 \leq s \leq t \leq T$ defines an evolution operator on $\overline{A(0)}$ and moreover satisfies

$$|U(t,s)x - U(t,s)y| \leq e^{\omega\,(t-s)}\, |x - y|$$

for $0 \leq s \leq t \leq T$ and $x,\ y \in \overline{dom\,(A(0))}$.

Moreover, we have the following lemma.

Lemma 1.2 If $x \in \hat{D}$, then there exists a constant L such that

$$|U(s + r, s)x - U(\hat{s} + r, \hat{s})x| \leq L\,\rho(|s - \hat{s}|)$$

for $r \geq 0$ and $s,\ \hat{s},\ s + r,\ \hat{s} + r \leq T$.

Proof: Let $a_k = |x_k - \hat{x}_k|$ where

$$x_k = \Pi_{i=1}^k (I - \lambda\, A(s + i\,\lambda))^{-1} x \quad \text{and} \quad \hat{x}_k = \Pi_{i=1}^k (I - \lambda\, A(\hat{s} + i\,\lambda))^{-1} x.$$

Then, for $\lambda = \frac{r}{n}$

$$
\begin{aligned}
a_k &= |J_\lambda(s + k\,\lambda)x_{k-1} - J_\lambda(\hat{s} + k\,\lambda)\hat{x}_{k-1}| \\
&\le |J_\lambda(s + k\,\lambda)x_{k-1} - J_\lambda(\hat{s} + k\,\lambda)x_{k-1}| + |J_\lambda(\hat{s} + k\,\lambda)x_{k-1} - J_\lambda(\hat{s} + k\,\lambda)\hat{x}_{k-1}| \\
&\le \lambda C \rho\,(|s - \hat{s}|) + (1 - \lambda\omega)^{-1}\, a_{k-1}
\end{aligned}
$$

for some C. Thus, we obtain

$$
|a_k| \le \frac{e^{2\omega r} - 1}{2\omega}\, \rho(|s - \hat{s}|),
$$

and letting $\lambda \to 0$ we obtain the desired result. □

3.1 DS-approximation and integral solution

In order to discuss the convergence of the sequence $\{x_i\}$ to the solution of (2.1) we introduce the following notions.

Definition 2.1 Given $s \in [0, T]$ and $x_0 \in D \cap \overline{dom\,(A(s))}$, $u_\lambda(t)$ is said to be a DS–approximation of (2.1) if

$$
u_\lambda(t) = x_i^\lambda, \quad t \in (t_{i-1}^\lambda, t_i^\lambda]
$$

where for some $\alpha > 0$ the sequence of $\{t_i^\lambda, x_i^\lambda, y_i^\lambda, \epsilon_i^\lambda\}$ in $R \times D_\alpha \times X \times X$ satisfies

$$
\begin{aligned}
& s = t_0^\lambda < t_1^\lambda < \cdots < t_i^\lambda < \cdots < t_{N_\lambda}^\lambda = T \\
& x_i^\lambda \in dom\,(A(t_i^\lambda)) \\
& y_i^\lambda = \frac{x_i^\lambda - x_{i-1}^\lambda}{t_i^\lambda - t_{i-1}^\lambda} - \epsilon_i^\lambda \in A(t_i^\lambda)x_i^\lambda, \;\; 1 \le i \le N_\lambda \qquad (2.10) \\
& d_\lambda = \max\,(t_i^\lambda - t_{i-1}^\lambda) \to 0, \quad \textstyle\sum_{i=1}^{N_\lambda} (t_i^\lambda - t_{i-1}^\lambda)\,|\epsilon_i^\lambda| \to 0 \;\text{ as } \lambda \to 0.
\end{aligned}
$$

Definition 2.2 A continuous function $u(t) : [s, T] \to X$ in is said to be a mild solution of (2.1) on $[s, T]$ if there exists a DS–approximation $u_\lambda(\cdot)$ such that $\lim_{n\to\infty} |u_\lambda(t) - u(t)| = 0$ uniformly on $[s, T]$. If for $\alpha > 0$ $u_\lambda(t) \in D_\alpha,\; t \in [s, T]$ then we say that the mild solution is confined to D_α.

We next introduce the notion of integral solution that plays an important role in characterizing the mild solution and establishing the uniqueness of the mild solution.

Definition 2.3 A continuous function $u : [s, T] \to X$ is said to be an integral solution on $[s, T]$ of (2.1) if there exists a constant $\beta > 0$ such that for $\omega = \omega_\beta$ the following integral inequality is satisfied.

$$
|u(t) - x| - |u(\tau) - x| \le \int_\tau^t \langle y, u(\sigma) - x\rangle_+ + \omega\,|u(\sigma) - x| + C\,|f(\sigma) - f(r)|\,d\sigma \qquad (2.11)
$$

for $s \le \tau \le t \le T,\ r \in [s,T]$ and $[x,y] \in A(r)$ with $x \in D_\beta$, where $C = L(|x|)K(|y|)$.

Definition 2.4 An operator $U(t,s),\ 0 \le s \le t \le T$ of nonlinear operators from D into itself is called a nonlinear evolution operator on D if

$$U(t,s)x = U(t,r)U(r,s)x \quad \text{and} \quad U(t,t)x = x \quad \text{for } x \in D,\ 0 \le s \le r \le t \le T$$

$$t \to U(t,s)x \in X \text{ is continuous for each } s \ge 0 \text{ and } x \in D.$$

Let $\{t_j^\mu, x_j^\mu, y_j^\mu, \epsilon_j^\mu\}$ be a DS–approximation sequence in $R \times D_\alpha \times X \times X$ starting from $x(\hat{s}) = \hat{x}_0$ that satisfies

$$\hat{s} = t_0^\mu < t_1^\mu < \cdots < t_j^\mu < \cdots < t_{N_\mu}^\mu = T$$

$$x_j^\mu \in dom\,(A(t_j^\mu))$$

$$y_j^\mu = \frac{x_j^\mu - x_{j-1}^\mu}{t_j^\mu - t_{j-1}^\mu} - \epsilon_j^\mu \in A(t_j^\mu)x_j^\mu,\ \ 1 \le j \le N_\mu \tag{2.12}$$

$$d_\mu = \max\,(t_j^\mu - t_{j-1}^\mu) \to 0, \quad \textstyle\sum_{j=1}^{N_\mu} (t_j^\mu - t_{j-1}^\mu)\,|\epsilon_j^\mu| \to 0 \ \text{ as } \mu \to 0.$$

Define

$$h_i^\lambda = t_i^\lambda - t_{i-1}^\lambda,\ \ 1 \le i \le N_\lambda \quad \text{and} \quad h_j^\mu = t_j^\mu - t_{j-1}^\mu,\ \ 1 \le j \le N_\mu$$

Then we have

$$(2.13) \qquad \begin{aligned} &x_i^\lambda - h_i^\lambda y_i^\lambda - x_{i-1}^\lambda = h_i^\lambda \epsilon_i^\lambda,\ \ y_i^\lambda \in A(t_i^\lambda)x_i^\lambda, \quad 1 \le i \le N_\lambda \\ &x_j^\mu - h_j^\mu y_j^\mu - x_{j-1}^\mu = h_j^\mu \epsilon_j^\mu,\ \ y_j^\mu \in A(t_j^\mu)x_j^\mu, \quad 1 \le j \le N_\mu. \end{aligned}$$

We now discuss the estimate of $a_{i,j} = |x_i^\lambda - x_j^\mu|$ due to Kobayasi, Kobayashi, Oharu [KKO]. We define

$$(2.14) \qquad \begin{aligned} &\alpha_{i,j} = \frac{h_j^\mu}{h_i^\lambda + h_j^\mu},\ \ \beta_{i,j} = \frac{h_i^\lambda}{h_i^\lambda + h_j^\mu},\ \ \gamma_{i,j} = \frac{h_i^\lambda h_j^\mu}{h_i^\lambda + h_j^\mu}, \\ &c_{i,j}(\sigma) = \left((t_i^\lambda - t_j^\mu - \sigma)^2 + d_\lambda(t_i^\lambda - s) + d_\mu(t_j^\mu - \hat{s})\right)^{\frac{1}{2}}, \quad d_{i,j} = |f(t_i^\lambda) - f(t_j^\mu)|. \end{aligned}$$

We start with the following technical lemma which is essential for establishing the main estimate.

Lemma 2.2 For $1 \le i \le N_\lambda,\ 1 \le j \le N_\mu$ and $\sigma \in$ we have

$$\alpha_{i,j}\,c_{i-1,j}(\sigma) + \beta_{i,j}\,c_{i-1,j}(\sigma) \le c_{i,j}(\sigma)$$

Proof: Since $\alpha_{i,j} + \beta_{i,j} = 1$ by Cauchy Schwarz inequality we have

$$I = \alpha_{i,j}\,c_{i-1,j}(\sigma) + \beta_{i,j}\,c_{i-1,j}(\sigma) \le (\alpha_{i,j}\,c_{i-1,j}^2(\sigma) + \beta_{i,j}\,c_{i-1,j}(\sigma))^{\frac{1}{2}}.$$

Since $t_{i-1}^\lambda = t_i^\lambda - h_i^\lambda$ and $t_{j-1}^\mu = t_j^\mu - h_j^\mu$

$$(t_{i-1}^\lambda - t_j^\mu - \sigma)^2 = (t_i^\lambda - t_j^\mu - \sigma)^2 - 2h_i^\lambda(t_i^\lambda - t_j^\mu - \sigma) + (h_i^\lambda)^2$$

$$(t_i^\lambda - t_{j-1}^\mu - \sigma)^2 = (t_i^\lambda - t_j^\mu - \sigma)^2 + 2h_j^\mu(t_i^\lambda - t_j^\mu - \sigma) + (h_j^\mu)^2.$$

Thus,

$$I^2 \leq \frac{1}{h_i^\lambda + h_j^\mu}[h_j^\mu((t_{i-1}^\lambda - t_j^\mu - \sigma)^2 + d_\lambda(t_{i-1}^\lambda - s) + d_\mu(t_j^\mu - \hat{s}))$$

$$+h_i^\lambda((t_i^\lambda - t_{j-1}^\mu - \sigma)^2 + d_\lambda(t_i^\lambda - s) + d_\mu(t_{j-1}^\mu - \hat{s}))]$$

$$= (t_i^\lambda - t_j^\mu - \sigma)^2 + d_\lambda(t_i^\lambda - s) + d_\mu(t_j^\mu - \hat{s}) + \gamma_{i,j}\,(h_i^\lambda - d_\lambda + h_j^\mu - d_\mu) \leq c_{i,j}^2(\sigma)$$

where we used the fact that $h_i^\lambda \leq d_\lambda$ and $h_j^\mu \leq d_\mu$. □

Next, we prove the uniform bound of $|x_i^\lambda|,\ 1 \leq i \leq N_\lambda$.

Lemma 2.3 Let $x_i^\lambda,\ 1 \leq i \leq N_\lambda$ be the solution to (2.10) and either $(C.1)$ or $(C.2)$ holds. Then there exists $M = M(T, x_0, \alpha)$ such that $|x_i^\lambda| \leq M$ for $1 \leq i \leq N_\lambda$.

Proof: We drop the index λ for simplicity of our expositions in the proof. For $\omega = \omega_\alpha$ from (2.4) we have

$$(1 - \omega h_i)\,|x_i - u| \leq |x_i - h_i y_i - u| + h_i|v| + h_i|f(t_i) - f(r)|\,L(|u|)K(|v|)$$

for $[u, v] \in A(r)$. Since $x_i - h_i y_i = x_{i-1} + h_i\epsilon_i$ we obtain from (2.6.b)

$$(1 - \omega h_i)\,|x_i - u| \leq |x_{i-1} - u| + h_i(|v| + |\epsilon_i| + |f(t_i) - f(r)|\,L(|u|)K(|v|)$$

for $r \in [0, T]$ and $[u, v] \in A(r)$. Multiplying this by $\Pi_{k=1}^{i-1}(1 - \omega h_i)$ and then summing up this in i we have

(2.15)

$$|x_i - u| \leq \Pi_{k=1}^{i}(1 - \omega h_i)^{-1}[|x_0 - u| + (t_i - s)\,|v| + \textstyle\sum_{k=1}^{i} h_k\,(|\epsilon_k| + |f(t_k) - f(r)|\,L(|u|)K(|v|)]$$

But since for $\delta > 0$

$$(1 - h)^{-1} \leq e^{(1+\delta)h}, \quad \text{for } 0 \leq h \leq \frac{\delta}{1 + \delta}$$

assuming $\omega d_\lambda \leq \frac{1}{2}$ it follows that for $\delta = 1$

$$(1 - \omega h_i)^{-1} \leq e^{2\omega\, h_i}.$$

Thus we have

$$|x_i - u| \leq e^{2\omega T}\,[|x_0 - u| + T|v| + \textstyle\sum_{k=1}^{i} h_k\,(|\epsilon_k| + |f(t_k) - f(r)|\,L(|u|)K(|v|)]$$

Since f is continuous

$$\textstyle\sum_{k=1}^{i} h_k\,|f(t_k) - f(r)| \to \displaystyle\int_s^T |f(t) - f(r)|\,dt \quad \text{as } n \to \infty$$

which completes the proof. □

The following lemma shows the uniform bound of $|y_i^\lambda|,\ 1 \leq i \leq N_\lambda$ for the sequence $(t_i^\lambda, x_i^\lambda, y_i^\lambda)$ satisfying (2.8).

Lemma 2.4 Suppose the sequence $(t_i^\lambda, x_i^\lambda, y_i^\lambda, \epsilon_i^\lambda)$ satisfy (2.10) with $\sum_{i=1}^{N_\lambda} \epsilon_i^\lambda \leq M_1$ for some constant M_1 independent of λ (especially $\epsilon_i^\lambda = 0$) and $x_0^\lambda = x_0 \in dom\,(A(s))$. Then there exists a constant $M_2 = M_2(T, x_0, \alpha)$ independent of λ such that $|y_i^\lambda| \leq M_2,\ 1 \leq i \leq N_\lambda$.

Proof: From (2.10) we have

$$y_i^\lambda \in A(t_i^\lambda)x_i^\lambda \quad \text{and} \quad y_i^\lambda = \frac{x_i^\lambda - x_{i-1}^\lambda}{h_i^\lambda} - \epsilon_i^\lambda$$

and thus from (2.6.b) with $x_1 = x_i^\lambda,\ x_2 = x_{i-1}^\lambda,\ y_1 = y_i^\lambda,\ y_2 = y_{i-1}^\lambda$ and $\lambda = h_i^\lambda$ we have

$$(1 - h_i^\lambda \omega)\, |y_i^\lambda + \epsilon_i^\lambda| \leq |\epsilon_i^\lambda| + |y_{i-1}^\lambda| + |f(t_i^\lambda) - f(t_{i-1}^\lambda)| L(|x_{i-1}^\lambda|)(1 + |y_{i-1}^\lambda|).$$

and thus

$$(2.16) \qquad (1 - h_i^\lambda \omega)\, |y_i^\lambda| \leq 2|\epsilon_i^\lambda| + |y_{i-1}^\lambda| + |f(t_i^\lambda) - f(t_{i-1}^\lambda)| L(|x_{i-1}^\lambda|)(1 + |y_{i-1}^\lambda|).$$

If we define $a_i = \Pi_{k=1}^i (1 - h_k^\lambda \omega)\, |y_i^\lambda|$ then multiplying (2.16) by $\Pi_{k=1}^{i-1}(1 - h_k^\lambda \omega)$ we have

$$a_i \leq (1 + b_i)\, a_{i-1} + b_i + 2\,\epsilon_i \leq e^{b_i}\, a_{i-1} + b_i + 2\,\epsilon_i$$

where

$$b_i = L(M)\, |f(t_i^\lambda) - f(t_{i-1}^\lambda)|.$$

Thus we obtain the estimate

$$a_i \leq exp\,(\textstyle\sum_{k=1}^i b_k)\,(a_0 + \sum_{k=1}^i (b_k + 2\,\epsilon_k))$$

where $y_0^\lambda \in A(s)x_0$. Since f is of bounded variation on $[0, T]$ this estimate implies that $|y_i^\lambda|$ is uniformly bounded. □

We define the the modulus $\rho(\cdot)$ of continuity of f by

$$\rho(\sigma) = \max\,\{|f(t) - f(s)|;\ |t - s| \leq \sigma \text{ and } t, s \in [0, T]\}$$

Then $\rho : [0, T] \to R^+$ is bounded, nondecreasing and lim $\rho(\sigma) \to 0$ as $\sigma \to 0^+$. The following inequality plays an important role in the proof of the main estimate.

$$(2.17) \qquad \rho(r) \leq c^{-1}\rho(T)|r - r'| + \rho(\delta) \quad \text{for } \ r \in [0, T]$$

where $0 < c < \delta < T$ and $0 \leq r' < \delta - c$. In fact, if $r \leq \delta$ then $\rho(r) \leq \rho(\delta)$ and thus (2.17) holds. If $r > \delta$ and $r' < \delta - c$ then $c < \delta - r' < r - r'$ and thus $\rho(r) \leq \rho(T) \leq \frac{r - r'}{c}\,\rho(T)$ which implies (2.17).

Now, we are ready to prove the fundamental estimate due to [KKO] in the following theorem.

Theorem 2.5 (Kobayashi-Kobayashi-Oharu) Let $s, \hat{s} \in [0,T]$, $x_0 \in D \cap \overline{dom\,(A(s))}$ and $\hat{x}_0 \in D \cap \overline{dom\,(A(\hat{s}))}$. Assume that for $\alpha > 0$ the sequences $\{t_i^\lambda, x_i^\lambda, y_i^\lambda, \epsilon_i^\lambda\}$ and $\{t_j^\mu, x_j^\mu, y_j^\mu, \epsilon_j^\mu\}$ in $R \times D_\alpha \times X \times X$ satisfy (2.10) and (2.12) and $x_0^\lambda = x_0,\ x_0^\mu = \hat{x}_0$, respectively and that $(C.1)$ holds or $(C.2)$ holds with $\epsilon_i^\lambda = \epsilon_j^\mu = 0$. Then for $0 \le |\sigma| < \delta < T,\ 0 < c < \delta - |\sigma|$, if $d_\lambda, d_\mu < \delta - |\sigma| - c$ then there exists a constant $\tilde{M} = \tilde{M}(T, x_0, \hat{x}_0, \alpha, [u,v])$ such that

$$\omega_{i,j}|x_i^\lambda - x_j^\mu| \le |x_0 - u| + |\hat{x}_0 - u| + c_{i,j}(s - \hat{s})\,(|v| + \tilde{M}\rho(T))$$

$$+ \textstyle\sum_{k=1}^{i} h_k^\lambda |\epsilon_k^\lambda| + \sum_{l=1}^{j} h_l^\mu |\epsilon_l^\mu| \tag{2.18}$$

$$+\tilde{M}(t_j^\mu - \hat{s})\,(c^{-1}\rho(T) c_{i,j}(\sigma) + \rho(\delta))$$

for $1 \le i \le N_\lambda$ and $1 \le j \le N_\mu$, where $r \in [0,T],\ [u,v] \in A(r)$, and

$$\omega_{i,j} = \Pi_{k=1}^{i}\,(1 - \omega h_k^\lambda)\,\Pi_{l=1}^{j}\,(1 - \omega h_l^\mu). \tag{2.19}$$

Proof: From (2.15)

$$\omega_{i,0}\,|x_i^\lambda - x_0^\mu| \le \omega_{i,0}\,(|x_i^\lambda - u| + |x_0^\mu - u|) \le |x_0^\lambda - u| + |x_0^\mu - u| + (t_i^\lambda - s)|v|$$

$$+ \textstyle\sum_{k=1}^{i} h_k^\lambda\,(|\epsilon_k^\lambda| + |f(t_k^\lambda) - f(r)|\,L(|u|))K(|v|)$$

Let $L(|u|)K(|v|) \le \tilde{M}$. Since

$$|f(t_k^\lambda) - f(r)| \le \rho(|t_k^\lambda - r|) \le \rho(T)$$

it follows that

$$\omega_{i,0} a_{i,0} \le |x_0^\lambda - u| + |x_0^\mu - u| + (t_i^\lambda - s)(|v| + \tilde{M}\rho(T)) + \textstyle\sum_{k=1}^{i} h_k^\lambda\,|\epsilon_k^\lambda|.$$

and so (2.18) is satisfied for $1 \le i \le N_\lambda$ and $j = 0$.

Obviously, the same argument is applied to show that (2.18) holds for the case $i = 0$ and $1 \le j \le N_\mu$. If we prove that suppose (2.18) holds for the pairs $(i, j-1)$ and $(i-1, j)$ then (2.18) holds for the pair (i,j), then by induction (2.18) holds for every pair (i,j). To this end, we first prove the following relation between $a_{i-1,j},\ a_{i,j-1}$ and $a_{i,j}$.

$$\omega_{i,j}\,a_{i,j} \le \alpha_{i,j}(\omega_{i-1,j}a_{i-1,j} + h_i^\lambda|\epsilon_i^\lambda|) + \beta_{i,j}(\omega_{i,j-1}a_{i,j-1} + h_j^\mu|\epsilon_j^\mu|) + \tilde{M}\gamma_{i,j}d_{i,j}. \tag{2.20}$$

where $\alpha_{i,j},\ \beta_{i,j},\ \gamma_{i,j}$ are defined in (2.14) and $d_{i,j} \le \rho(|t_i^\lambda - t_j^\mu|)$. From (2.6.a) with $\lambda = h_i^\lambda,\ x_1 = x_i^\lambda$ and $\mu = h_j^\mu,\ x_2 = x_j^\mu$ we have

$$(h_i^\lambda + h_j^\mu - \omega h_i^\lambda h_j^\mu)\,|x_i^\lambda - x_j^\mu|$$

$$\le h_i^\lambda\,|x_j^\mu - h_j^\mu y_j^\mu - x_i^\lambda| + h_j^\mu\,|x_i^\lambda - h_i^\lambda y_i^\lambda - x_j^\mu| + \tilde{M}\,h_i^\lambda h_j^\mu |f(t_i^\lambda) - f(t_j^\mu)|.$$

where we assumed $L(M(T,\hat{x}_0,\alpha))K(M_2(T,\hat{x}_0,\alpha)) \le \tilde{M}$. Substituting (2.13) into this and then dividing by $h_i^\lambda + h_j^\mu$, we obtain

$$(1 - \omega\,\gamma_{i,j})\,a_{i,j} \le \alpha_{i,j}a_{i-1,j} + \beta_{i,j}a_{i,j-1} + \gamma_{i,j}\,(|\epsilon_i^\lambda| + |\epsilon_j^\mu| + \tilde{M}\,d_{i,j})$$

Multiplying the both side of this inequality by $\omega_{i,j}$ we have

$$(1-\omega\,\gamma_{i,j})\omega_{i,j}\,a_{i,j} \le (1-h_i^\lambda\omega)\omega_{i-1,j}\alpha_{i,j}a_{i-1,j} + (1-h_j^\mu\omega)\omega_{i,j-1}\beta_{i,j}a_{i,j-1}$$
$$+\omega_{i,j}\gamma_{i,j}(|\epsilon_i^\lambda| + |\epsilon_j^\mu| + \tilde{M}d_{i,j}).$$

Since

$$0 < \omega_{i,j} \le \max\{1-h_i^\lambda\omega, 1-h_j^\mu\omega\} \le 1-\omega\gamma_{i,j}$$

and $\gamma_{i,j} = h_i^\lambda\alpha_{i,j} = h_j^\mu\beta_{i,j}$, devision of this by $1-\omega\gamma_{i,j}$ yields (2.20). Substituting the estimates of $a_{i-1,j}$ and $a_{i,j-1}$ by (2.18) and the induction hypothesis, we have

$$\omega_{i,j}\,a_{i,j} \le |x_0^\lambda - u| + |x_0^\mu - u| + (\alpha_{i,j}c_{i-1,j}(s-\hat{s}) + \beta_{i,j}c_{i,j-1}(s-\hat{s})\,(|v| + \tilde{M}\rho(T))$$

$$(2.21) \qquad \begin{aligned} &+\textstyle\sum_{k=1}^{i} h_k^\lambda\,|\epsilon_k^\lambda| + \sum_{l=1}^{j} h_l^\mu\,|\epsilon_l^\mu| \\ &+\tilde{M}(t_j^\mu - \hat{s})(c^{-1}\rho(T)\alpha_{i,j}c_{i-1,j}(\sigma) + \alpha_{i,j}\rho(\delta)) \\ &+\tilde{M}(t_{j-1}^\mu - \hat{s})(c^{-1}\rho(T)\beta_{i,j}c_{i,j-1}(\sigma) + \beta_{i,j}\rho(\delta)) + \tilde{M}\gamma_{i,j}d_{i,j}. \end{aligned}$$

It thus follows from Lemma 2.2 and (2.21) that

$$(2.22) \qquad \begin{aligned} \omega_{i,j}\,a_{i,j} &\le |x_0^\lambda - u| + |x_0^\mu - u| + c_{i,j}(s-\hat{s})\,(|v| + \tilde{M}\rho(T)) \\ &+\textstyle\sum_{k=1}^{i} h_k^\lambda\,|\epsilon_k^\lambda| + \sum_{l=1}^{j} h_l^\mu\,|\epsilon_l^\mu| + \tilde{M}(t_j^\mu - \hat{s})(c^{-1}\rho(T)c_{i,j}(\sigma) + \rho(\delta)) \\ &+\tilde{M}\,(-h_j^\mu\beta_{i,j}(c^{-1}\rho(T)c_{i,j-1}(\sigma) + \rho(\delta)) + \gamma_{i,j}d_{i,j}) \end{aligned}$$

We show that the last term of (2.22) is less than or equal to zero. In fact, if $r = |t_i^\lambda - t_j^\mu|$ and $r' = |\sigma - h_j^\mu|$ then it follows that

$$r' \le |\sigma| + h_j^\mu \le |\sigma| + d_\mu \le \delta - c \;\text{ and }\; |r - r'| \le |t_i^\lambda - t_j^\mu + h_j^\mu - \sigma| \le c_{i,j-1}(\sigma)$$

and thus (2.17) implies

$$\begin{aligned} \gamma_{i,j}d_{i,j} = h_j^\mu\beta_{i,j}d_{i,j} &\le h_j^\mu\beta_{i,j}\rho(|t_i^\lambda - t_j^\mu|) \\ &\le h_j^\mu\beta_{i,j}(c^{-1}\rho(T)c_{i,j-1}(\sigma) + \rho(\delta)) \end{aligned}$$

Hence (2.18) holds if (2.18) holds for the pairs $(i, j-1)$ and $(i-1, j)$. □

Next, we show the existence of DS–approximation under range condition (R).

Lemma 2.6 Let $A(t)$ be a quasi-dissipative operator satisfying $H1$ and relaxed range condition

$$(R) \qquad \begin{aligned} &\text{for for each } \beta \ge 0 \text{ and all } x \in D_alpha, \\ &\liminf_{\lambda\to 0^+} \tfrac{1}{\lambda}\, d((I - \lambda\,A(t))\,(dom\,(A(t)) \cap D_{(1-a\lambda)^{-1}(\alpha+b)}, \end{aligned}$$

where $d(S,x)$ denotes the distance between a set S and a point x in X. and let $x_0 \in D \cap \overline{dom\,(A(s))}$ and $\epsilon > 0$. Then there exist $\{t_i\}$, $\{x_i\}$ and $\{y_i\}$ which satisfy (1) – (3), where $[x_i, y_i] \in A(t_i)$ with $x_i \in dom\,(A(t_i)) \cap D_{\psi(t_i - s, \varphi(x_0))}$, $i \geq 1$:

$$(1) \quad s = t_0 < t_1 < \cdots < t_i < \cdots < t_N = T$$

$$(2) \quad t_i - t_{i-1} \leq \epsilon$$

$$(3) \quad |x_i - x_{i-1} - (t_i - t_{i-1})\, y_i| \leq \epsilon\,(t_i - t_{i-1})$$

Proof: We let $\alpha = \psi(T - s, \varphi(x_0))$, $\omega = \omega_\alpha$ and may assume that $2\omega\epsilon \leq 1$. For each $x \in D_\beta \cap \overline{dom\,(A(s))}$, from condition (R) we can choose $\delta \in (0, \epsilon]$ such that there exist $x_\delta \in D_{\psi(\delta,\beta)} \cap dom\,(A(t+\delta))$ and $y_\delta \in A(t+\delta)x_\delta$ such that

$$|x_\delta - x - \delta\, y_\delta| \leq \epsilon\, \delta \tag{2.23}$$

For each $x \in D_\beta$ we define $\delta(x)$ as the least upper bound of $\delta > 0$ that satisfies (2.23). Note that $\psi(\delta, \psi(\tau, \beta)) = \psi(\tau + \delta, \beta)$ for $\tau,\ \delta \geq 0$.

We can select the sequence $\{t_i, x_i, y_i\}$ satisfying (1)–(3) as follows. At each t_i, from the definition of $\delta(x_i)$, we can select $x_{i+1} \in D_{\psi(\delta,\varphi(x_i))} \cap dom\, A(t_i + \delta)$, $y_i \in A(t_i + \delta)$ and $h_{i+1} = t_{i+1} - t_i > \frac{\delta(x_i)}{2}$. If we can show that $t_N \geq T$ then the proof is completed. Suppose $\lim_{i \to \infty}\ t_i = a < T$. It will be shown at the end of the proof that for all $i \geq j \geq k$

(2.24)

$$\omega_{i,j}\, |x_i - x_j| \leq (t_i - t_j)\,(|y_k| + \tilde{M}\rho(a - t_k))$$

$$+\epsilon(t_i - t_k) + \epsilon(t_j - t_k) + \tilde{M} \sum_{n=k+1}^{i} h_n\, |f(t_n) - f(t_k)| + \tilde{M} \sum_{n=k+1}^{j} h_n\, |f(t_n) - f(t_k)|$$

where and $0 < c < \delta$ are arbitrary. It thus follows that

$$\limsup_{i,j \to \infty}\ |x_i - x_j|$$

$$\leq e^{4\omega(a - t_k)}(2\epsilon(a - t_k) + 2\tilde{M} \int_{t_k}^{a} |f(s) - f(t_k)|\, ds) \to 0 \ \text{ as } k \to \infty.$$

Hence $\{x_i\}$ is a Cauchy sequence. Let $x = \lim_{i \to \infty} x_i$. Since φ is lower semi-continuous we have

$$\varphi(x) \leq \liminf_{n \to \infty}\ \varphi(x_n) \leq \psi(\textstyle\sum_{i=k+1}^{\infty} h_i, \varphi(x_k))$$

for all $k \geq 1$. From condition (R), there exist $\mu \in (0, \frac{\epsilon}{2}]$ and $[x_\mu, y_\mu] \in A(a + \mu)x_\mu$ satisfying

$$|x_\mu - x - \mu\, y_\mu| \leq \frac{\epsilon}{2}\, \mu \quad \text{and} \quad \varphi(x_\mu) \leq \psi(\mu, \varphi(x)) \tag{2.25}$$

Now, we choose $k \geq 1$ so that

$$h_{k+1} < \tfrac{\mu}{2}, \quad \textstyle\sum_{i=k+1}^{\infty}\ h_i \leq \tfrac{\epsilon}{2}$$

$$\textstyle\sum_{i=k+1}^{\infty}\ h_i\, |y_\mu| \leq \tfrac{\epsilon}{4}\, \mu \quad \text{and} \quad |x_k - x| \leq \tfrac{\epsilon}{4}\, \mu.$$

Then from (2.25), we have for $\mu + \sum_{i=k+1}^{\infty} h_i \le \epsilon$

$$|x_\mu - x_k - (\mu + \sum_{i=k+1}^{\infty} h_i) y_\mu| \le |x_\mu - x - \mu\, y_\mu| + |x_k - x| + \sum_{i=k+1}^{\infty} h_i\, |y_\mu|$$

$$\le \epsilon\, \mu \le (\mu + \sum_{i=k+1}^{\infty} h_i)\, \epsilon$$

and

$$\varphi(x_\mu) \le \psi(\mu, \varphi(x)) \le \psi(\mu, \psi(\sum_{i=k+1}^{\infty} h_i, \varphi(x_k)) \le \psi(\mu + \sum_{i=k+1}^{\infty} h_i, \varphi(x_k))$$

Hence, by the definition of $\delta(x_k)$

$$\mu + \sum_{i=k+1}^{\infty} h_i \le \delta(x_k)$$

However, since $\frac{\delta(x_k)}{2} < h_{k+1} < \frac{\mu}{2}$ we have $\delta(x_k) < \mu$ and $\mu + \sum_{i=k+1}^{\infty} h_i \le \mu$. This is a contradiction and therefore $t_N = T$ for some N.

Finally, we prove (2.24) for all $i \ge j \ge k$. The proof is very similar to the one for Theorem 2.5 and we use the same notation as in the proof of Theorem 2.5. Setting $\lambda = \mu$, $s = r = t_k$, $u = x_k$ and $v = y_k$, from (2.15) we have

$$\omega_{i,k}\, |x_i - x_k| \le (t_i - t_k)(|y_k| + \tilde{M} \rho(a - t_k)) + (t_i - t_k)\, \epsilon + \tilde{M} \sum_{n=k+1}^{i} h_n\, |f(t_n) - f(t_k)|.$$

Hence (2.24) holds for $j = k$. Also, it is self-evident that (2.24) holds for $i = j$. Now, let $i > j > k$ and assume that $a_{i-1,j} = |x_{i-1} - x_j|$ and $a_{i,j-1}$ satisfy (2.24). Then, if we show that $a_{i,j} = |x_i - x_j|$ satisfies (2.24), then by induction, (2.24) holds for all $i \ge j \ge k$. By the arguments leading to (2.20) in the proof of Theorem 2.5 we have

$$\omega_{i,j}\, a_{i,j} \le \alpha_{i,j}(\omega_{i-1,j} a_{i-1,j} + h_i\, \epsilon) + \beta_{i,j}(\omega_{i,j-1} a_{i,j-1} + h_j\, \epsilon) + \tilde{M} \gamma_{i,j}\, |f(t_i) - f(t_j)|.$$

Substituting the estimates of $a_{i-1,j}$ and $a_{i,j-1}$ by (2.24) into this, we obtain (2.24) for $a_{i,j}$ since

$$\alpha_{i,j}(t_{i-1} - t_j) + \beta_{i,j}(t_i - t_{j-1}) = t_i - t_j$$

and

$$|f(t_i) - f(t_j)| \le |f(t_i) - f(t_k)| + |f(t_j) - f(t_k)|. \square$$

Similarly, we can show the existence of DS–approximation under range condition (3.3) and continuity condition (3.3).

Lemma 2.7 Let $A(t)$ be a quasi-dissipative operator satisfying (3.2) and range condition (3.3), and let $x_0 \in D \cap dom\,(A(s))$ and $\epsilon > 0$. Then there exist $\{t_i\}$, $\{x_i\}$ and $\{y_i\}$ which satisfy (1) – (3), where $[x_i, y_i] \in A(t_i)$ with $x_i \in dom\,(A(t_i)) \cap D_{\psi(t_i - s, \varphi(x_0))}$, $i \ge 1$:

(1) $s = t_0 < t_1 < \cdots < t_i < \cdots < t_N = T$

(2) $t_i - t_{i-1} \le \epsilon$

(3) $x_i - x_{i-1} - (t_i - t_{i-1})\, y_i = 0$

Proof: We can select the sequence $\{t_i, x_i, y_i\}$ satisfying (1)–(3) as follows. By the range condition $(R.1)$ at each t_i, we can select $t_{i+1} - t_i = \min\{\epsilon, \lambda(x_i)\}$ and $[x_{i+1}, y_{i+1}] \in A(t_{i+1})$ such that

$$x_i - x_{i-1} - (t_i - t_{i-1})\, y_i = 0 \quad \text{and} \quad \varphi(x_{i+1}) - \varphi(x_i) \le (t_{i+1} - t_i)\, g(\varphi(x_{i+1})).$$

Suppose $\lim_{i\to\infty}\, t_i = a < T$. Then it follows from the proof of Lemma 2.6 that $\{x_i\}$ is a Cauchy sequence. Let $x = \lim_{i\to\infty} x_i$. Then $x \in D$ and $\lim_{i\to\infty} \lambda(x_i) = 0$. But since $\lambda(\cdot)$ is lower semicontinuous, it follows that $\lambda(x) = 0$, which is a contradiction. Hence $t_N = T$ for some N. □

Now we show the existence of the mild solution to (2.1).

Theorem 2.8 Let $A(t),\ t \in [0,T]$ satisfy $(A.1) - (A.2)$ and assume that either $(R) - (C.1)$ or $(R.1) - (C.2)$ hold. For the case of $(C.2)$ we assume that $x_0 \in D \cap dom\,(A(s))$ and f is of bounded variation. Then we have
(1) For $s \in [0,T]$ and $x_0 \in D \cap \overline{dom\,(A(s))}$ there exists an $\alpha > 0$ such that there exists a DS–approximation sequence $\{t_i^\lambda, x_i^\lambda, y_i^\lambda, \epsilon_i^\lambda\}$ satisfying (2.10) . Under condition $(R.1)$ we have $\epsilon_i^\lambda = 0,\ 1 \le i \le N_\lambda$.
(2) Every DS–approximate sequence $u_\lambda(t)$ in D_α converges to a continuous function $u(t; s, x_0) : [s,T] \to X$ uniformly on $[s,T]$, and $u(t,\cdot) \in D_\alpha \cap \overline{dom\,(A(t))}$. Moreover, if $\overline{dom\,(A(t))}$ is independent of $t \in [0,T]$, then $s \to u(t; s, x_0) \in X$ is continuous.
(3) If $x_0 \in dom\,(A(s))$ and $f(\cdot)$ is of bounded variation then $t \to u(t; x_0) \in X$ is Lipschitz continuous on $[s,T]$.

Proof: The existence of DS–approximation sequence follows from Lemmas 2.6 and 2.7. Moreover, it follows from Lemma 2.4 that $|y_i^\lambda|$ is uniformly bounded in λ and $1 \le i \le N_\lambda$ for the case of $(C.2)$. We apply Theorem 2.5 with $\hat{s} = s$ and $\hat{x}_0 = x_0$. Let $t \in (s,T]$ and assume that $t \in (t_{k_\lambda-1}^\lambda, t_{k_\lambda}^\lambda]$ and $t \in (t_{j_\mu-1}^\mu, t_{j_\mu}^\mu]$. By the definition of $c_{i,j}(\cdot)$ $c_{k_\lambda,j_\mu}(0) \to 0$ as $\lambda,\ \mu \to 0$ since $t_{k_\lambda}^\lambda \to t$ as $\lambda \to 0$ and $t_{j_\mu}^\mu \to t$ as $\mu \to 0$. As shown in the proof of Lemma 2.3, if $d_\lambda\omega_\alpha,\ d_\mu\omega_\alpha \le \frac{1}{2}$ then

$$\omega_{i,j}^{-1} \le e^{4\omega_\alpha(T-s)} = C$$

Since $u_\lambda(t) - u_\mu(t) = x_{k_\lambda}^\lambda - x_{j_\mu}^\mu$ it follows from Theorem 2.5 that

$$\lim_{\lambda,\ \mu\to 0} |u_\lambda(t) - u_\mu(t)| \le C\,(2\,|x_0 - u| + \tilde{M}(T-s)\rho(\delta))$$

for all $u \in D_\alpha \cap dom\,(A(s))$ and $\delta > 0$, where we set $\sigma = 0$. Since $\lim \rho(\delta) = 0$ as $\delta \to 0^+$ it follows that $\lim_{\lambda,\ \mu\to 0}\, |u_\lambda(t) - u_\mu(t)| = 0$ uniformly on $[s,T]$. Note that $u_\lambda(t) \in dom\,(A(t_{k_\lambda-1}^\lambda))$ and $t_{k_\lambda-1}^\lambda \to t^-$ as $\lambda \to 0$. Thus, it follows from (A.2) that $u(t) \in \overline{dom\,(A(t))}$. Since $\varphi(\cdot)$ is lower semi-continuous $u(t) \in D_\alpha,\ t \in [s,T]$.

Next, we prove the continuity of $u(t; s, \cdot)$. Let $t,\ \tau \in [s,T]$ and $t \in (t_{k_\lambda-1}^\lambda, t_{k_\lambda}]$ and $t \in (t_{j_\lambda-1}^\lambda, t_{j_\lambda}]$. Since $d_\lambda \to 0$ it follows that $t_{k_\lambda} \to t$ and $t_{j_\lambda} \to \tau$ and thus $c_{i,j}(0) \to |t - \tau|$. Thus from Theorem 2.5 with $\lambda = \mu,\ s = \hat{s},\ t_j^\mu = t_j^\lambda,\ x_j^\mu = x_j^\lambda$ and $\sigma = 0$ we obtain

$$|u(t) - u(\tau)| = \lim_{\lambda\to 0} |u_\lambda(t) - u_\lambda(\tau)|$$

$$\le C\,(2\,|x_0 - u| + |t-\tau|(|v| + \tilde{M}\rho(T)) + \tilde{M}(\tau - s)(c^{-1}\rho(T)\,|t-\tau| + \rho(\delta))$$

for all $[u,v] \in A(s)$ and $0 < c < \delta < T$. For $\epsilon > 0$ we take $u \in D_\alpha \cap dom\,(A(s))$ such that $2C\,|x_0 - u| \le \frac{\epsilon}{4}$ and let $\delta > 0$ such that $C\rho(\delta) \le \frac{\epsilon}{4}$. Then if we set $c = \frac{\delta}{2}$ and choose

$$|t-\tau| \le \min\left\{\frac{\epsilon}{4C\,(|v| + \tilde{M}\rho(T))}, \frac{\epsilon\delta}{8C\,(T-s)\rho(T)}\right\}$$

then we have $|u(t) - u(\tau)| \le \epsilon$ and thus $u(\cdot)$ is uniformly continuous on $[s,T]$. Similarly, for fixed $t \in [0,T]$ and $x_0 \in X$ we have

$$|u(t;s,x_0) - u(t;\hat{s},x_0)| \le C\,(2|x_0-u| + |s-\hat{s}|(|v| + \tilde{M}\rho(T)) + \tilde{M}(t-s)c^{-1}\rho(T)\,|s-\hat{s}| + \rho(\delta))$$

for $[u,v] \in A(r)$. Thus, if $\overline{dom\,(A(t))}$ is independent of $t \in [0,T]$, then $s \to u(t;s,x_0) \in X$ is continuous.

Assume that $x_0 \in dom\,(A(s))$ and $f(\cdot)$ is of bounded variation. We prove that $t \to u(t) \in X$ is Lipschitz continuous. It follows from Lemma 2.5 that $|y_i^\lambda| \le M_1$ and $y_i^\lambda \in dom\,(A(t_i^\lambda))x_i^\lambda$ for $1 \le i \le N_\lambda$. Letting $s = t_{i_\lambda}^\lambda$ and $u = x_{i_\lambda}^\lambda$ and $\sigma = 0$ where $s_0 \in (t_{i_\lambda - 1}^\lambda, t_{i_\lambda}^\lambda]$, it follows from Theorem 2.5 that

$$|u(t) - u(\tau)| = \lim_{\lambda \to 0} |u_\lambda(t) - u_\lambda(\tau)|$$

$$\le C\,(|t-\tau|(M_1 + \tilde{M}\rho(T)) + \tilde{M}(\tau - s_0)c^{-1}\rho(T_0 - s_0)\,|t-\tau| + \rho(\delta))$$

for all $0 < c < \delta < T$ and $s_0 \in [s,\tau)$ and $T_0 \in (t,T]$. Since there $f(\cdot)$ is of bounded variation there exists a constant L such that $|u(t) - u(\tau)| \le L\,|t-\tau|$ for $s \le \tau \le t \le T$. □

Next we prove the uniqueness of the mild solution.

Theorem 2.9 Let $A(t),\ t \in [0,T]$ satisfy either (3.2)–(3.3). For $\alpha > 0$ let $u : [s,T] \to X$ be a mild solution of (2.1) on $[s,T]$ confined to D_α. For the case of $(C.2)$ we assume that the sequence $\{y_i^\lambda\}$ defined by (2.10) is bounded in X uniformly in λ and $1 \le i \le N_\lambda$. Then we have
(1) The mild solution u is an integral solution of (2.1) on $[s,T]$.
(2) If v is an integral solution of (2.1) on $[s,T]$ then there exists $\omega = \omega_\alpha$ such that

$$|v(t) - u(t)| \le e^{\omega\,(t-s)}|v(0) - u(0)|$$

(3) The mild solution is unique.

Proof: First we show that the mild solution is an integral solution. Since

$$\langle y, x\rangle_- - \langle z, x\rangle_+ \le \langle y - z, x\rangle_-$$

it follows from (2.5) that for $[x,y] \in A(r)$

$$\text{(2.26)}\quad \langle y_i^\lambda, x_i^\lambda - x\rangle_- - \langle y, x_i^\lambda - x\rangle_+ \le \langle y_i^\lambda - y, x_i^\lambda - x\rangle_- \le \omega_\alpha|x_i^\lambda - x| + C\,|f(t_i^\lambda) - f(r)|$$

where $C = L(|x|)K(|y|)$. Thus

$$\langle y_i^\lambda, x_i^\lambda - x\rangle_- \le \langle y, x_i^\lambda - x\rangle_+ + \omega_\alpha|x_i^\lambda - x| + C\,|f(t_i^\lambda) - f(r)|.$$

Since $h_i^\lambda y_i^\lambda = (x_i^\lambda - x) - (x_{i-1}^\lambda - x) - h_i^\lambda \epsilon_i^\lambda$, the left side of this is estimated by

$$\langle h_i^\lambda y_i^\lambda, x_i^\lambda - x\rangle_- = |x_i^\lambda - x| + \langle -(x_{i-1}^\lambda - x) - h_i^\lambda \epsilon_i^\lambda, x_i^\lambda - x\rangle_- \geq |x_i^\lambda - x| - |x_{i-1}^\lambda - x| - h_i^\lambda |\epsilon_i^\lambda|$$

where we used the fact that $\langle \alpha\, x + y, x\rangle_- = \alpha|x| + \langle y, x\rangle_-$. Hence, we have

$$|x_i^\lambda - x| - |x_{i-1}^\lambda - x| \leq h_i^\lambda\, (\omega_\alpha |x_i^\lambda - x| + \langle y, x_i^\lambda - x\rangle_+ + C\,|f(t_i^\lambda) - f(r)| + |\epsilon_i^\lambda|).$$

Summing up this in i from $i = j+1$ to $i = k$, we obtain

$$|x_k^\lambda - x| - |x_j^\lambda - x| \leq \int_{t_j^\lambda}^{t_k^\lambda} (\omega\,|u_\lambda(\sigma) - x| + \langle y, u^\lambda(\sigma) - x\rangle_+)\, d\sigma$$

$$+ \textstyle\sum_{i=j+1}^k h_i^\lambda\, (|f(t_i^\lambda) - f(r)| + |\epsilon_i^\lambda|)$$

Let $s \leq \tau \leq t \leq T$ and let $t_k^\lambda \to \tau$ and $t_j^\lambda \to t$ as $\lambda \to 0$. By Theorem 2.8 and the upper-semicontinuity of $\langle \cdot, \cdot\rangle_+$ we obtain (2.11), letting $\lambda \to 0$.

Next we show the assertion (2). Since $v : [s, T] \to X$ is an integral solution of (2.1) on $[s, T]$, there exist $\omega = \omega(\alpha)$ and $C = L(|x|)K(|y|)$ such that

$$|v(t) - x| - |v(\tau) - x| \leq \int_\tau^t \omega\,|v(\sigma) - x| + \langle y, v(\sigma) - x\rangle_+ + C\,|f(\sigma) - f(r)|\, d\sigma$$

for $s \leq \tau \leq t \leq T$ and $[x, y] \in A(r),\ r \in [s, T]$ with $x \in D_\alpha$. Since $[x_i^\lambda, y_i^\lambda] \in A(t_i^\lambda)$ and $x_i^\lambda \in D_\alpha$, it follows that

$$|v(t) - x_i^\lambda| - |v(\tau) - x_i^\lambda| \leq \int_\tau^t \omega\,|v(\sigma) - x_i^\lambda| + \langle y_i^\lambda, v(\sigma) - x_i^\lambda\rangle_+ + C\,|f(\sigma) - f(t_i^\lambda)|\, d\sigma$$

where $C = \sup_\lambda\, L(|x_i^\lambda|)K(|y_i^\lambda|)$. Since $h_i^\lambda y_i^\lambda = (x_i^\lambda - v(\sigma)) - (x_{i-1}^\lambda - v(\sigma)) - h_i^\lambda \epsilon_i^\lambda$ and $\langle \alpha\, x + y, x\rangle_+ = \alpha\,|x| + \langle y, x\rangle_+$

$$\langle h_i^\lambda y_i^\lambda, v(\sigma) - x_i^\lambda\rangle_+ = -|v(\sigma) - x_i^\lambda| + \langle -(x_{i-1}^\lambda - v(\sigma) - h_i^\lambda \epsilon_i^\lambda, v(\sigma) - x_i^\lambda\rangle_+$$

$$\leq -|v(\sigma) - x_i^\lambda| + |v(\sigma) - x_{i-1}^\lambda| + h_i^\lambda |\epsilon_i^\lambda|$$

Thus, we have

$$(|v(t) - x_i^\lambda| - |v(\tau) - x_i^\lambda|)h_i^\lambda$$

$$\leq \int_\tau^t (-|v(\sigma) - x_i^\lambda| + |v(\sigma) - x_{i-1}^\lambda| + h_i^\lambda\, (\omega\,|v(\sigma) - x_i^\lambda| + C\,|f(\sigma) - f(t_i^\lambda)| + |\epsilon_i^\lambda|)\, d\sigma.$$

Summing up the both sides of this in i from $i = j+1$ to $i = k$, we obtain

$$\int_{t_j^\lambda}^{t_k^\lambda} (|v(t) - u_\lambda(\xi)| - |v(\tau) - u_\lambda(\xi|)\, d\xi$$

$$\leq \int_\tau^t (-|v(\sigma) - u_\lambda(t_k^\lambda)| + |v(\sigma) - u_\lambda(t_j^\lambda)| + \int_{t_j^\lambda}^{t_k^\lambda} \omega\,|v(\sigma) - u_\lambda(\xi)|)\, d\xi\, d\sigma$$

$$+ \int_\tau^t \textstyle\sum_{i=j+1}^k h_i^\lambda\, (C\,|f(\sigma) - f(t_i^\lambda)| + |\epsilon_i^\lambda|)\, d\sigma.$$

We now take any pair $\rho,\ \eta$ such that $s \leq \eta \leq \rho \leq T$ and choose the sequences $t_j^\lambda \to \eta$ and $t_k^\lambda \to \rho$ as $\lambda \to 0$. Letting $\lambda \to 0$, we obtain

$$\begin{aligned} &\int_\eta^\rho (|v(t) - u(\xi)| - |v(\tau) - u(\xi)|)\, d\xi + \int_\tau^t (|v(\sigma) - u(\rho)| - |v(\sigma) - u(\eta)|)\, d\sigma \\ &\leq \int_\tau^t \int_\eta^\rho (\omega\, |v(\sigma) - u(\xi)| + C\, |f(\sigma) - f(\xi)|)\, d\xi\, d\sigma \end{aligned} \tag{2.27}$$

For $h > 0$ we define the function $F_h : [s, T - h] \to R^+$ by

$$F_h(t) = h^{-2} \int_t^{t+h} \int_t^{t+h} |v(\sigma) - u(\xi)|\, d\sigma\, d\xi.$$

Then (2.27) implies that $F_h(\cdot)$ satisfies

$$\frac{d}{dt} F_h(t) \leq \omega\, F_h(t) + C h^{-2} \int_t^{t+h} \int_t^{t+h} |f(\sigma) - f(\xi)|\, d\sigma\, d\xi.$$

and by Gronwall's inequality

$$F_h(t) \leq e^{w(t-s)} (F_h(s) + \int_s^t C e^{\omega(t-\tau)}\, (h^{-2} \int_\tau^{\tau+h} \int_\tau^{\tau+h} |f(\sigma) - f(\xi)|\, d\sigma\, d\xi)\, d\tau.$$

Letting $h \to 0^+$ and by the continuity of u and v on $[s, T]$, we obtain the desired estimate.

Finally the uniqueness of the mild solution follows from the assertions (1) and (2). □

Corollary 2.10 Let $A(t)$ satisfy $(A.1) - (A.2)$ and $(R.1) - (C.2)$. For $x \in D$ and every sequence in $D \cap dom\,(A(s))$ such that $|x_n - x| \to 0$ as $n \to \infty$, the limit $\lim_{n\to\infty} u(t, s, x_n)$ in X exists and belongs to $D \cap \overline{dom\,(A(t))}$ and the limit is independent of the choice of convergent sequences $\{x_n\}$. With no confusion we denote such a limit function by $u(t; s, x)$. Then $u(t; s, x)$ is the unique integral solution to (2.1) and satisfies

$$|u(t; s, x) - u(t; s, \hat{x})| \leq e^{\omega(t-s)}\, |x - \hat{x}|$$

for $0 \leq s \leq t \leq T$ and $x,\ \hat{x} \in D \cap \overline{dom\,(A(s)}$.

Proof: For $x \in D \cap \overline{dom\,(A(s))}$ we assume that $\{x_n\}$ is a sequence in $D \cap dom\,(A(s))$ such that $|x_n - x| \to 0$ as $n \to \infty$. Then it follows from the proof of Theorem 2.8 that $u(t; s, x_n)$ is confined to D_α for some $\alpha > 0$ and thus from Theorem 2.9 it is the integral solution to (2.1) where ω can be chosen to be independent of n in (2.11). It follows from Theorem 2.9 that

$$|u(t; s, x_n) - u(t; s, x_m)| \leq e^{\omega(t-s)}\, |x_n - x_m| \to 0 \quad \text{as } n, m \to \infty.$$

for every sequence in $D \cap dom\,(A(s))$ such that $|x_n - x| \to 0$ as $n \to \infty$. Thus $\{u(t; s, x_n)\}$ is a Cauchy sequence in $C(s, T; X)$ and thus has the unique limit. Let $\{\hat{x}_n\}$ be any other convergent sequence to x. Then

$$|u(t; s, x_n) - u(t; s, \hat{x}_m)| \leq e^{\omega(t-s)}\, |x_n - \hat{x}_m| \to 0 \quad \text{as } n, m \to \infty.$$

Hence the two sequences $u(t; s, x_n)$ and $u(t; s, \hat{x}_m)$ converge to the same limit $u(t; s, x)$ in X. The limit function $u(t; s, x)$ is an integrable solution on $[s, T]$ since $\langle \cdot, \cdot \rangle_+$ is upper-semicontinuous. □

Define the nonlinear operators $U(t, s) : D \cap \overline{dom\,(A(s))} \to D \cap \overline{dom\,(A(t))}$ by

$$U(t, s)x = u(t; s, x) \tag{2.28}$$

where $u(t; s, x)$ is the integral solution to (2.1) defined in the sense of Theorem 2.8 for case $(R) - (C.1)$ and of Corollary 2.10 for case (3.2)–(3.3). Then, we have the following theorem.

Theorem 2.11 Let $A(t),\ t \in [0, T]$ satisfy (3.2)–(3.3). Then the family of operators $U(t, s)$ generated by $A(t)$ via (2.28) defines an evolution operator on D in the sense of Definition 2.4 and there exist a constant ω such that

$$|U(t, s)x - U(t, s)\hat{x}| \le e^{\omega(t-s)}\, |x - \hat{x}| \tag{2.29}$$

for $0 \le s \le t \le T$ and $x,\ \hat{x} \in D \cap \overline{dom\,(A(s)}$. Moreover, there exists a constant $\tilde{C}$ such that

$$|U(t + s, s)x - U(t + \hat{s}, \hat{s})\hat{x}| \le e^{\omega t}\, |x - \hat{x}| + \int_0^t \tilde{C} e^{\omega(t-\tau)}\, |f(\tau + s) - f(\tau + \hat{s})|\, d\tau \tag{2.30}$$

for $x \in D \cap \overline{dom\,(A(s)},\ \hat{x} \in D \cap \overline{dom\,(A(\hat{s})}$ in case (C.1) and $x \in D \cap dom\,(A(s),\ \hat{x} \in D \cap dom\,(A(\hat{s})$ in case (C.2), respectively.

Proof: The well-posedness and continuity of U and (2.29) follow from Theorems 2.8–2.9 and Corollary 2.10. For the semigroup property we let $t^{\lambda}_{k_\lambda} \to t$ and $t^{\mu}_{j_\mu} \to t$ and note that from Lemma 2.5 with $s = t^{\lambda}_{k'_\lambda} \to t^-$ we obtain

$$|u(t; s, x) - u(t; r, \tilde{x})| = \lim_{\lambda \to,\ \mu \to 0} |u_\lambda(t; s, x) - u_\mu(t, r, \tilde{x})|$$

$$\le C\,(2|\tilde{x} - u| + \tilde{M}(t - r)\rho(\delta)),$$

where $\tilde{x} = u(r; s, x)$, for all $u \in D_\alpha \cap dom\,(A(r))$ and $0 < \delta < T$. This implies $u(t; s, x) = u(t; r, \tilde{x})$ and hence the semigroup property.

For the estimate (2.30) it follows from (2.27) that

$$\frac{d}{dt} G_h(t) \le \omega\, G_h(t) + \tilde{C} h^{-2} \int_t^{t+h} \int_t^{t+h} |f(\sigma + s) - f(\xi + \hat{s})|\, d\sigma\, d\xi)$$

where

$$G_h(t) = h^{-2} \int_t^{t+h} \int_t^{t+h} |u(\sigma + s; s, x) - u(\xi + \hat{s}; \hat{s}, \hat{x})|\, d\sigma\, d\xi.$$

By Gronwall's inequality

$$G_h(t) \le e^{\omega t} G_h(0) + \int_0^t \tilde{C} e^{\omega(t-\tau)}\, h^{-2} \int_\tau^{\tau+h} \int_\tau^{\tau+h} |f(\sigma + s) - f(\xi + \hat{s})|\, d\sigma\, d\xi.$$

Letting $h \to 0^+$ and the continuity of u in t we obtain the estimate (2.30). $\square$.

Suppose the range condition $(R.1)$ is strengthen by $(R.1)$ holding for all $0 < \delta \leq \delta_0$ independent of $u^0 \in D$, then from (A.1) $(I - \lambda A(t))^{-1} : D \to D \times dom\,(A(t))$ is locally Lipschitz and the squence $\{x_k\}$ defined by

$$x_k = \Pi_{i=1}^k (I - h_i A(t_i))^{-1} x$$

for any sequence $\{h_i\}$ in $(0, \delta_0]$ defines a DS-approximation, confinded in D_α. Thus, from Theorems 2.8–2.9 and Corollary 2.10 we have the product formula

$$U(t,s)x = \lim \Pi_{k=1}^{[\frac{t-s}{h}]} \left(I - h\,A(s + k\,\frac{t-s}{h})\right)^{-1} x \quad \text{as } \ h \to 0^+. \tag{2.30}$$

Corollary 2.12 Let C be a closed convex subset of X.
(1) $C \subset R(I - \lambda\,A(t))$ for $0 < \lambda \leq \delta$ and $t \geq 0$
(2) $(I - \lambda\,A(t))^{-1} \subset C$
(3) there exists a a continuous function f in X which is of bounded variation on any bounded interval $[0, T]$ and a monotone increasing function $L : R^+ \to R^+$ such that for all $x_1 \in dom\,(A(t)) \cap C,\ x_2 \in dom\,(A(s)) \cap C$ and $y_1 \in A(t)x_1,\ y_2 \in A(s)x_2$

$$(1 - \lambda\omega)|x_1 - x_2| \leq |x_1 - x_2 + \lambda\,(y_1 - y_2)| + \lambda\,|f(t) - f(s)|L(|x_2|)K(|y_2|).$$

Then

$$U(t,s)x = \lim \Pi_{k=1}^{[\frac{t-s}{h}]} \left(I - h\,A(s + k\,\frac{t-s}{h})\right)^{-1} x \quad \text{as } \ h \to 0^+$$

defines a unique integral solution in C. Here, $U(t,s) : \overline{dom\,(A(s))} \cap C \to \overline{dom\,(A(t))} \cap C$ is continuous in $t \in [s, \infty)$ and satisfy

$$|U(t,s)x - U(t,s)y| \leq e^{\omega(t-s)}\,|x - y| \quad \text{for } \ x, y \in X.$$

Proof: We let $D_\alpha = C$ for $\alpha \geq 0$, $\varphi(x) = I_C$, the indicator function of C and $g = 0$. Then condition $(R.1) - (C.2)$ holds and therefore the corollary follows from Theorems 2.8–2.10.

3.2 Applications

We consider the Cauchy problem of the form

$$\frac{d}{dt}u(t) = A(t, u(t))u(t) \quad u(s) = u_0 \tag{1.1}$$

where $A(t,u),\ t \in [0,T]$ is a maximal dissipative linear operator in a Banach space X for each u belonging to D. Define the nonlinear operator $\mathcal{A}(t)$ in X by $\mathcal{A}(t)u = A(t,u)u$. We assume that $dom\,(A(t,u))$ is independent of $u \in D$ and for each $\alpha \in R$ there exists an $\omega_\alpha \in R$ such that

$$\langle A(t,u)x_1 - A(s,u)x_2, x_1 - x_2\rangle_- \leq \omega_\alpha\,|x_1 - x_2| + |f(t) - f(s)|L(|x_2|)K(|A(s,u)x_2|) \tag{1.2}$$

for $u \in D_\alpha$ and $x_1 \in dom\,(A(t,u)),\ x_2 \in dom\,(A(s,u))$. Moreover, $\mathcal{A}(t)$ satisfies (3.2), i.e.,

$$(1.3)\quad (1-\lambda\,\omega_\alpha)\,|u_1-u_2| \le |(u_1-u_2)-\lambda\,(\mathcal{A}(t)u_1-\mathcal{A}(s)u_2)|+\lambda\,|f(t)-f(s)|L(|u_2|)K(|\mathcal{A}(s)u_2|)$$

for $u_1 \in dom\,(\mathcal{A}(t)) \cap D_\alpha,\ u_2 \in dom\,(\mathcal{A}(s)) \cap D_\alpha$. We consider the finite difference approximation of (1.1); for sufficiently small $\lambda > 0$ there exists a family $\{u_i^\lambda\}$ in D such that

$$(1.4)\qquad \begin{aligned} &\frac{u_i^\lambda - u_{i-1}^\lambda}{h_i^\lambda} = A(t_i^\lambda, u_{i-1}^\lambda)u_i^\lambda \quad \text{with} \quad u_0^\lambda = u_0 \\ &\frac{\varphi(u_i^\lambda) - \varphi(u_{i-1}^\lambda)}{h_i^\lambda} \le a\,\varphi(u_i^\lambda) + b. \end{aligned}$$

Then, it follows from Theorems 2.5–2.7 that if the sequence

$$(1.5)\qquad \epsilon_i^\lambda = A(t_i^\lambda, u_i^\lambda)u_i^\lambda - A(t_i^\lambda, u_{i-1}^\lambda)u_i^\lambda \ \text{ satisfy } \ \textstyle\sum_{i=1}^{N_\lambda} h_i^\lambda\,|\epsilon_i^\lambda| \to 0 \ \text{ as } \ \lambda \to 0^+,$$

then (1.1) has the unique integrable solution. We have the following theorem.

Theorem 5.0 Assume (1.2) holds and for each $\alpha \in R$ there exist a $c_\alpha \ge 0$ such that

$$(1.6)\qquad |(A(t,u)u - A(t,v)u| \le c_\alpha\,|u-v|, \quad \text{for } u,\ v \in dom\,(\mathcal{A}(t)) \cap D_\alpha.$$

Let f be of bounded variation and $u_0 \in dom\,(\mathcal{A}(s)) \cap D$. Then (1.3) and (1.5) are satisfied and thus (1.1) has a unique integrable solution u and $\lim_{\lambda \to 0^+} u_\lambda = u$ uniformly on $[s,T]$.

Proof: If $y_i^\lambda = (h_i^\lambda)^{-1}(u_i^\lambda - u_{i-1}^\lambda)$, then we have

$$\begin{aligned} y_{i+1}^\lambda - y_i^\lambda &= A(t_{i+1}^\lambda, u_i^\lambda)u_{i+1}^\lambda - A(t_i^\lambda, u_i^\lambda)u_i^\lambda \\ &\quad + A(t_i^\lambda, u_i^\lambda)u_i^\lambda - A(t_i^\lambda, u_{i-1}^\lambda)u_i^\lambda \end{aligned}$$

and thus from (1.2) and (1.6)

$$(1-\lambda\omega_\alpha)\,|y_{i+1}^\lambda| \le (1+\lambda c_\alpha)\,|y_i^\lambda| + |f(t_{i+1}^\lambda) - f(t_i^\lambda)|L(|u_i^\lambda|)K(|y_i^\lambda|)$$

with $y_0^\lambda = A(s,u^0)u^0 = \mathcal{A}(s)u_0$. Thus by the same arguments as in the proof of Lemma 2.4, we obtain $|y_i^\lambda| \le M$ for some constant M and therefore from (1.6)

$$\textstyle\sum_{i=1}^{N_\lambda} |\epsilon_i^\lambda|\lambda \le MT\,\lambda \to 0 \quad \text{as} \ \ \lambda \to 0^+.$$

We also note that (1.3) follows from (1.2) and (1.6).

3.3 Navier Stokes Equation, Revisited

We consider the incompressible Navier-Stokes equations (5.1). We use exactly the same notation as in Section.. Define the evolution operator $A(t,u)$ by $w = A(t,u)v \in H$, where

$$(5.1)\qquad (w,\phi)_H + \nu\,\sigma(u,\phi) + b(u,v,\phi) - (f(t),\phi)_H = 0$$

for $\phi \in V$, with $dom\,(A(t)) = dom\,(A_0)$. We let $D = V$ and define the functional as below.

Theorem 5.1 The evolution operator $(A(t,u),\varphi,D,H)$ defined above satisfies the conditions (1.2)–(1.5) with $g(r) = b$, a suitably chosen positive constant.

Proof: For $u_1,\ u_2 \in dom\,(A_0)$.

$$(A(t,v)u_1 - A(s,v)u_2, u_1 - u_2)_H + \nu\,|u_1 - u_2|_V^2 = (f(t) - f(s), u_1 - u_2)$$

since $b(u, v_1 - v_2, v_1 - v_2) = 0$, which implies (1.2). The existence of $u^\delta \in V$ for the equation: $\delta^{-1}(u^\delta - u^0) = A(t,u^0)u^\delta$ for $u^0 \in V$, $\delta > 0$ and $t \in [0,T]$ follows from *Step 2.* of Section NS and we have

$$\frac{|u^\delta|_H^2 - |u^0|_H^2}{\delta} + \nu\,|u^\delta|_V^2 \le \frac{1}{\nu}\,|f(t)|_{V^*}^2. \tag{5.2}$$

We also have the estimate of $|u^\delta|_V$.

$$\frac{1}{2\delta}\,(|u^\delta|_V^2 - |u^0|_V^2) + \frac{\nu}{2}\,|A_0u^\delta|^2 \le \frac{27M_1^4}{4\nu^3}|u^0|_H^2|u^0|_V^2|u^\delta|_V^2 + \frac{1}{\nu}\,|Pf(t)|_H^2. \tag{5.3}$$

for the two dimensional case. Multiplying the both side of (5.2) by $|u^\delta|_H^2 + |u^0|_H^2$, we have

$$\frac{|u^\delta|_H^4 - |u^0|_H^4}{\delta} + \nu\,(|u^\delta|_H^2 + |u^0|_H^2)|u^\delta|_V^2 \le \frac{1}{\nu}\,(|u^\delta|_H^2 + |u^0|_H^2)\,|f(t)|_{V^*}^2$$

Since $s \to log(1+s)$ is concave

$$c_0\nu^4\,\frac{log(1+|u^\delta|_V^2) - log(1+|u^0|_V^2)}{\delta}$$

$$\le c_0\nu^4\delta^{-1}\,\frac{|u^\delta|_V^2 - |u^0|_V^2}{1+|u_0|_V^2} \le 2\nu\,\frac{|u^0|_H^2|u^0|_V^2|u^\delta|_V^2 + 2\nu^{-1}|Pf(t)|_H^2}{1+|u^0|_V^2}$$

where we set $c_0 = \dfrac{4}{27M_1^4}$. Thus, if we define

$$\varphi(u) = 2\,|u|_H^2 + c_0\nu^4\,log(1+|u|_V^2)$$

then for every $\delta > 0$

$$\frac{\varphi(u^\delta) - \varphi(u^0)}{\delta} \le b$$

for some constant $b \ge 0$, since $|u_i^\lambda|_H$ is uniformly bounded. For (1.5) if $y_i^\lambda = (h_i^\lambda)^{-1}(u_i^\lambda - u_{i-1}^\lambda)$, then

$$(y_{i+1}^\lambda - y_i^\lambda, y_{i+1}^\lambda) + \nu h_{i+1}^\lambda\,|y_{i+1}^\lambda|^2 + h_i^\lambda\,b(y_i^\lambda, u_i^\lambda, y_{i+1}^\lambda) + (f(t_{i+1}^\lambda) = f(t_i^\lambda), y_{i+1}^\lambda)$$

Note that

$$\begin{aligned} |b(y_i^\lambda, u_i^\lambda, y_{i+1}^\lambda)| &\le M_1|u_i^\lambda|_V\,|y_i^\lambda|_H^{\frac{1}{2}}|y_i^\lambda|_V^{\frac{1}{2}}|y_{i+1}^\lambda|_H^{\frac{1}{2}}|y_{i+1}^\lambda|_V^{\frac{1}{2}} \\ &\le \frac{\nu}{2}\,|y_{i+1}^\lambda|_V^2 + \frac{\nu}{2}\,|y_{i+1}^\lambda|_V^2 + \frac{M_1^2|u_i^\lambda|_V^2}{8\nu}\,|y_{i+1}^\lambda|_V^2 + \frac{M_1^2|u_i^\lambda|_V^2}{8\nu}\,|y_{i+1}^\lambda|_V^2. \end{aligned} \tag{5.4}$$

For simplicity of our discussions we assume $h_i^\lambda = h$ Then we have

(5.5) $(1-h\,\omega_\alpha)\,|y_{i+1}^\lambda|_H^2+h\nu\;|y_{i+1}^\lambda|_V^2+\leq(1+h\,\omega_\alpha)\,|y_{i+1}^\lambda|_H^2+\nu\,|y_{i+1}^\lambda|_V^2+|f(t_{i+1}^\lambda-f(t_i^\lambda)|_H|y_{i+1}^\lambda|_H$

which implies that $|y_i^\lambda|_H \leq M$ for some constant independent of $h > 0$.

Note that

$$|A(t,u)u - A(t,v)u|_H = \sup_{|phi|_H \leq 1} |b(u,u,\phi) - b(v,u,\phi)| \leq |u-v|_V^{\frac{1}{2}}|u-v|_H^{\frac{1}{2}}|u|_V^{\frac{1}{2}}|A_0 u|_H^{\frac{1}{2}}$$

and from (5.3)

$$\frac{\nu}{2}\sum_{i=1}^N h_i^\lambda |A_0 u_i^\lambda|_H^2 \leq C$$

for some constant C independent of N and h_i^λ. Since $|y_i^\lambda|_H \leq M$, $|u_i^\lambda - u_{i-1}^\lambda| \leq M\,h_i^\lambda$. Thus (1.5) holds and (1.1) has a unique integrable solution u and $\lim_{\lambda\to 0^+} u_\lambda = u$ uniformly on $[s,T]$.

Next, we consider the three dimensional problem ($d = 3$). We show that there exists a locally defined solution and a global solution exists when the data $(u_0, f(\cdot))$ are small.

We have the corresponding estimate of (5.3):

$$\frac{1}{2\delta}\,(|u^\delta|_V^2 - |u^0|_V^2) + \frac{\nu}{2}\,|A_0 u^\delta|^2 \leq \frac{27M_2^4}{4\nu^3}|u^0|_V^4|u^\delta|_V^2 + \frac{1}{\nu}\,|Pf(t)|_H^2. \tag{5.6}$$

We define the functional φ by

$$\varphi(u) = 1 - (1 + |u|_V^2)^{-1}$$

Since $s \to 1-(1+s)^{-1}$ is concave and $|u_i^\lambda|_H$ is uniformly bounded, it follows from (5.2) and (5.6) that for every $\delta > 0$

$$\frac{\varphi(u^\delta) - \varphi(u^0)}{\delta} \leq (\frac{27M_2^4}{4\nu^3}|u^0|_V^4|u^\delta|_V^2 + \frac{1}{\nu}\,|Pf(t)|_H^2)(1+|u^0|^2)^{-2}. \tag{5.6}$$

Thus

$$\frac{\varphi(u_i^\lambda) - \varphi(u_{i-1}^\lambda)}{h_i^\lambda} \leq (\frac{27M_2^4}{4\nu^3}|u^0|_V^4|u^\delta|_V^2 + \frac{1}{\nu}\,|Pf(t)|_H^2)(1+|u^0|^2)^{-2} \leq b_i$$

where $b_i = (\frac{27M_2^4}{4\nu^3}\,|u_i^\lambda|_V^2 + \frac{c_1}{\nu}$ and from (5.3)

$$\sum_{i=1}^N h_i^\lambda\,|u_i^\lambda|_V^2 \leq \frac{1}{\nu}\,(|u_0|_H^2 + \frac{c_1 T}{\nu}).$$

The estimate (5.4) is replaced by

$$|b(y_i^\lambda, u_i^\lambda, y_{i+1}^\lambda)| \leq M_1|u_i^\lambda|_V\,|y_i^\lambda|_H^{\frac{1}{4}}|y_i^\lambda|_V^{\frac{3}{4}}|y_{i+1}^\lambda|_H^{\frac{1}{4}}|y_{i+1}^\lambda|_V^{\frac{3}{4}}$$

$$\leq \frac{\nu}{2}\,|y_{i+1}^\lambda|_V^2 + \frac{\nu}{2}\,|y_{i+1}^\lambda|_V^2 + \frac{3M_1^2|u_i^\lambda|_V^2}{32\nu}\,|y_{i+1}^\lambda|_V^2 + \frac{3M_1^2|u_i^\lambda|_V^2}{32\nu}\,|y_{i+1}^\lambda|_V^2.$$

Thus, if $h_i^\lambda = h$ then we obtain (5.5) and thus $|y_i^\lambda|_H \leq M$ for some constant independent of $h > 0$ and (1.5) holds. $\square$

3.4 Approximation theory

In this section we discuss the approximation theory of the mild solution to (3.1). Consider an evolution equation in X_n:

$$\frac{d}{dt}u_n(t) \in A_n(t)u_n(t) \quad t > s; \quad u_n(s) = x \tag{3.1}$$

where X_n is a linear closed subspace of X. We consider the family of approximating sequences $(A_n(t), dom\,(A_n(t)), \varphi, D)$ on X_n satisfying the uniform dissipativity: there exist constant $\omega = \omega_\alpha$ continous functions $f : [0,T] \to X$ and $L : R^+ \to R^+$ independent of $t,\ s \in [0,T]$ and n such that

$$(1 - \lambda\omega_\alpha)\,|x_1 - x_2| \le |x_1 - x_2 - \lambda\,(y_1 - y_2)| + \lambda\,|f(t) - f(s)|L(x_2)|K(|y_2|) \tag{3.2}$$

for all $x_1 \in D_\alpha \cap dom\,(A_n(t))$ $x_2 \in D_\alpha \cap dom\,(A_n(s))$and $y_1 \in A_n(t)x_1,\ y_2 \in A_n(s)x_2$, and the consistency:

$$\begin{aligned} &\text{for } \beta > 0,\ t \in [0,T],\ \text{and } [x,y] \in A(t) \text{ with } x \in D_\beta, \\ &\text{there exists } [x_n, y_n] \in A_n(t) \text{ with } x_n \in D_{\alpha(\beta)} \text{ such that} \\ &\qquad \lim\ |x_n - x| + |y_n - y| = 0 \text{ as } n \to \infty. \end{aligned} \tag{3.3}$$

where $\alpha(\beta) \ge \beta$ and $\alpha : R^+ \to R^+$ is an increasing function.

Lemma 3.1 Let $(A(t), X)$ satisfy (3.2)–(3.3) on $[0,T]$. For each $\lambda > 0$, we assume that $(t_i^\lambda, x_i^\lambda, y_i^\lambda, \epsilon_i^\lambda)$ satisfies

$$y_j^\lambda = \frac{x_i^\lambda - x_{i-1}^\lambda}{t_i^\lambda - t_{i-1}^\lambda} - \epsilon_i^\lambda \in A(t_i^\lambda)x_i^\lambda$$

with $t_0^\lambda = s$ and $x_0^\lambda = x$ and $x^\lambda \in D_\alpha$ for $0 \le i \le N_\lambda$. We assume that $x \in D \times dom\,(A(s))$, f is of bounded variation, and $|y_i^\lambda|$ is uniformly bounded in $1 \le i \le N_\lambda$ and $\lambda > 0$. Then the step function $u_\lambda(t; s, x)$ defined by $u_\lambda(t, s, x) = x_i^\lambda$ on $(t_{i-1}^\lambda, t_i^\lambda]$, satisfies

$$|u_\lambda(t;s,x) - u(t;s,x)| \le e^{2\omega(2t+d_\lambda)}\,(2|x-u| + d_\lambda\,(|v| + \tilde{M}\rho(T))$$

$$+\tilde{M}(T-s)(c^{-1}\rho(T)d_\lambda + \rho(\delta) + \delta_\lambda).$$

Proof: It follows from Lemms 2.6–2.7 that there exists a DS-approximation sequence $(t_j^\mu, x_j^\mu, y_j^\mu, \epsilon_j^\mu)$ as defined in (2.12). For the case of $(C.2) - (R.1)$, from Lemma 2.4 we have $|y_j^\mu| \le \tilde{K}$ for $1 \le j \le N_\mu$ uniformly in μ. It thus follows from Theorem 2.5 that there exists a constant $\tilde{M}$ such that

$$|u_\lambda(t;s,x) - u_\mu(t;s,x)| \le e^{2\omega(2t+d_\lambda)}\,(2|x-u| + d_\lambda\,(|v| + \tilde{M}\rho(T))$$

$$+\tilde{M}(T-s)(c^{-1}\rho(T)d_\lambda + \rho(\delta) + \delta_\lambda - \delta_\mu).$$

Theorem 3.2 Let $(A_n(t), dom\,(A_n(t)), \varphi, D)$ be approximating sequences satisfying (R) or $(R.1)$ (resp.) and (3.2)–(3.3) and we assume $(A(t), dom\,(A(t)), D, \varphi)$ satisfies (A.1)–(A.2) and (R) or $(R.1)$ (resp.). Then for every $x \in D \cap \overline{dom\,(A(s))}$ and $x_n \in D \cap X_n$ such that $\lim x_n = x$ as $n \to \infty$ we have

$$\lim |u_n(t; s, x_n) - u(t; s, x)| = 0, \quad \text{as } n \to \infty$$

uniformly on $[s, T]$, where $u(t; s, x)$ and $u_n(t; s, x)$ is the unique mild solution to (2.1) and (3.1), respectively.

Proof: Let $[x_i, y_i] \in A(t_i)$ and $x_i \in D_\alpha$ for $i = 1,\ 2$. From (3.3) we can choose $[x_i^n, y_i^n] \in A^n(t_i),\ i = 1, 2$ with $x_i \in D_{\alpha'}$ such that $|x_i^n - x_i| + |y_i^n - y_i| \to 0$ as $n \to \infty$ for $i = 1,\ 2$. Thus, letting $n \to \infty$ in (3.2), we obtain $(C.1)$ or $(C.2)$. Let $(t_i^\lambda, x_i^\lambda, y_i^\lambda)$ be a DS–approximation sequence of (2.1). We assume that there exits a $\beta > 0$ such that $x_i^\lambda \in D_\beta$ for all λ and $1 \le i \le N_\lambda$. By the consistency (3.3) for any $\epsilon > 0$ there exists an integer $n = n(\epsilon)$ such that for $n \ge n(\epsilon)$

$$|x_i^{\lambda,n} - x_i^\lambda| \le \epsilon, \quad |y_i^{\lambda,n} - y_i^\lambda| \le \epsilon \tag{3.4}$$

and $x_i^{\lambda,n} \in D_\alpha$ for $1 \le i \le N_\lambda$.

$$\begin{aligned} &\textstyle\sum_{i=1}^{N_\lambda} |x_i^{\lambda,n} - x_{i-1}^{\lambda,n} - h_i^\lambda\, y_i^{\lambda,n}| \\ &\quad \le \textstyle\sum_{i=1}^{N_\lambda} |x_i^\lambda - x_{i-1}^\lambda - h_i^\lambda\, y_i^\lambda| + (N_\lambda + T)\,\epsilon + |x_n - x| = \delta_{n,\lambda,\epsilon} \end{aligned} \tag{3.5}$$

By Theorems 2.5 and 2.8 that

$$\begin{aligned} |u_{\lambda,n}(t, x_n) - u_n(t; x_n)| &\le e^{2\omega(2t+d_\lambda)}\,(2|x_n - u_n| + d_\lambda\,(|v_n| + \tilde{M}\rho(T)) \\ &\quad + \tilde{M}(T - s)(c^{-1}\rho(T)d_\lambda + \rho(\delta) + \delta_{\lambda,n,\epsilon}) \end{aligned} \tag{3.6}$$

for all $0 < c < \delta < T$ and $[u_n, v_n] \in A_n(s)$ with $u_n \in D_\alpha$. From the definition of function $u_\lambda,\ u_{\lambda,n}$ and (3.4)

$$|u_\lambda(t; s, x) - u_{\lambda,n}(t; s, x_n)| \le \epsilon, \quad t \in (s, T],\ \ n \ge n(\epsilon)$$

Thus, we have

$$\begin{aligned} |u_n(t; s, x_n) - u(t; , s, x)| &\le e^{2\omega(2t+d_\lambda)}\,(2|x_n - u_n| + 2|x - u| + d_\lambda\,(|v_n| + |v| + 2\tilde{M}\rho(T)) \\ &\quad + 2\tilde{M}(T - s)(c^{-1}\rho(T)d_\lambda + \rho(\delta)) + \delta_{\lambda,n,\epsilon}) + \epsilon \end{aligned}$$

for $t \in (s, T],\ \ n \ge n(\epsilon)$, where $[u, v] \in A(s)$ with $u \in D_\beta$. From the consistency (3.3) we can take $[u_n, v_n] \in A_n(s)$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$. It thus follows that

$$\begin{aligned} \lim\nolimits_{n\to\infty} |u_n(t; s, x_n) - u(t; s, x)| &= e^{2\omega(2t+d_\lambda)}\,(4|x - u| + 2d_\lambda\,(|v| + \tilde{M}\rho(T)) \\ &\quad + 2\tilde{M}(T - s)(c^{-1}\rho(T)d_\lambda + \rho(\delta)) + \delta_{\lambda,\epsilon}) + \epsilon \end{aligned}$$

for $t \in [s,T]$, where $\delta_{\lambda,\epsilon} = (N_\lambda + T)\epsilon$. Now, letting $\epsilon \to 0^+$ and then $\lambda \to 0^+$, we obtain

$$\lim_{n\to\infty} |u_n(t;s,x_n) - u(t;s,x)| = e^{4\omega t}\,(4|x-u| + \rho(\delta)).$$

Since $u \in D_\beta \cap dom\,(A(s))$ and $\delta > 0$ are arbitrary it follows that $\lim_{n\to\infty} |u_n(t;s,x_n) - u(t;s,x)| = 0$ uniformly on $[s,T]$. □

The following theorems give the equivalent characterization of the consistency condition (3.3).

Theorem 3.2 Let (A_n, X_n), $n \ge 1$ and (A,X) be dissipative operators satisfying the condition

$$\overline{dom\,(A_n)} \subset R(I - \lambda\,A_n) \quad \text{and} \quad \overline{dom\,(A)} \subset R(I - \lambda\,A)$$

and set $J^n_\lambda = (I - \lambda\,A_n)^{-1}$ and $J_\lambda = (I - \lambda\,A)^{-1}$ for $\lambda > 0$. Also, let B be the operator that has $\{[J_\lambda x, \lambda^{-1}(J_\lambda x - x)] : x \in \overline{dom\,(A)},\ 0 < \lambda < \omega^{-1}\}$ as its graph. Then the following statements (i) and (ii) are equivalent.

(i) $B \subset \lim_{n\to\infty} A_n$ (i.e., for all $[x,y] \in B$ there exists a sequence $\{(x_n,y_n)\}|$ such that $[x_n,y_n] \in A_n$ and $\lim\,|x_n - x| + |y_n - y| \to 0$ as $n \to \infty$.)

(ii) $\overline{dom\,(A)} \subset \lim_{n\to\infty} \overline{dom\,(A_n)}$ (equivalently, for all $x \in dom\,(A)$ there exists a sequence $\{x_n\}$ such that $x_n \in dom\,(A_n)$ and $\lim\ |x_n - x| \to 0$ as $n \to \infty$.) and for all $x_n \in \overline{dom\,(A_n)}$ and $x \in \overline{dom\,(A)}$ with $x = \lim_{n\to\infty} x_n$, we have $\lim_{n\to\infty} J^n_\lambda x_n \to J_\lambda x$ for each $0 < \lambda < \omega^{-1}$.

In particular, if $\overline{dom\,(A)} \subset \overline{dom\,(A_n)}$ for all n, then the above are equivalent to

(iii) $\lim_{n\to\infty} J^n_\lambda x \to J_\lambda x$ for all $0 < \lambda < \omega^{-1}$ and $x \in \overline{dom\,(A)}$.

Proof: $(i) \to (ii)$. Assume (i) holds. Then it is easy to prove that $\overline{dom\,(B)} \subset \lim_{n\to\infty} \overline{dom\,(A_n)}$. Since $J_\lambda x \to x$ as $lambda \to 0^+$ for all $x \in \overline{dom\,(A)}$, it follows that $\overline{dom\,(B)} = \overline{dom\,(A)}$. Thus, the first assertion of (ii) holds. Next, we let $x_n \in \overline{dom\,(A_n)}$, $x \in \overline{dom\,(A)}$ and $\lim_{n\to\infty} x_n = x$. From (i), we have $I - \lambda\,B \subset \lim_{n\to\infty}\ I - \lambda\,A_n$. Thus, $R(I - \lambda\,B) \subset \lim_{n\to\infty}\ R(I - \lambda\,A_n)$. Since $x = J_\lambda x - \lambda\{\lambda^{-1}(J_\lambda x - x)\} \in (I - \lambda\,B)J_\lambda x$ for $x \in \overline{dom\,(A)}$, we have $\overline{dom\,(A)} \subset R(I - \lambda\,B)$. From (i) we can choose $[u_n, v_n] \in A_n$ such that $\lim\,|u_n - J_\lambda x)| + |v_n - \lambda^{-1}(J_\lambda x - x)| \to 0$ as $n \to \infty$. If $z_n = J^n_\lambda x_n$, then there exists a $[z_n, y_n] \in A_n$ such that $x_n = z_n - \lambda y_n$. It thus follows from the dissipativity of A_n that

$$|u_n - z_n| \le |u_n - z_n - \lambda(v_n - y_n)| = |u_n - \lambda v_n - x_n| \to |J_\lambda x - (J_\lambda x - x) - x| = 0$$

as $n \to \infty$. Hence, we obtain

$$\lim J_\lambda x_n = \lim z_n = \lim u_n = J_\lambda x \quad \text{as } n \to \infty.$$

$(ii) \to (i)$. If $[u,v] \in B$, then from the definition of B, there exist $x \in \overline{dom\,(A)}$ and $0 < \lambda < \omega^{-1}$ such that $u = J_\lambda x$ and $v = \lambda^{-1}(J_\lambda x - x)$. Since $x \in \overline{dom\,(A)} \subset \lim_{n\to\infty} \overline{dom\,(A_n)}$ we can choose $x_n \in \overline{dom\,(A_n)}$ such that $\lim x_n = x$. Thus from the assumption, we have $\lim J^n_\lambda x_n = j_\lambda x = u$ and $\lim \lambda^{-1}(J^n_\lambda x_n - x_n) = \lambda^{-1}(J_\lambda x - x) = v$ as $n \to \infty$. Hence we obtain $[u,v] \in \lim_{n\to\infty}\ A_n$ and thus $B \subset \lim_{n\to\infty}\ A_n$.

Finally, if $\overline{dom\,(A)} \subset \overline{dom\,(A_n)}$, then $(ii) \to (iii)$ is obvious. Conversely, if (iii) holds, then

$$|J^n_\lambda x_n - J_\lambda x| \le |x_n - x| + |J^n_\lambda x - J_\lambda x| \to 0 \quad \text{as } n \to \infty$$

when $\lim x_n = x$, and thus (ii) holds. $\square$

Theorem 3.3 Let $(A_n, X_n),\ n \geq 1$ and (A, X) be m-dissipative operators, i.e.,

$$X_n = R(I - \lambda\, A_n) \quad \text{and} \quad X = R(I - \lambda\, A)$$

and set $J_\lambda^n = (I - \lambda\, A_n)^{-1}$ and $J_\lambda = (I - \lambda\, A)^{-1}$ for $0 < \lambda < \omega^{-1}$. Then the following statements are equivalent.

(i) $A = \lim_{n\to\infty}\ A_n$

(ii) $A \subset \lim_{n\to\infty}\ A_n$

(iii) For all $x_n,\ x \in X$ such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty}\ J_\lambda^n x_n \to J_\lambda x$ for each $0 < \lambda < \omega^{-1}$.

(iv) For all $x \in X$ and $0 < \lambda < \omega^{-1}$ $\lim_{n\to\infty}\ J_\lambda^n x \to J_\lambda x$.

(v) For some $0 < \lambda_0 < \omega^{-1}$, $\lim_{n\to\infty}\ J_\lambda^n x \to J_\lambda x$ for all $x \in X$.

Proof: $(i) \to (ii)$ and $(iv) \to (v)$ are obvious. $(ii) \to (iii)$ follows from the proof of $(i) \to (ii)$ in Theorem 3.2.

$(v) \to (ii)$. If $[x, y] \in A$, then from (v), $\lim J_{\lambda_0}^n(x - \lambda_0\, y) = J_{\lambda_0}(x - \lambda_0\, y) = x$ and $\lim \lambda_0^{-1}(J_{\lambda_0}^n(x - \lambda_0\, y) - (x - \lambda_0\, y)) = y$ as $n \to \infty$. Since $\lambda_0^{-1}(J_{\lambda_0}^n(x - \lambda_0\, y) - (x - \lambda_0\, y)) \in A_n J_{\lambda_0}^n(x - \lambda_0\, y)$, $[x, y] \in \lim_{n\to\infty}\ A_n$ and thus (ii) holds.

$(ii) \to (i)$. It suffices to show that $\lim_{n\to\infty}\ A_n \subset A$. If $[x, y] \in \lim_{n\to\infty}\ A_n$ then there exists a $[x_n, y_n] \in A_n$ such that $\lim |x_n - x| + |y_n - y| = 0$ as $n \to \infty$. Since $x_n - \lambda\, y_n \to x - \lambda\, y$ and $n \to \infty$, it follows from (iii) that $x_n = J_\lambda^n(x_n - \lambda\, y_n) \to J_\lambda(x - \lambda\, y)$ as $n \to \infty$ and hence $x = J_\lambda(x - \lambda\, y) \in dom\,(A)$. But, since

$$\lambda^{-1}(x - (x - \lambda\, y)) = \lambda^{-1}(J_\lambda(x - \lambda\, y) - (x - \lambda\, y)) \in AJ_\lambda(x - \lambda\, y) = Ax$$

we have $[x, y] \in A$ and thus (i) holds. $\square$

The following corollary is an immediate consequence of Theorems 3.1 and 3.3.

Corollary 3.4 Let $(A_n(t), X_n)$ and $(A(t), X)$ be m-dissipative operators for $t \in [0, T]$, (i.e.,

$$X_n = R(I - \lambda\, A_n(t)) \quad \text{and} \quad X = R(I - \lambda\, A(t))$$

and $(C.2)$ is satisfied), and let $U_n(t, s),\ U(t, s)$ be the nonlinear semigroups generated by $A_n(t),\ A(t)$, respectively. We set $J_\lambda^n(t) = (I - \lambda\, A_n(t))^{-1}$ and $J_\lambda(t) = (I - \lambda\, A(t))^{-1}$ for $0 < \lambda < \omega^{-1}$. Then, if

$$J_{\lambda_0}^n x_n \to J_{\lambda_0} x \quad \text{as } \ n \to \infty$$

for all sequence $\{x_n\}$ satisfying $x_n \to x$ as $n \to \infty$, then

$$|U_n(t, s)x_n - U(t, s)x| \to 0 \quad \text{as } \ n \to \infty$$

where the convergence is uniform on arbitrary bounded subintervals.

The next corollary is an extension of the Trotter-Kato theorem on the convergence of linear semigroups to the nonlinear evolution operators.

Corollary 3.5 Let $(A_n(t), X)$ be m-dissipative operators and $\{U_n(t, s),\ t \geq s \geq 0\}$ be the semigroups generated by $A_n(t)$. We set $J_\lambda^n(t) = (I - \lambda\, A_n(t))^{-1}$. For some $)\lambda_0 < \omega^{-1}$, we assume that there exists lim $J^n_{\lambda_0}(t)x$ exits as $n \to \infty$ for all $x \in X$ and $t \geq 0$ and denote the limit by $J_{\lambda_0}x$. Then we have

(i) The operator $A(t)$ defined by the graph $\{[J_{\lambda_0}(t)x, \lambda_0^{-1}(J_{\lambda_0}(t)x - x)] : x \in X\}$ is an m-dissipative operator. Hence $(A(t), X)$ generates a nonlinear semigroup $U(t,s)$ on X.

(ii) For every $x \in \overline{R(J_{\lambda_0}(s))}$ $(= \overline{dom\,(A(s))})$, there exist a $x_n \in dom\,(A_n(s))$ such that lim $x_n = x$ and lim $U(t,s)x_n = U(t,s)x$ for $t \geq s \geq 0$ as $n \to \infty$. Moreover, the above convergence holds for every $x_n \in \overline{dom\,(A_n(s))} = X$ satisfying lim $x_n = x$, and the convergence is uniform on arbitrary bounded intervals.

Proof: (i) Let $[u_i, v_i] \in A(t_i),\ i = 1,2$. Then from the definition of $A(t)$ we have

$$u_i = J_{\lambda_0}(t_i)x_i \quad \text{and} \quad v_i = \lambda_0^{-1}\left(J_{\lambda_0}(t_i)x_i - x_i\right)$$

for $x_i,\ i = 1,2$. By the assumption

$$\lim J^n_{\lambda_0}(t_i)x_i = J_{\lambda_0}(t_i)x_i \qquad \lim \lambda_0^{-1}(J^n_{\lambda_0}(t_i)x_i - x_i) = \lambda_0^{-1}(J_{\lambda_0}(t_i)x_i - x_i)$$

as $n \to \infty$ for $i = 1,2$. Since $\lambda_0^{-1}(J^n_{\lambda_0}(t_i)x_i - x_i) \in A_n(t_i)J^n(J^n_{\lambda_0}(t_i)x_i)$, it follows from (C.2) that

$$(1 - \lambda\omega)\,|J^n_{\lambda_0}(t_1)x_1 - J^n_{\lambda_0}(t_2)x_2|$$

$$\leq |J^n_{\lambda_0}(t_1)x_1 - J^n_{\lambda_0}(t_2)x_2 - \lambda\,(\lambda_0^{-1}(J^n_{\lambda_0}(t_1)x_1 - x_1) - \lambda_0^{-1}(J^n_{\lambda_0}(t_2)x_2 - x_2)|$$

$$+\lambda\,|f(t) - f(s)|L(|J^n_{\lambda_0}(t_2)x_2|)(1 + |\lambda_0^{-1}(J^n_{\lambda_0}(t_2)x_s - x_2)|).$$

Letting $n \to \infty$, we obtain

$$(1 - \lambda\omega)\,|u_1 - u_2| \leq |u_1 - u_2 - \lambda\,(v_1 - v_2)| + \lambda\,|f(t) - f(s)|L(|u_2|)(1 + |v_2|).$$

Hence, $(A(t), X)$ is dissipative. Next, from the definition of $A(t)$, for every $x|inX$ we have $x = J_{\lambda_0}(t)x - \lambda_0(\lambda_0^{-1}(J_{\lambda_0}(t)x - x)) \in (I - \lambda_0\,A(t))J_{\lambda_0}x$, which implies $J_{\lambda_0}(t)x = (I - \lambda\,A(t))^{-1}x$. Hence, we obtain $R(I - \lambda_0\,A(t)) = X$ and thus $A(t)$ is m-dissipative.

(ii) Since $A(t)$ is an m-dissipative operator and $J_{\lambda_0}x = (I - \lambda_0\,A(t))^{-1}$ and $\overline{dom\,(A(t))} = \overline{R((J_{\lambda_0}(t))}$, it follows from Corollary 3.4 that $U_n(t,s)x_n \to U(t,s)x$ as $n \to \infty$ if $x \in \overline{dom\,(A(s))}$ and $x_n \to x$ as $n \to \infty$. □

3.5 Chernoff Theorem

In this section we discuss the Chernoff theorem for the evolution equation (2.1).

Lemma 2.12 Let $\{T_\rho(t)\},\ t \in [0,T]$ for $\rho > 0$ be a family of mapping from D into itself satisfying

$$T_\rho(t) : D_\beta \to D_{\psi(\rho,\beta)},$$

for $t \in [0,T]$ and $\beta \geq 0$,

$$|T_\rho(t)x - T_\rho(t)y| \leq (1 + \omega_\alpha\rho)\,|x - y|$$

for $x,\ y \in D_\alpha$, and

$$|A_\rho(t)x - A_\rho(s)x| \le L(|x|)(1 + |A_\rho(s)x|)\,|f(t) - f(s)|$$

for $x \in D$ and $t,\ s \in [0,T]$, where

$$A_\rho(t)x = \frac{1}{\rho}(T_\rho(t)x - x), \quad x \in D$$

Then, the evolution operator $A_\rho(t),\ t \in [0,T]$ satisfies $(C.2)$.

Proof: For $x_1,\ x_2 \in D_\alpha$

$$|x_1 - x_2 - \lambda\,(A_\rho(t)x_1 - A_\rho(s)x_2)|$$

$$\ge (1 + \frac{\lambda}{\rho})\,|x_1 - x_2| - \frac{\lambda}{\rho}\,(|T_\rho(t)x_1 - T_\rho(t)x_2|) + |T_\rho(t)x_2 - T_\rho(s)x_2|$$

$$\ge (1 - \lambda\omega_\alpha)\,|x_1 - x_2| - \lambda\,L(|x_2|)(1 + |A_\rho(s)x_2|)\,|f(t) - f(s)|.\square$$

Lemma 2.14 Let X_0 be a closed convex subset of a Banach space X and $\alpha \ge 1$. Suppose $C(t) : X_0 \to X_0,\ t \in [0,T]$ be a family of evolution operators satisfying

$$|C(t)x - C(t)y| \le \alpha\,|x - y| \tag{2.30}$$

for $x,\ y \in X_0$, and

$$|C(t)x - C(s)x| \le L(|x|)(\delta + |(C(s) - I)x|)\,|g(t) - g(s)|. \tag{2.31}$$

where $\delta > 0$ and $g : [0,T] \to X$ is continuous. Then, there exists a unique function $u = u(t;x) \in C^1(0,T;X_0)$ satisfying

$$\frac{d}{dt}u(t) = (C(t) - I)u(t), \quad u(0) = x, \tag{2.32}$$

and we have

$$|u(t;x) - u(t;y)| \le e^{(\alpha-1)t}\,|x - y|. \tag{2.33}$$

Proof: It suffices to prove that there exists a unique function $u \in C(0,T;X_0)$ satisfying

$$u(t) = e^{-t}x + \int_0^t e^{-(t-s)}C(s)u(s)\,ds, \quad t \in [0,T]. \tag{2.34}$$

First, note that if $v(t) : [0,T] \to X_0$ is continuous, then for $t \in [0,T]$

$$e^{-t}x + \textstyle\int_0^t e^{-(t-s)}C(s)v(s)\,ds$$

$$= (1 - \lambda)x + \lambda\left(\frac{\int_0^t e^{-(t-s)}C(s)v(s)\,ds}{\int_0^t e^{-(t-s)}\,ds}\right) \in X_0$$

where

$$\lambda = \int_0^t e^{-(t-s)}\, ds,$$

since X_0 is closed and convex. Define a sequence of functions $u_n \in C(0,T;X_0)$ by

$$u_n(t) = e^{-t}x + \int_0^t e^{-(t-s)}C(s)u_{n-1}(s)\, ds$$

with $u_0(t) = x \in X_0$. By induction we can show that

$$|u_{n+1}(t) - u_n(t)| \le \frac{(\alpha t)^n}{n!}\int_0^t |C(s)x - x|\, ds$$

and thus $\{u_n\}$ is a Cauchy sequence in $C(0,T;X_0)$. It follows that u_n converges to a unique limit $u \in C(0,T;X_0)$ and u satisfies (2.34). Also, it is easy to show that there exists a unique continuous function that satisfies (2.34). Moreover since for $x,\ y \in X_0$

$$e^t\,|u(t;x) - u(t;y)| \le \int_0^t \alpha\, e^s|u(s;x) - u(s;y)|\, ds,$$

by Gronwall's inequality we have (2.33). Since $s \to C(s)u(s) \in X_0$ is continuous, $u \in C^1(0,T;X_0)$ satisfies (2.32). □

Theorem 2.15 Let $\alpha \ge 1$ and let $C(t) : D_\beta \to D_\beta,\ t \in [0,T]$ be a family of evolution operators satisfying (2.30)–(2.31). Assume that X_β is a closed convex subset of X and g is Lipschitz continuous with Lipschitz constant L_g. Then, if we let $u(t) = u(t;x)$ be the solution to

$$\frac{d}{dt}u(t;x) = (C(t) - I)u(t;x), \quad u(0;x) = x,$$

then there exist some constants M_τ such that for $t \in [0,\tau]$

(2.35)

$$|u(t;x) - \Pi_{i=1}^n C_i x| \le \alpha^n e^{(\alpha-1)t}\,[(n-\alpha t)^2 + \alpha t]^{\frac{1}{2}}\,(C_{n,\tau} + |C(0)x - x|)$$

$$+L_g L(M_\tau)\int_0^t e^{(\alpha-1)(t-s)}\,[(n-s) - \alpha(t-s)^2 + \alpha(t-s)]^{\frac{1}{2}}\,K(|C(s)u(s) - u(s)|)\, ds$$

where $C_i = C(i),\ i \ge 0$ and $C_{n,\tau} = L(|x|)L_g(\delta + |C(0)x - x|)\,\max(n,\tau)$.

Proof: Note that

$$u(t;x) - x = \int_0^t e^{s-t}(C(s)u(s;x) - x)\, ds \tag{2.36}$$

Thus, from (2.30)–(2.31)

$$|u(t;x) - x| \le \int_0^t e^{s-t}(|C(s)u(s;x) - C(0)u(s,x)| + |C(0)u(s;x) - C(0)x| + |C(0)x - x|)\, ds$$

$$\le \int_0^t e^{s-t}(L(|u(s,x)|)|g(s) - g(0)|K(|C(0)x - x|) + |C(0)x - x| + \alpha\,|u(s;x) - x|)\, ds$$

where $K(s) = \delta + s$. Or, equivalently

$$e^t\,|u(t;x) - x| \le \int_0^t e^s(L(M)|g(s) - g(0)|K(|C(0)x - x|) + |C(0)x - x| + \alpha\, e^s|u(s;x) - x|)\,ds$$

since $|u(t,x)| \le M$ on $[0,\tau]$ for some $M = M_\tau$. Hence by Gronwall's inequality

$$(2.37)\quad |u(t;x) - x| \le \int_0^t e^{(\alpha-1)(t-s)}(L(M)|g(s) - g(0)|K(|C(0)x - x|) + |C(0)x - x|)\,ds.$$

It follows from (2.36) that

$$u(t;x) - \Pi_{i=1}^n C_i x = e^{-t}(x - \Pi_{i=1}^n C_i x) + \int_0^t e^{s-t}(C(s)u(s;x) - \Pi_{i=1}^n C_i x)\,ds.$$

Thus, by assumption

$$|u(t) - \Pi_{i=1}^n C_i x| \le e^{-t}|x - \Pi_{i=1}^n C_i x| + \int_0^t e^{s-t}(|C(s)u(s) - C_n u(s)| + |C_n u(s) - C_n \Pi_{i=1}^{n-1} C_i x|)\,ds$$

$$\le e^{-t}\alpha^n\delta_n + \int_0^t e^{s-t}(L(M)|g(s) - g(n)|\,|C(s)x - x| + \alpha\,|u(s) - \Pi_{i=1}^{n-1} C_i x|\,ds$$

where $\delta_n = \sum_{i=1}^n |C_i x - x|$. If we define

$$\varphi_n(t) = \alpha^{-n} e^t |u(t,x) - \Pi_{i=1}^n C_i x|$$

then

$$\varphi_n(t) \le \delta_n + \int_0^t (M e^s \alpha^{-n}|g(s) - g(n)| + \varphi_{n-1}(s))\,ds$$

By induction in n we obtain from (2.36)–(2.37)

(2.38)

$$\varphi_n(t) \le \sum_{k=0}^{n-1} \frac{\delta_{n-k} t^k}{k!} + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}\alpha s e^{\alpha s}\,ds\,(L(|x|)L_g \tau K(|C(0)x - x|) + |C(0)x - x|)$$

$$+ L(M)\alpha^{-n}\sum_{k=0}^{n}\int_0^t e^s \alpha^k \frac{(t-s)^k}{k!}\,|C(s)u(s) - u(s)|\,|g(s) - g(n-k)|\,ds$$

on $t \in [0,\tau]$, where we used

$$\int_0^t e^{(\alpha-1)s}\,ds \le \alpha t e^{-t} e^{\alpha t}.$$

Since

$$\int_0^t (t-s)^{n-1} s^{k+1}\,ds = t^{k+n+1}\frac{(k+1)!(n-1)!}{(k+n+1)!}.$$

we have

$$\int_0^t (t-s)^{n-1} s e^{\alpha s}\, ds = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \int_0^t (t-s)^{n-1} s^{k+1}\, ds$$

$$= (n-1)! \sum_{k=0}^{\infty} \frac{(k+1)\alpha^k t^{k+n+1}}{(k+n+1)!} = (n-1)! \sum_{k=n+1}^{\infty} \frac{(k-n)\alpha^{k-1} t^k}{k!}.$$

Note that from (2.31)

$$|C(k)x - x| \le L(|x|) L_g k K(|C(0)x - x|) \quad \text{for } 0 \le k \le n.$$

Let $C_{n,\tau} = L(|x|)L_g K(|C(0)x - x|) max(n, \tau)$. Then we have

(2.39)

$$\sum_{k=0}^{n-1} \frac{\delta_{n-k} t^k}{k!} + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} \alpha s e^{\alpha s}\, ds\, (L(|x|) L_g \tau K(|C(0)x - x|) + |C(0)x - x|)$$

$$\le \left(\sum_{k=0}^{\infty} \frac{|k-n| \alpha^k t^k}{k!} \right) (C_{n,\tau} + |C(0)x - x|) \le C e^{\alpha t} \left[(n - \alpha t)^2 + \alpha t \right]^{\frac{1}{2}} (C_{n,\tau} + |C(0)x - x|),$$

where we used

$$\sum_{k=0}^{\infty} \frac{|k-n| \alpha^k t^k}{k!} \le \frac{\alpha t}{2} \left(\sum_{k=0}^{\infty} \frac{|k-n|^2 \alpha^k t^k}{k!} \right)^{\frac{1}{2}} \le e^{\alpha t} [(n - \alpha t)^2 + \alpha t]^{\frac{1}{2}}$$

Moreover, we have

(2.40)

$$\begin{aligned} &\sum_{k=0}^{n} e^s \alpha^k \frac{(t-s)^k}{k!} |g(s) - g(n-k)| \\ &\le L_g e^s \sum_{k=0}^{n} \frac{\alpha^k (t-s)^k}{k!} |s - (n-k)| \\ &\le L_g e^s e^{\frac{\alpha(t-s)}{2}} \left(\sum_{k=0}^{\infty} \frac{|s-(n-k)|^2 \alpha^k (t-s)^k}{k!} \right)^{\frac{1}{2}} \\ &\le L_g e^s e^{\alpha(t-s)} \left[(n-s) - \alpha(t-s)^2 + \alpha(t-s) \right]^{\frac{1}{2}} \end{aligned}$$

Hence (2.35) follows from (2.36)–(2.40). □

Theorem 2.16 Let $\{T_\rho(t)\},\ t \in [0, T]$ for $\rho > 0$ be a family of mapping from D into itself satisfying

(2.41)
$$T_\rho(t) : D_\beta \to D_{\psi(\rho,\beta)},$$

for $t \in [0,T]$ and $\beta \geq 0$,

$$|T_\rho(t)x - T_\rho(t)y| \leq (1+\omega_\alpha\rho)\,|x-y| \tag{2.42}$$

for $x,\ y \in D_\alpha$, and

$$|A_\rho(t)x - A_\rho(s)x| \leq L(|x|)(1+|A_\rho(s)x|)\,|f(t)-f(s)| \tag{2.43}$$

for $x \in D$ and $t,\ s \in [0,T]$, where

$$A_\rho(t)x = \frac{1}{\rho}(T_\rho(t)x - x), \quad x \in D$$

Assume that D_β is a closed convex subset in X for each $\beta \geq 0$ and f is Lipschitz continuous on $[0,T]$ with Lipschitz constant L_f. Then, if we let $u_\rho(t) = u(t;x)$ be the solution to

$$\frac{d}{dt}u(t;x) = A_\rho(t)u(t;x), \quad u(0;x) = x \in D_\alpha, \tag{2.44}$$

then there exist constant M and $\omega = \omega_\alpha$ such that

$$|u_\rho(t) - \Pi_{k=1}^{[\frac{t}{\rho}]}T_\rho(k\rho)x| \leq\leq e^{\omega t}\left[(1+\omega t)^2\rho + \omega t\rho + t\right]^{\frac{1}{2}}\left(e^{\omega t}\left(L(|x|L_f t(1+|A_\rho(0)x|) + |A_\rho(0)x|\right)\right.$$

$$\left. +L_f L(M)\int_0^t e^{\omega(t-\sigma)}(1+|A_\rho(\sigma)u_\rho(\sigma)|)\,d\sigma\right)\sqrt{\rho}.$$

for $x \in dom\,(A(0))$ and $t \in [0,T]$.

Proof: Let $\alpha = 1+\omega\rho$ and we let $C(t) = T_\rho(\rho t)$, $g(t) = f(\rho t)$ and $\delta = \rho$. Then, $C(t)$ satisfies (2.30)–(2.31). Next, note that $|u_\rho(t)| \leq M$. It thus follows from Lemma 2.14 and Theorem 2.15 that

$$|u_\rho(t) - \Pi_{k=1}^{[\frac{t}{\rho}]}T_\rho(k\rho)x| \leq e^{2\omega t}\left[(1+\omega t)^2\rho + \omega t\rho + t\right]^{\frac{1}{2}}\sqrt{\rho}\,((|A_\rho(0)x| + L(|x|)L_f t(1+|A_\rho(0)x|))$$

$$+L_f L(M)e^{\omega t}\int_0^t e^{\omega(t-\sigma)}[(1+\omega(t-\sigma))^2\rho + \omega(t-\sigma)\rho + (t-\sigma)]^{\frac{1}{2}}\sqrt{\rho}(1+|A_\rho(\sigma)u_\rho(\sigma)|)\,d\sigma.$$

where we set $s = \frac{\sigma}{\rho}$, since $L_g = L_f\rho$. $\square$

Theorem 2.17 Assume that $(A(t), dom\,(A(t)), D, \varphi)$ satisfies $(A.1)-(A.2)$ and either $(R)-(C.1)$ or $(2.3)-(C.2)$ and that D_β is a closed convex subset in X for each $\beta \geq 0$ and f is Lipschitz continuous on $[0,T]$. Let $\{T_\rho(t)\},\ t \in [0,T]$ for $\rho > 0$ be a family of mapping from D into itself satisfying (2.41)–(2.43) and assume the consistency

for $\beta \geq 0,\ t \in [0,T],$ and $[x,y] \in A(t)$ with $x \in D_\beta$

there exists $x_\rho \in D_\alpha(\beta)$ such that $\lim\ |x_\rho - x| + |A_\rho(t)x_\rho - y| = 0$ as $\rho \to 0$.

Then, for $0 \leq s \leq t \geq T$ and $x \in D \cap \overline{dom\,(A(s))}$

$$|\Pi_{k=1}^{[\frac{t-s}{\rho}]}T_\rho(k\rho+s)x - u(t;s,x)| \to 0 \quad \text{as} \quad \rho \to 0^+ \tag{2.45}$$

uniformly on $[0,T]$, where $u(t;s,x)$ is the mild solution to (2.1), defined in Theorem 2.10.

Proof: Without loss of generality we can assume that $s=0$. It follows from Lemma 2.4 and Lemma 2.12 that

$$|A_\rho(t)u(t)| \le e^{(\alpha-1)\,t+L(M)\,|f|_{BV(0,T)}}\,(|A_\rho(0)x| + L(M)\,|f|_{BV(0,T)}) \tag{2.46}$$

on $[0,T]$. Note that $u_\rho(t)$ satisfies

$$\frac{u_\rho(i\lambda)-u_\rho((i-1)\lambda)}{\lambda} = A_\rho(i\lambda)u_\rho(i\lambda)+\epsilon_i^\lambda$$

where

$$\epsilon_i^\lambda = \frac{1}{\lambda}\int_{(i-1)\lambda}^{i\lambda} (A_\rho(s)u_\rho(s)-A_\rho(i\lambda)u_\rho(i\lambda))\,ds.$$

Since $A_\rho(t):[0,T]\times X\to X$ and $t\to u_\rho(t)\in X$ are Lipschitz continuous, it follows that

$$\textstyle\sum_{i=1}^{N_\lambda}\lambda\,|\epsilon_i^\lambda|\to 0 \quad \text{as} \quad \lambda\to 0^+.$$

Hence $u_\rho(t)$ is the integral solution to (2.44). It thus follows from Theorem 3.2 that

$$\lim\,|u_\rho(t)-u(t;s,x)|_X\to 0 \quad \text{as} \quad \rho\to 0$$

for $x\in D\cap dom\,(A(s))$. Now, from Theorem 2.16 and (2.46)

$$|\Pi_{k=1}^{[\frac{t-s}{\rho}]}T_\rho(k\rho+s)x-u_\rho(t)|\le M\sqrt{\rho}$$

for $x\in D_\beta\cap dom\,(A(s))$. Thus, (2.45) holds for all for $x\in D_\beta\cap dom\,(A(s))$. By the continuity of the right-hand side of (2.45) with respect to the initial condition $x\in X$ (2.45) holds for all $x\in D_\beta\cap \overline{dom\,(A(s))}$. $\square$

The next corollary follows from Theorem 2.17 and is an extension of the Chernoff theorem of the autonomous nonlinear semigroups to the evolution case.

Corollary 2.18 (Chernoff Theorem Let C be a closed convex subset of X. We assume that the evolution operator $(A(t),X)$ satisfies (A.1) with Lipschtz continuous f and

$$C\subset R(I-\lambda\,A(t)) \quad \text{and} \quad (I-\lambda A(t))^{-1}\in C$$

for $0<\lambda\le\delta$ and $t\ge 0$. Let $\{T_\rho(t)\},\ t\ge 0$ for $\rho>0$ be a family of mapping from C into itself satisfying

$$|T_\rho(t)x-T_\rho(t)y|\le(1+\omega\rho)\,|x-y|$$

and

$$|A_\rho(t)x-A_\rho(s)x|\le L(|x|)(1+|A_\rho(s)x|)\,|f(t)-f(s)|$$

for $x,\ y\in C$. If $A(t)\subset\lim_{\rho\to 0^+}A_\rho(t)$, or equivalently

$$(I-\lambda\,\frac{T_\rho(t)-I}{\rho})^{-1}x\to(I-\lambda\,A(t))^{-1}x$$

for all $x \in C$ and $0 < \lambda \leq \delta$, then for $x \in C$ and $0 \leq s \leq t$

$$|\Pi_{k=1}^{[\frac{t-s}{\rho}]} T_\rho(s+k\rho)x - U(t,s)x| \to 0 \quad \text{as} \ \ \rho \to 0^+,$$

where $U(t,s)$ is the nonlinear semigroup generated by $A(t)$ and the convergence is uniform on arbitrary bounded intervals.

Corollary 2.19 (Chernoff Theorem) Assume that $(A(t), X)$ is m-dissipative and satisfy (A.1) with Lipschitz continuous f, and that $\overline{dom\,(A(t))}$ are independent of $t \in [0,T]$ and convex. Let $\{T_\rho(t)\},\ t \geq 0$ for $\rho > 0$ be a family of mapping from X into itself satisfying

$$|T_\rho(t)x - T_\rho(t)y| \leq (1+\omega\rho)\,|x-y|$$

for $x,\ y \in X$, and

$$|A_\rho(t)x - A_\rho(s)x| \leq L(|x|)(1+|A_\rho(s)x|)\,|f(t)-f(s)|$$

If $A(t) \subset \lim_{\rho\to 0^+}\ A_\rho(t)$, or equivalently

$$(I - \lambda_0\,\frac{T_\rho(t)-I}{\rho})^{-1}x \to (I-\lambda_0\,A(t))^{-1}x$$

for all $x \in X$, $t \geq 0$ and some $0 < \lambda_0 < \omega^{-1}$, then for $x \in X$ and $t \geq s \geq 0$

$$|\Pi_{k=1}^{[\frac{t-s}{\rho}]} T_\rho(s+k\rho)x - U(t,s)x| \to 0 \quad \text{as} \ \ \rho \to 0^+,$$

where $U(t,s)$ is the nonlinear semigroup generated by $A(t)$ and the convergence is uniform on arbitrary bounded intervals.

Theorem 2.20 Let $(A(t), X)$ be m-dissipative subsets of $X \times X$ and satisfy (A.1) with Lipschitz continuous f and assume that $\overline{dom\,(A(t))}$ are independent of $t \in [0,T]$ and convex. Let $A_\lambda(t) = \lambda^{-1}(J_\lambda(t) - I)$ for $\lambda > 0$ and $t \in [0,T]$ and $u(t;s,x) = U(t,s;A_\lambda)x$ be the solution to

$$\frac{d}{dt}u(t) = A_\lambda(t)u(t), \quad u(s) = x \in \overline{dom\,(A(s))}$$

Then, we have

$$U(t,s)x = \lim_{\lambda\to 0^+} \Pi_{k=1}^{[\frac{t}{\lambda}]} J_\lambda(s+k\,\lambda) = \lim_{\lambda\to 0^+} U(t,s;A_\lambda)x.$$

3.6 Operator Splitting

Theorem 3.1 Let X and X^* be uniformly convex and let $A_n,\ n \geq 1$ and A be m-dissipative subsets of $X \times X$. If for all $[x,y] \in A^0$ there exists $[x_n,y_n] \in A_n$ such that $|x_n - x| + |y_n - y| \to 0$ as $n \to \infty$, then

$$S_n(t)x_n \to S(t)x \quad \text{as} \ \ n \to \infty$$

for every sequence $x_n \in \overline{dom\,(A_n)}$ satisfying $x_n \to x \in \overline{dom\,(A)}$, where the convergence is uniform on arbitrary bounded intevals.

Proof: It follows from Theorem 1.4.2–1.4.3 that for $x \in dom\,(A)$, $S(t)x \in dom\,(A^0)$, $\frac{d^+}{dt}S(t)x = A^0S(t)x$, and $t \to A^0S(t)x \in X$ is continuous except a countable number of values $t \geq 0$. Hence if we define $x_i^\lambda = S(t_i^\lambda)x,\ t_i^\lambda = i\,\lambda$, then

$$\frac{x_i^\lambda - x_{i-1}^\lambda}{\lambda} - A^0x_i^\lambda = \epsilon_i^\lambda = \lambda^{-1}\int_{t_{i-1}^\lambda}^{t_i^\lambda} (A^0(t)x(t) - A(t_i^\lambda)x(t_i^\lambda))\,dt$$

where

$$\textstyle\sum_{i=1}^{N_\lambda} \lambda\,|\epsilon_i^\lambda| \leq \displaystyle\int_0^T |A^0S(t)x - A^0S(([\frac{t}{\lambda}]+1)\lambda|\,dt.$$

Since $|A^0S(t)x - A^0S(([\frac{t}{\lambda}]+1)\lambda)x| \to 0$ a.e. $t \in [0,T]$ as $\lambda \to 0^+$, by Lebesgue dominated convergence theorem $\sum_{i=1}^{N_\lambda} \lambda\,|\epsilon_i^\lambda| \to 0$ as $\lambda \to 0$. Hence it follows from the proof of Theorem 2.3.2 that $|S_n(t)x_n - S(t)x| \to 0$ as $\lambda \to 0$ for all $x_n \in dom\,(A)$ satisfying $x_n \to x \in dom\,(A)$. The theorem follows from the fact that $S(t)$ and $S_n(t)$ are of cotractions. $\square$

Theorem 3.2 Let X and X^* be uniformly convex and let A and B be two m-dissipative subests of $X \times X$. Assume that $A+B$ is m-dissipative and let $S(t)$ be the semigroup generated by $A+B$. Then we have

$$S(t)x = \lim_{\rho\to 0^+} \left((I-\rho\,A)^{-1}(I-\rho\,B)^{-1}\right)^{[\frac{t}{\rho}]}x$$

for $x \in \overline{dom\,(A)\cap dom\,(B)}$, uniformly in any t-bounded intervals. $\square$

Proof: Define $T_\rho = J_\rho^AJ_\rho^B$ and let $x_\rho = x - \rho\,b$ where $b \in Bx$. Then, since $J_\rho^B(x-\rho\,b) = x$

$$\frac{T_\rho x_\rho - x_\rho}{\rho} = \frac{J_\rho^Ax - x}{\rho} + b.$$

Hence $\rho^{-1}(T_\rho x_\rho - x_\rho) \to A^0x + b$ as $\rho \to 0^+$. If we choose $b \in Bx$ such that $A^0x + b = (A+B)^0x$, then $\rho^{-1}(T_\rho x_\rho - x_\rho) \to (A+B)^0x$ as $\rho \to 0^+$. Thus, the theorem follows from Theorem 3.1 and Corollay 2.3.6. $\square$

Theorem 3.3 Let X and X^* be uniformly convex and let A and B be two m-dissipative subests of $X \times X$. Assume that $A+B$ is m-dissipative and let $S(t)$ be the semigroup generated by $A+B$. Then we have

$$S(t)x = \lim_{\rho\to 0^+} \left((2(I-\frac{\rho}{2}\,A)^{-1} - I)(2(I-\frac{\rho}{2}\,B)^{-1} - I)\right)^{[\frac{t}{\rho}]}x$$

for $x \in \overline{dom\,(A)\cap dom\,(B)}$, uniformly in any t-bounded intervals.
Proof: Define $T_{2\rho} = (2J_\rho^A - I)(2J_\rho^B - I)$ and let $x_\rho = x - \rho\,b$ where $b \in Bx$. Then, since $J_\rho^B(x-\rho\,b) = x$, it follows that $2J_\rho^B(x_\rho) - x_\rho = x + \rho\,b$ and

$$\frac{T_{2\rho}x_\rho - x_\rho}{2\rho} = \frac{2J_\rho^A(x+\rho\,b) - (x+\rho\,b) - (x-\rho\,b)}{2\rho} = \frac{J_\rho^A(x+\rho\,b) - x}{\rho}.$$

If we define the subset E of $X\times X$ by $Ex = Ax+b$, then E is m-dissipative and $J_\rho^A(x+\rho\,b) = J_\rho^Ex$. Thus, it follows from Theorem 1.6 that $\rho^{-1}(T_\rho x_\rho - x_\rho) \to (A+B)^0x$ as $\rho \to 0^+$ and hence the theorem follows from Theorem 3.1 and Corollary 2.3.6. $\square$

Theorem 3.4 Let X and X^* be uniformly convex and let A and B be two m-dissipative subests of $X \times X$. Assume that A, B are single-valued and $A+B$ is m-dissipative. Let $S_A(t)$, $S_B(t)$ and $S(t)$ be the semigroups generated by A, B and $A+B$, respectively. Then we have

$$S(t)x = \lim_{\rho\to 0^+} \left(S_A(\rho)S_B(\rho)\right)^{[\frac{t}{\rho}]} x$$

for $x \in \overline{dom\,(A) \cap dom\,(B)}$, uniformly in any t-bounded intervals.

Proof: Clearly $T_\rho = S_A(\rho)S_B(\rho)$ is nonexpansive on $C = \overline{dom\,(A) \cap dom\,(B)}$. We show that $\lim_{\rho\to 0^+} h^{-1}(T_\rho x - x) = Ax + B^0x$ for every $dom\,(A) \cap dom\,(B)$. Note that

$$\frac{T_\rho x - x}{\rho} = \frac{S_A(\rho)x - x}{\rho} + \frac{S_A(\rho)S_B(\rho)x - S_A(\rho)x}{\rho}.$$

Since A is single-valued, it follows from Theorem 1.4.3 that $\lim_{\rho\to 0^+} h^{-1}(S_A(\rho)x - x) = Ax$. Thus, it suffices to show that

$$y_\rho = \frac{S_A(\rho)S_B(\rho)x - S_A(\rho)x}{\rho} \to B^0x \quad \text{as} \ \ \rho \to 0^+.$$

Since $S_A(\rho)$ is nonexpansive and B is dissipative, it follows from Theorem 1.4.3 that

(3.1) $$|y_\rho| \le |\frac{S_B(\rho)x - x}{\rho}| \le |B^0x| \quad \text{for all } \rho > 0.$$

On the other hand,

(3.2) $$\langle \frac{S_A(\rho)u - u}{\rho} + \frac{S_B(\rho)x - x}{\rho} - \frac{S_A(\rho)x - x}{\rho} - y_\rho, F(u - S_B(\rho)x)\rangle \ge 0$$

since $S_A(\rho) - I$ is dissipative and F is singled-valued. We choose a subsequence $\{y_{\rho_n}\}$ that converges weakly to y in X. It thus follows from (3.2) that since A is single-valued

$$\langle Au + B^0x - Ax - y, F(u - x)\rangle \ge 0.$$

Since A is maximal dissipstive, this implies that $y = B^0x$ and thus y_{ρ_n} converges weakly to B^0x. Since X is uniformly convex, (3.2) implies that y_{ρ_n} converges strongly to B^0x as $\rho_n \to 0^+$. Now, since $C = A+B$ is m-dissipative and $C^0x = Ax + B^0x$, the theorem follows from Corollary 2.3.6. $\square$