In this course we discuss the well-posededness of the evolution equations in Banach spaces. Such problems arise in PDEs dynamics and functional equations. We develop the linear and nonlinear theory for the corresponding solution semigroups. The lectures include for example the Hille-Yosiida theory, Lumer-Philiips theory for linear semigroup and Crandall-Liggett theory for nonlinear conrtractive semigroup and Crandall-Pazy theory for nonlinear evolution equations. Especially, (numerical) approximation theory for PDE solutions are discussed based on Trotter-Kato theory and Takahashi-Oharu theory, Chernoff theory and the operator splitting method. The theory and its applications are examined and demonstrated using many motivated PDE examples including linear dynamics (e.g. heat, wave and hyperbolic equations) and nonlinear dynamics (e.g. nonlinear diffusion, conservation law, Hamilton-Jacobi and Navier-Stokes equations). A new class of PDE examples are formulated and the detailed applications of the theory is carried out.

The lecture also covers the basic elliptic theory via Lax-Milgram, Minty-Browder theory and convex optimization. Functional analytic methods are also introduced for the basic PDEs theory.

The students are expected to have the basic knowledge in real and functional analysis and PDEs.

Lecture notes will be provided. Reference book: "Evolution equations and Approximation" K. Ito and F. Kappel, World Scientific.

1 Linear Cauchy problem and C_0-semigroup theory

In this section we discuss the Cauchy problem of the form

$$\frac{d}{dt}u(t) = Au(t) + f(t), \quad u(0) = u_0 \in X$$

in a Banach space X, where $u_0 \in X$ is the initial condition and $f \in L^1(0,T;X)$. Such problems arise in PDE dynamics and functional equations.

We construct the mild solution $u(t) \in C(0,T;X)$:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\,ds \tag{1.1}$$

where a family of bounded linear operator $\{S(t),\ t \geq 0\}$ is C_0-semigroup on X.

Definition (C_0 semigroup) (1) Let X be a Banach space. A family of bounded linear operators $\{S(t),\ t \geq 0\}$ on X is called a strongly continuous (C_0) semigroup if

$$S(t+s) = S(t)S(s) \text{ for } t,\ s \geq 0 \text{ with } S(0) = I$$

$$|S(t)\phi - \phi| \to 0 \text{ as } t \to 0^+ \quad \text{for all } \phi \in X.$$

(2) A linear operator A in X defined by

$$A\phi = \lim_{t \to 0^+} \frac{S(t)\phi - \phi}{t} \tag{1.2}$$

with

$$dom(A) = \{\phi \in X : \text{the strong limit of } \lim_{t\to 0^+} \frac{S(t)\phi - \phi}{t} \text{ in } X \text{ exists}\}.$$

is called the infinitesimal generator of the C_0 semigroup $S(t)$.

In this section we present the basic theory of the linear C_0-semigroup on a Banach space X. The theory allows to analyze a wide class of the physical and engineering dynamics using the unified framework. We also present the concrete examples to demonstrate the theory. There is a necessary and sufficient condition (Hile-Yosida Theorem) for a closed, densely defined linear A in X to be the infinitesimal generator of the C_0 semigroup $S(t)$. Moreover, we will show that the mild solution $u(t)$ satisfies

$$\langle u(t), \psi \rangle = \langle u_0, \psi \rangle + \int (\langle x(s), A^*\psi \rangle + \langle f(s), \psi \rangle \, ds \tag{1.3}$$

for all $\psi \in dom\,(A^*)$.

Examples (1) For $A \in \mathcal{L}(X)$, define a sequence of linear operators in X

$$S_N(t) = \sum_k \frac{1}{k!}(At)^k.$$

Then

$$|S_N(t)| \le \sum \frac{1}{k!}(|A|t)^k \le e^{|A|\,t}$$

and

$$\frac{d}{dt} S_N(t) = A S_{N-1}(t)$$

Since

$$S(t) = e^{At} = \lim_{N\to\infty} S_N(t), \tag{1.4}$$

in the operator norm, we have

$$\frac{d}{dt} S(t) = AS(t) = S(t)A.$$

(2) Consider the the hyperbolic equation

$$u_t + u_x = 0, \quad u(0,x) = u_0(x) \text{ in } (0,1). \tag{1.5}$$

Define the semigroup $S(t)$ of translations on $X = L^2(0,1)$ by

$$[S(t)u_0](x) = \tilde{u}_0(x-t), \quad \text{where} \quad \tilde{u}_0(x) = 0, x \le 0, \quad \tilde{u}_0 = u_0 \text{ on } [0,1]. \tag{1.6}$$

Then, $\{S(t),\ t \ge 0\}$ is a C_0 semigroup on X. If we define $u(t,x) = [S(t)u_0](x)$ with $u_0 \in H^1(0,1)$ with $u_0(0) = 0$ satisfies (1.8) a.e.. The generator A is given by

$$A\phi = -\phi' \text{ with } dom(A) = \{\phi \in H^1(0,1) \text{ with } \phi(0) = 0\}.$$

In fact

$$\frac{S(t)u_0 - u_0}{t} = \frac{\tilde{u}_0(x-t) - \tilde{u}_0}{t} = -u_0'(x), \quad \text{a.e. } x \in (0,1).$$

if $u_0 \in dom(A)$. Thus, $u(t) = S(t)u_0$ satisfies the Cauchy problem $\frac{d}{dt}u(t) = Au(t)$ if $u_0 \in dom(A)$.

On the other hand if we apply the operator exponential formula (1.4) for this A,

$$u(t,x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} u_0^k(x) t^k = u_0(x-t)$$

for $u_0 \in C^\infty(0,1)$, which coincides with (1.6). That is, the solution semigroup $S(t)$ is the extension of the operator exponential formula.
(3) Let $X_t \in R^d$ is a Markov process, i.e.

$$E^x[g(X_{t+h})|\mathcal{F}_t] = E^{0,X_t}[g(X_h)].$$

for all $g \in X = L^2(R^n)$. Define the linear operator $S(t)$ by

$$(S(t)u_0)(x) = E^{0,x}[u_0(X_t)], \quad t \geq 0, \ \ u_0 \in X.$$

The semigroup property of $S(t)$ follows from the Markov property, i.e.

$$S(t+s)u_0 = E^{0,x}[u_0(X_{t+s})] = E[E[u_0(X_{t+s}^{0,x})|\mathcal{F}_t] = E[E^{0,X_t^{0,x}} u_0(X_s)]] = E[(S(t)u_0)(X_s)] = S(s)(S(t)u_0).$$

The strong continuity follows from that $X_t^{0,x} - x$ is a.s for all $x \in R^n$. If $X_t = B_t$ is a Brownian motion, then the semigroup $S(t)$ is defined by

$$[S(t)u_0](x) = \frac{1}{(\sqrt{2\pi t}\sigma)^n} \int_{R^n} e^{-\frac{|x-y|^2}{2\sigma^2\,t}} u_0(y)\,dy, \tag{1.7}$$

and $u(t) = S(t)u_0$ satisfies the heat equation.

$$u_t = \frac{\sigma^2}{2}\Delta u, \quad u(0,x) = u_0(x) \text{ in } L^2(R^n). \tag{1.8}$$

1.1 Finite difference in time

Let A be closed, densely defined linear operator $dom(A) \to X$. We use the finite difference method in time to construct the mild solution (1.1). For a stepsize $\lambda > 0$ consider a sequence $\{u^n\}$ in X generated by

$$\frac{u^n - u^{n-1}}{\lambda} = Au^n + f^{n-1}, \tag{1.9}$$

with

$$f^{n-1} = \frac{1}{\lambda}\int_{(n-1)\lambda}^{n\lambda} f(t)\,dt.$$

Assume that for $\lambda > 0$ the resolvent operator

$$J_\lambda = (I - \lambda\, A)^{-1}$$

is bounded. Then, we have the product formula:

$$u^n = J_\lambda^n u_0 + \sum_{k=0}^{n-1} J_\lambda^{n-k} f^k \, \lambda. \tag{1.10}$$

In order to $u^n \in X$ is uniformly bounded in n for all $u_0 \in X$ (with $f = 0$), it is necessary that

$$|J_\lambda^n| \leq \frac{M}{(1-\lambda\omega)^n} \text{ for } \lambda\omega < 1, \tag{1.11}$$

for some $M \geq 1$ and $\omega \in R$.

Hille's Theorem Define a piecewise constant function in X by

$$u_\lambda(t) = u^{k-1} \text{ on } [t_{k-1}, t_k)$$

Then,

$$\max_{t\in[0,T]} |u_\lambda - u(t)|_X \to 0$$

as $\lambda \to 0^+$ to the mild solution (1.1). That is,

$$S(t)x = \lim_{n\to\infty} (I - \frac{t}{n}A)^{[\frac{t}{n}]}x$$

exists for all $x \in X$ and $\{S(t),\ t \geq 0\}$ is the C_0 semigoup on X and its generator is A, where $[s]$ is the largest integer less than $s \in R$.

Proof: First, note that

$$|J_\lambda| \leq \frac{M}{1-\lambda\omega}$$

and for $x \in dom(A)$

$$J_\lambda x - x = \lambda J_\lambda Ax,$$

and thus

$$|J_\lambda x - x| = |\lambda J_\lambda Ax| \leq \frac{\lambda}{1-\lambda\omega}|Ax| \to 0$$

as $\lambda \to 0^+$. Since $dom(A)$ is dense in X it follows that

$$|J_\lambda x - x| \to 0 \text{ as } \lambda \to 0^+ \text{ for all } x \in X.$$

Define the linear operators $T_\lambda(t)$ and $S_\lambda(t)$ by

$$S_\lambda(t) = J_\lambda^k \text{ and } T_\lambda(t) = J_\lambda^{k-1} + \frac{t-t_k}{\lambda}(J_\lambda^k - J_\lambda^{k-1}), \text{ on } (t_{k-1}, t_k].$$

Then,

$$\frac{d}{dt}T_\lambda(t) = AS_\lambda(t), \text{ a.e. in } t \in [0,T].$$

Thus,

$$T_\lambda(t)u_0 - T_\mu(t)u_0 = \int_0^t \frac{d}{ds}(T_\lambda(s)T_\mu(t-s)u_0)\,ds = \int_0^t (S_\lambda(s)T_\mu(t-s) - T_\lambda(s)S_\mu(t-s))Au_0\,ds$$

Since

$$T_\lambda(s)u - S_\lambda(s)u = \frac{s-t_k}{\lambda} T_\lambda(t_{k-1})(J_\lambda - I)u \text{ on } s \in (t_{k-1}, t_k].$$

By the bounded convergence theorem

$$|T_\lambda(t)u_0 - T_\mu(t)u|_X \to 0$$

as $\lambda,\ \mu \to 0^+$ for all $u \in dom(A^2)$. Thus, the unique limit defines the linear operator $S(t)$ by

$$S(t)u_0 = \lim_{\lambda \to 0^+} S_\lambda(t)u_0. \tag{1.12}$$

for all $u_0 \in dom(A^2)$. Since

$$|S_\lambda(t)| \le \frac{M}{(1-\lambda\omega)^{[t/n]}} \le Me^{\omega t}$$

and $dom(A^2)$ is dense, (1.12) holds for all $u_0 \in X$. Moreover, we have

$$S(t+s)u = \lim_{\lambda \to 0^+} J_\lambda^{n+m} = J_\lambda^n J_\lambda^m u = S(t)S(s)u$$

and $\lim_{t \to 0^+} S(t)u = \lim_{t \to 0^+} J_t u = u$ for all $u \in X$. Thus, $S(t)$ is the C_0 semigroup on X. Moreover, $\{S(t), t \ge 0\}$ is in the class $G(M, \omega)$, i.e.,

$$|S(t)| \le Me^{\omega t}.$$

Note that

$$T_\lambda(t)u_0 - u_0 = A\int_0^t S_\lambda u_0\, ds.$$

Since $\lim_{\lambda \to 0^+} T_\lambda(t)u_0 = \lim_{\lambda \to 0^+} S_\lambda(t)u_0 = S(t)u_0$ and A is closed, we have

$$S(t)u_0 - u_0 = A\int_0^t S(s)u_0\, ds, \quad \int_0^t S(s)u_0\, ds \in dom(A).$$

If B is a generator of $\{S(t),\ t \ge 0\}$, then

$$Bx = \lim_{t \to 0^+} \frac{S(t)x - x}{t} = Ax$$

if $x \in dom(A)$. Conversely, if $u_0 \in dom(B)$, then $u_0 \in dom(A)$ since A is closed and $t \to S(t)u$ is continuous at 0 for all $u \in X$ and thus

$$\frac{1}{t} A \int_0^t S(s)u_0\, ds = Au_0 \text{ as } t \to 0^+.$$

Hence

$$Au_0 = \frac{S(t)u_0 - u_0}{t} = Bu_0$$

That is, A is the generator of $\{S(t),\ t \ge 0\}$.

Similarly, we have

$$\sum_{k=0}^{n-1} J_\lambda^{n-k} f^k = \int_0^t S_\lambda(t-s)f(s)\,ds \to \int_0^t S(t-s)f(s)\,ds \text{ as } \lambda \to 0^+$$

by the Lebesgue dominated convergence theorem. □

The following theorem states the basic properties of C_0 semigroups:

Theorem (Semigroup) (1) There exists $M \geq 1$, $\omega \in R$ such that $S \in G(M,\omega)$ class, i.e.,

$$|S(t)| \leq M\,e^{\omega t}, \quad t \geq 0. \tag{1.13}$$

(2) If $x(t) = S(t)x_0$, $x_0 \in X$, then $x \in C(0,T;X)$
(3) If $x_0 \in dom\,(A)$, then $x \in C^1(0,T;X) \cap C(0,T;dom(A))$ and

$$\frac{d}{dt}x(t) = Ax(t) = AS(t)x_0.$$

(4) The infinitesimal generator A is closed and densely defined. For $x \in X$

$$S(t)x - x = A\int_0^t S(s)x\,ds. \tag{1.14}$$

(5) $\lambda > \omega$ the resolvent is given by

$$(\lambda\,I - A)^{-1} = \int_0^\infty e^{-\lambda s}S(s)\,ds \tag{1.15}$$

with estimate

$$|(\lambda\,I - A)^{-n}| \leq \frac{M}{(\lambda-\omega)^n}. \tag{1.16}$$

Proof: (1) By the uniform boundedness principle there exists $M \geq 1$ such that $|S(t)| \leq M$ on $[0,t_0]$ For arbitrary $t = k\,t_0 + \tau$, $k \in N$ and $\tau \in [0,t_0)$ it follows from the semigroup property we get

$$|S(t)| \leq |S(\tau)||S(t_0|^k \leq Me^{k\,log\,|S(t_0)|} \leq Me^{\omega\,t}$$

with $\omega = \frac{1}{t_0}\log|S(t_0)|$.
(2) It follows from the semigroup property that for $h > 0$

$$x(t+h) - x(t) = (S(h) - I)S(t)x_0$$

and for $t - h \geq 0$

$$x(t-h) - x(t) = S(t-h)(I - S(h))x_0$$

Thus, $x \in C(0,T;X)$ follows from the strong continuity of $S(t)$ at $t = 0$.
(3)–(4) Moreover,

$$\frac{x(t+h)-x(t)}{h} = \frac{S(h)-I}{h}S(t)x_0 = S(t)\frac{S(h)x_0 - x_0}{h}$$

and thus $S(t)x_0 \in dom(A)$ and

$$\lim_{h\to 0^+} \frac{x(t+h)-x(t)}{h} = AS(t)x_0 = Ax(t).$$

Similarly,

$$\lim_{h\to 0^+} \frac{x(t-h)-x(t)}{-h} = \lim_{h\to 0^+} S(t-h)\frac{S(h)\phi-\phi}{h} = S(t)Ax_0.$$

Hence, for $x_0 \in dom(A)$

$$S(t)x_0 - x_0 = \int_0^t S(s)Ax_0\,ds = \int_0^t AS(s)x_0\,ds = A\int_0^t S(s)x_0\,ds \qquad (1.17)$$

If $x_n \in don(A) \to x$ and $Ax_n \to y$ in X, we have

$$S(t)x - x = \int_0^t S(s)y\,ds$$

Since

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t S(s)y\,ds = y$$

$x \in dom(A)$ and $y = Ax$ and hence A is closed. Since A is closed it follows from (1.17) that for $x \in X$

$$\int_0^t S(s)x\,ds \in dom(A)$$

and (1.14) holds. For $x \in X$ let

$$x_h = \frac{1}{h}\int_0^h S(s)x\,ds \in dom(A)$$

Since $x_h \to x$ as $h \to 0^+$, $dom(A)$ is dense in X.
(5) For $\lambda > \omega$ define $R_t \in \mathcal{L}(X)$ by

$$R_t = \int_0^t e^{-\lambda s}S(s)\,ds.$$

Since $A - \lambda I$ is the infinitesimal generator of the semigroup $e^{\lambda t}S(t)$, from (1.14)

$$(\lambda I - A)R_t x = x - e^{-\lambda t}S(t)x.$$

Since A is closed and $|e^{-\lambda t}S(t)| \to 0$ as $t \to \infty$, we have $R = \lim_{t\to\infty} R_t$ satisfies

$$(\lambda I - A)R\phi = \phi.$$

Conversely, for $\phi \in dom(A)$

$$R(A-\lambda I)\phi = \int_0^\infty e^{-\lambda s}S(s)(A-\lambda I)\phi = \lim_{t\to\infty} e^{-\lambda t}S(t)\phi - \phi = -\phi$$

Hence

$$R = \int_0^\infty e^{-\lambda s} S(s)\, ds = (\lambda\, I - A)^{-1}$$

Since for $\phi \in X$

$$|Rx| \le \int_0^\infty |e^{-\lambda s} S(s)x| \le M \int_0^\infty e^{(\omega-\lambda)s} |x|\, ds = \frac{M}{\lambda - \omega} |x|,$$

we have

$$|(\lambda\, I - A)^{-1}| \le \frac{M}{\lambda - \omega}, \quad \lambda > \omega.$$

Note that

$$(\lambda\, I - A)^{-2} = \int_0^\infty e^{-\lambda t} S(t)\, ds \int_0^\infty e^{\lambda\, s} S(s)\, ds = \int_0^\infty \int_0^\infty e^{-\lambda(t+s)} S(t+s)\, ds\, dt$$

$$= \int_0^\infty \int_t^\infty e^{-\lambda \sigma} S(\sigma)\, d\sigma\, dt = \int_0^\infty \sigma e^{-\lambda \sigma} S(\sigma)\, d\sigma.$$

By induction, we obtain

$$(\lambda\, I - A)^{-n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} S(t)\, dt. \tag{1.18}$$

Thus,

$$|(\lambda\, I - A)^{-n}| \le \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(\lambda-\omega)t}\, dt = \frac{M}{(\lambda - \omega)^n}. \square$$

We then we have the necessary and sufficient condition:

Hile-Yosida Theorem A closed, densely defined linear operator A on a Banach space X is the infinitesimal generator of a C_0 semigroup of class $G(M, \omega)$ if and only if

$$|(\lambda\, I - A)^{-n}| \le \frac{M}{(\lambda - \omega)^n} \quad \text{for all } \lambda > \omega \tag{1.19}$$

Proof: The sufficient part follows from the previous Theorem. In addition, we describe the Yosida construction. Define the Yosida approximation $A_\lambda \in \mathcal{L}(X)$ of A by

$$A_\lambda = \frac{J_\lambda - I}{\lambda} = A J_\lambda. \tag{1.20}$$

Define the Yosida approximation:

$$S_\lambda(t) = e^{A_\lambda t} = e^{-\frac{t}{\lambda}} e^{J_\lambda \frac{t}{\lambda}}.$$

Since

$$|J_\lambda^k| \le \frac{M}{(1 - \lambda\omega)^k}$$

we have

$$|S_\lambda(t)| \le e^{-\frac{t}{\lambda}} \sum_{k=0}^\infty \frac{1}{k!} |J_\lambda^k| (\frac{t}{\lambda})^k \le M e^{\frac{\omega}{1-\lambda\omega} t}.$$

Since

$$\frac{d}{ds}S_\lambda(s)S_{\hat{\lambda}}(t-s) = S_\lambda(s)(A_\lambda - A_{\hat{\lambda}})S_{\hat{\lambda}}(t-s),$$

we have

$$S_\lambda(t)x - S_{\hat{\lambda}}(t)x = \int_0^t S_\lambda(s)S_{\hat{\lambda}}(t-s)(A_\lambda - A_{\hat{\lambda}})x\,ds$$

Thus, for $x \in dom(A)$

$$|S_\lambda(t)x - S_{\hat{\lambda}}(t)x| \le M^2 te^{\omega t}\,|(A_\lambda - A_{\hat{\lambda}})x| \to 0$$

as $\lambda,\ \hat{\lambda} \to 0^+$. Since $dom(A)$ is dense in X this implies that

$$S(t)x = \lim_{\lambda\to 0^+} S_\lambda(t)x \text{ exist for all } x \in X,$$

defines a C_0 semigroup of $G(M,\omega)$ class. The necessary part follows from (1.18) □

Theorem (Mild solution) (1) If for $f \in L^1(0,T;X)$ define

$$x(t) = x(0) + \int_0^t S(t-s)f(s)\,ds,$$

then $x(t) \in C(0,T;X)$ and it satisfies

$$x(t) = A\int_0^t x(s)\,ds + \int_0^t f(s)\,ds. \tag{1.21}$$

(2) If $Af \in L^1(0,T;X)$ then $x \in C(0,T;dom(A))$ and

$$x(t) = x(0) + \int_0^t (Ax(s) + f(s))\,ds.$$

(3) If $f \in W^{1,1}(0,T;X)$, i.e. $f(t) = f(0) + \int_0^t f'(s)\,ds$, $\frac{d}{dt}f = f' \in L^1(0,T;X)$, then $Ax \in C(0,T;X)$ and

$$A\int_0^t S(t-s)f(s)\,ds = S(t)f(0) - f(t) + \int_0^t S(t-s)f'(s)\,ds. \tag{1.22}$$

Proof: Since

$$\int_0^t\int_0^\tau S(t-s)f(s)\,dsd\tau = \int_0^t (\int_s^t S(\tau-s)d\,\tau)f(s)\,ds,$$

and

$$A\int_0^t S(s)\,ds = S(t) - I$$

we have $x(t) \in dom(A)$ and

$$A\int_0^t x(s)\,ds = S(t)x - x + \int_0^t S(t-s)f(s)\,ds - \int_0^t f(s)\,ds.$$

and we have (1.21).
(2) Since for $h > 0$

$$\frac{x(t+h)-x(t)}{h} = \int_0^t S(t-s)\frac{S(h)-I}{h}f(s)\,ds + \frac{1}{h}\int_t^{t+h} S(t+h-s)f(s)\,ds$$

if $Af \in L^1(0,T;X)$

$$\lim_{h\to 0^+}\frac{x(t+h)-x(t)}{h} = \int_0^t S(t-s)Af(s)\,ds + f(t)$$

a.e. $t \in (0,T)$. Similarly,

$$\frac{x(t-h)-x(t)}{-h} = \int_0^{t-h} S(t-h-s)\frac{S(h)-I}{h}f(s)\,ds + \frac{1}{h}\int_{t-h}^{t} S(t-s)f(s)\,ds$$

$$\to \int_0^t S(t-s)Af(s)\,ds + f(t)$$

a.e. $t \in (0,T)$.
(3) Since

$$\frac{S(h)-I}{h}x(t) = \frac{1}{h}(\int_0^h S(t+h-s)f(s)\,ds - \int_t^{t+h} S(t+h-s)f(s)\,ds$$

$$+\int_0^t S(t-s)\frac{f(s+h)-f(s)}{h}\,ds,$$

letting $h \to 0^+$, we obtain(1.22). $\square$

It follows from Theorems the mild solution

$$x(t) = S(t)x(0) + \int_0^t S(t-s)f(s)\,ds$$

satisfies

$$x(t) = x(0) + A\int_0^t x(s) + \int_0^t f(s)\,ds.$$

Note that the mild solution $x \in C(0,T;X)$ depends continuously on $x(0) \in X$ and $f \in L^1(0,T;X)$ with estimate

$$|x(t)| \le M(e^{\omega t}|x(0)| + \int_0^t e^{\omega(t-s)}|f(s)|\,ds).$$

Thus, the mild solution is the limit of a sequence $\{x_n\}$ of strong solutions with $x_n(0) \in dom(A)$ and $f_n \in W^{1,1}(0,T;X)$, i.e., since $dom(A)$ is dense in X and $W^{1,1}(0,T;X)$ is dense in $L^1(0,T;X)$,

$$|x_n(t) - x(t)|_X \to 0 \text{ uniformly on } [0,T]$$

for

$$|x_n(0)-x(0)|_X \to 0 \text{ and } |f_n - f|_{L^1(0,T;X)} \to 0 \text{ as } n \to \infty.$$

Moreover, the mild solution $x \in C(0,T:X)$ is a weak solution to the Cauchy problem

$$\frac{d}{dt}x(t) = Ax(t) + f(t) \tag{1.23}$$

in the sense of (1.3), i.e., for all $\psi \in dom(A^*)$ $\langle x(t), \psi\rangle_{X\times X^*}$ is absolutely continues and

$$\frac{d}{dt}\langle x(t), \psi\rangle = \langle x(t), \psi\rangle + \langle f(t), \psi\rangle \text{ a.e. in } (0,T).$$

If $x(0) \in dom(A)$ and $Af \in L^1(0,T;X)$, then $Ax \in C(0,T;X)$, $x \in W^{1,1}(0,T;X)$ and

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \text{ a.e. in } (0,T)$$

If $x(0) \in dom(A)$ and $f \in W^{1,1}(0,T;X)$, then $x \in C(0,T;dom(A)) \cap C^1(0,T;X)$ and

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \text{ everywhere in } [0,T].$$

1.2 Weak-solution and Ball's result

Let A be a densely defined, closed linear operator on a Banach space X. Consider the Cauchy equation in X:

$$\frac{d}{dt}u = Au + f(t), \tag{1.24}$$

where $u(0) = x \in X$ and $f \in L^1(0,\tau;X)$ is a weak solution to of (1.24) if for every $\psi \in$dom(A^*) the function $t \to \langle u(t), \psi\rangle$ is absolutely continuous on $[0,\tau]$ and

$$\frac{d}{dt}\langle u(t), \psi\rangle = \langle u(t), A^*\psi\rangle + \langle f(t), \psi\rangle, \quad \text{a.e. in } [0,\tau]. \tag{1.25}$$

It has been shown that the mild solution to (1.24) is a weak solution.

Lemma B.1 Let A be a densely defined, closed linear operator on a Banach space X. If $x,\ y \in X$ satisfy $\langle y, \psi\rangle = \langle x, A^*\psi\rangle$ for all $\psi \in$dom(A^*), then $x \in$ dom(A) and $y = Ax$.

Proof: Let $G(A) \subset X \times X$ denotes the graph of A. Since A is closed $G(A)$ is closed. Suppose $y \neq Ax$. By Hahn-Banach theorem there exist $z,\ z^* \in X^*$ such that $\langle Ax, z\rangle + \langle x, z^*\rangle = 0$ and $\langle y, z\rangle + \langle x, z^*\rangle \neq 0$. Thus $z \in$dom(A^*) and $z^* = A^*z$. By the condition $\langle y, z\rangle + \langle x, z^*\rangle = 0$, which is a contradiction. □

Then we have the following theorem.

Theorem (Ball) There exists for each $x \in X$ and $f \in L^1(0,\tau;X)$ a unique weak solution of (1.24)satisfying $u(0) = x$ if and only if A is the generator of a strongly continuous semigroup $\{T(t)\}$ of bounded linear operator on X, and in this case $u(t)$ is given by

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)\,ds. \tag{1.26}$$

Proof: Let A generate the strongly continuous semigroup $\{T(t)\}$ on X. Then, for some M, $|T(t)| \leq M$ on $t \in [0,\tau]$. Suppose $x \in$dom(A) and $f \in W^{1,1}(0,\tau;X)$. Then we have

$$\frac{d}{dt}\langle u(t),\psi\rangle = \langle Au(t)+f(t),\psi\rangle = \langle u(t),A^*\psi\rangle + \langle f(t),\psi\rangle.$$

For $(x,f) \in X \times L^1(0,\tau;X)$ there exists a sequence (x_n,f_n) in dom$(A) \times W^{1,1}(0,\tau;X)$ such that $|x_n - x|_X + |f_n - f|_{L^1(0,\tau;X)} \to 0$ as $n \to \infty$ If we define

$$u_n(t) = T(t)x_n + \int_0^t T(t-s)f_n(s)\,ds,$$

then we have

$$\langle u_n(t),\psi\rangle = \langle x,\psi\rangle + \int_0^t (\langle u_n(s),A^*\psi\rangle + \langle f_n(s),\psi\rangle)\,ds$$

and

$$|u_n(t)-u(t)|_X \leq M\,(|x_n - x|_X + \int_0^t |f_n(s)-f(s)|_X\,ds).$$

Passing limit $n \to]\infty$, we see that $u(t)$ is a weak solution of (1.24).

Next we prove that $u(t)$ is the only weak solution to (1.24) satisfying $u(0) = x$. Let $\tilde{u}(t)$ be another such weak solution and set $v = u - \tilde{u}$. Then we have

$$\langle v(t),\psi\rangle = \langle \int_0^t v(s)\,dt, A^*\psi\rangle$$

for all $\psi \in$dom(A^*) and $t \in [0,\tau]$. By Lemma B.1 this implies $z(t) = \int_0^t v(s)\,ds \in$dom$(A)$ and $\frac{d}{dt}z(t) = Az(t)$ with $z(0) = 0$. Thus $z = 0$ and hence $u(t) = \tilde{u}(t)$ on $[0,\tau]$.

Suppose that A such that (1.24) has a unique weak solution $u(t)$ satisfying $u(0) = x$. For $t \in [0,\tau]$ we define the linear operator $T(t)$ on X by $T(t)x = u(t) - u_0(t)$, where u_0 is the weak solution of (1.24) satisfying $u_(0) = 0$. If for $t = nT + s$, where n is a nonnegative integer and $s \in [0,\tau)$ we define $T(t)x = T(s)T(\tau)^n x$, then $T(t)$ is a semigroup. The map $\theta : x \to C(0,\tau;X)$ defined by $\theta(x) = T(\cdot)x$ has a closed graph by the uniform bounded principle and thus $T(t)$ is a strongly continuous semigroup. Let B be the generator of $\{T(t)\}$ and $x \in$dom(B). For $\psi \in$dom(A^*)

$$\frac{d}{dt}\langle T(t)x,\psi\rangle|_{t=0} = \langle Bx,\psi\rangle = \langle x,A^*\psi\rangle.$$

It follows from Lemma that $x \in$dom(A) and $Ax = Bx$. Thus dom$(B) \subset$dom(A). The proof of Theorem is completed by showing dom$(A) \subset$dom(B). Let $x \in$dom(A). Since for $z(t) = T(t)x$

$$\langle z(t),\psi\rangle = \langle \int_0^t z(s)\,dt, A^*\psi\rangle$$

it follows from Lemma that $\int_0^t T(s)x\,ds$ and $\int_0^t T(s)Ax\,ds$ belong to dom(A) and

$$\begin{aligned} T(t)x &= x + A\int_0^t T(s)x\,ds \\ T(t)Ax &= Ax + A\int_0^t T(s)Ax\,ds \end{aligned} \tag{1.27}$$

Consider the function

$$w(t) = \int_0^t T(s)Ax\,ds - A\int_0^t T(s)x\,ds.$$

It then follows from (1.27) that $z \in C(0,\tau;\ X)$. Clearly $w(0) = 0$ and it also follows from (1.27) that

$$\frac{d}{dt}\langle w(t),\psi\rangle = \langle w(t), A^*\psi\rangle. \tag{1.28}$$

for $\psi\ \in$dom(A^*). But it follows from our assumptions that (1.28) has the unique solution $w = 0$. Hence from (1.27)

$$T(t)x - x = A\int_0^t T(s)x\,ds$$

and thus

$$\lim_{t\to 0^+} \frac{T(t)x - x}{t} = Ax$$

which implies $x\ \in$dom(B). □

1.3 Lumer-Phillips Theorem

The condition (1.19) is very difficult to check for a given A in general. For the case $M = 1$ we have a very complete characterization.

Lumer-Phillips Theorem The followings are equivalent:
(a) A is the infinitesimal generator of a C_0 semigroup of $G(1,\omega)$ class.
(b) $A - \omega\, I$ is a densely defined linear m-dissipative operator,i.e.

$$|(\lambda\, I - A)x| \geq (\lambda - \omega)|x| \quad \text{for all } x \in don(A),\quad \lambda > \omega \tag{1.29}$$

and for some $\lambda_0 > \omega$

$$R(\lambda_0\, I - A) = X. \tag{1.30}$$

Proof: It follows from the m-dissipativity

$$|(\lambda_0\, I - A)^{-1}| \leq \frac{1}{\lambda_0 - \omega}$$

Suppose $x_n \in dom(A) \to x$ and $Ax_n \to y$ in X, the

$$x = \lim_{n\to\infty} x_n = (\lambda_0\, I - A)^{-1} \lim_{n\to\infty} (\lambda_0\, x_n - Ax_n) = (\lambda_0\, I - A)^{-1}(\lambda_0\, x - y).$$

Thus, $x \in dom(A)$ and $y = Ay$ and hence A is closed. Since for $\lambda > \omega$

$$\lambda\, I - A = (I + (\lambda - \lambda_0)(\lambda_0\, I - A)^{-1})(\lambda_0\, I - A),$$

if $\frac{|\lambda-\lambda_0|}{\lambda_0-\omega} < 1$, then $(\lambda\, I - A)^{-1} \in \mathcal{L}(X)$. Thus by the continuation method we have $(\lambda\, I - A)^{-1}$ exists and

$$|(\lambda\, I - A)^{-1}| \leq \frac{1}{\lambda - \omega}, \quad \lambda > \omega.$$

It follows from the Hile-Yosida theorem that (b) $\to$ (a).
(b) $\to$ (a) Since for $x^* \in F(x)$, the dual element of x, i.e. $x^* \in X^*$ satisfying $\langle x, x^*\rangle_{X\times X^*} = |x|^2$ and $|x^*| = |x|$

$$\langle e^{-\omega t}S(t)x, x^*\rangle \leq |x||x^*| = \langle x, x^*\rangle$$

we have for all $x \in dom(A)$

$$0 \geq \lim_{t\to 0^+}\langle \frac{e^{-\omega t}S(t)x - x}{t}, x^*\rangle = \langle (A - \omega\, I)x, x^*\rangle \text{ for all } x^* \in F(x).$$

which implies $A - \omega\, I$ is dissipative. $\square$

Theorem (Dissipative I) (1) A is a ω-dissipative

$$|\lambda\, x - Ax| \geq (\lambda - \omega)|x| \text{ for all } x \in dom\,(A).$$

if and only if (2) for all $x \in dom(A)$ there exists an $x^* \in F(x)$ such that

$$\langle Ax, x^*\rangle \leq \omega\, |x|^2. \tag{1.31}$$

(2) $\to$ (1). Let $x \in dom(A)$ and choose an $x^* \in F(0)$ such that $\langle A, x^*\rangle \leq 0$. Then, for any $\lambda > 0$,

$$\lambda\, |x|^2 = \lambda\, \langle x, x^*\rangle = \langle \lambda\, x - Ax + Ax, x^*\rangle \leq \langle \lambda\, x - Ax, x^*\rangle + \omega\; |x|^2 \leq |\lambda\, x - Ax||x| + \omega\, |x|^2,$$

which implies (1).
(1) $\to$ (2). Without loss of the generality one can assume $\omega = 0$. From (1) we obtain the estimate

$$\frac{1}{\lambda}(|x| - |x - \lambda\, Ax|) \leq 0$$

and

$$\langle Ax, x\rangle_- = -\lim_{\lambda\to 0^+}\frac{1}{\lambda}(|x| - |x - \lambda\, Ax|) \leq 0$$

which implies there exists $x^* \in F(x)$ such that (1.31) holds since $\langle Ax, x\rangle_- = \langle Ax, x^*\rangle$ for some $x^* \in F(x)$. $\square$

Thus, Lumer-Phillips theorem says that if m-diisipative, then (1.31) hold for all $x^* \in F(x)$.

Theorem (Dissipative II) Let A be a closed densely defined operator on X. If A and A^* are dissipative, then A is m-dissipative and thus the infinitesimal generator of a C_0-semigroup of contractions.

Proof: Let $y \in \overline{R(I - A)}$ be given. Then there exists a sequence $x_n \in dom(A)$ such that $y{=}x_n - Ax_n \to y$ as $n \to \infty$. By the dissipativity of A we obtain

$$|x_n - x_m| \leq |x_n - x_m - A(x_n - x_m)| \leq |y_{-}y_m|$$

Hence x_n is a Cauchy sequence in X. We set $x = lim_{n\to\infty}x_n$. Since A is closed, we see that $x \in dom\,(A)$ and $x - Ax = y$, i.e., $y \in R(I - A)$. Thus $R(I - A)$ is closed. Assume that $R(I - A) \neq X$. Then there exists an $x^* \in X^*$ such that

$$\langle (I - A)x, x^*) = 0 \text{ for all } x \in dom\,(A).$$

By definition of the dual operator this implies $x^* \in dom\,(A^*)$ and $(I - A)^* x^* = 0$. The dissipativity of A* implies $|x^*| < |x^* - A^* x^*| = 0$, which is a contradiction. $\square$

Example (revisited example 1)

$$A\phi = -\phi' \text{ in } X = L^2(0,1)$$

and for $\phi \in H^1(0,1)$

$$(A\phi, \phi)_X = -\int_0^1 \phi'(x)\phi\,dx = \frac{1}{2}(|\phi(0)|^2 - |\phi(1)|^2)$$

Thus, A is dissipative if and only if $\phi(0) = 0$, the in flow condition. Define the domain of A by

$$dom(A) = \{\phi \in H^1(0,1) : \phi(0) = 0\}$$

The resolvent equation is equivalent to

$$\lambda\, u + \frac{d}{dx} u = f$$

and

$$u(x) = \int_0^x e^{-\lambda\,(x-s)} f(s)\,ds,$$

and $R(\lambda\, I - A) = X$. By the Lumer-Philips theorem A generates the C_0 semigroup on $X = L^2(0,1)$.

Example (Conduction equation) Consider the heat conduction equation:

$$\frac{d}{dt}u = Au = \sum_{i,j} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b(x)\cdot\nabla u + c(x)u, \quad \text{in } \Omega.$$

Let $X = C(\Omega)$ and $dom(A) \subset C^2(\Omega)$. Assume that $a \in R^{n\times n} \in C(\Omega)$ $b \in R^{n,1}$ and $c \in R$ are continuous on $\bar{\Omega}$ and a is symmetric and

$$m\, I \le a(x) \le M\, I \text{ for } 0 < m \le M < \infty.$$

Then, if x_0 is an interior point of Ω at which the maximum of $\phi \in C^2(\Omega)$ is attained. Then,

$$\nabla\phi(x_0) = 0, \quad \sum_{ij} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) \le 0.$$

and thus

$$(\lambda\,\phi - A\phi)(x_0) \le \omega\,\phi(x_0)$$

where

$$\omega \le \max_{x\in\Omega} c(x).$$

Similarly, if x_0 is an interior point of Ω at which the minimum of $\phi \in C^2(\Omega)$ is attained, then

$$(\lambda\,\phi - A\phi)(x_0) \ge 0$$

If $x_0 \in \partial\Omega$ attains the maximum, then

$$\frac{\partial}{\partial \nu}\phi(x_0) \leq 0.$$

Consider the domain with the Robin boundary condition:

$$dom(A) = \{u \in \alpha(x)\, u(x) + \beta(x)\frac{\partial}{\partial \nu}u = 0 \text{ at } \partial\Omega\}$$

with $\alpha,\ \beta \geq 0$ and $\inf_{x\in\partial\Omega}(\alpha(x)+\beta(x)) > 0$. Then,

$$|\lambda\phi - A\phi|_X \geq (\lambda-\omega)|\phi|_X. \tag{1.32}$$

for all $\phi \in C^2(\Omega)$. It follows from the he Lax Milgram theory that

$$(\lambda_0\, I - A)^{-1} \in \mathcal{L}(L^2(\Omega), H^2(\Omega)),$$

assuming that coefficients (a, b, c) are sufficiently smooth. Let

$$dom(A) = \{(\lambda_0\, I - A)^{-1}C(\Omega)\}.$$

Since $C^2(\Omega)$ is dense in $dom(A)$, (1.32) holds for all $\phi \in dom(A)$, which shows A is dissipative.

Example (Advection equation and Mass transport equation) Consider the advection equation

$$u_t + \nabla\cdot(\vec{b}(x)u) = \nu\,\Delta u.$$

Let $X = L^1(\Omega)$. Assume

$$\vec{b} \in L^\infty(\Omega)$$

Let $\rho \in C^1(R)$ be a monotonically increasing function satisfying $\rho(0) = 0$ and $\rho(x) = sign(x),\ |x| \geq 1$ and $\rho_\epsilon(s) = \rho(\frac{s}{\epsilon})$ for $\epsilon > 0$. For $u \in C^1(\Omega)$

$$(Au, u) = \int_\Gamma (\nu\,\frac{\partial}{\partial n}u - n\cdot\vec{b}\,u, \rho_\epsilon(u))\,ds + (\vec{b}\,u - \nu\,\nabla u, \frac{1}{\epsilon}\rho'_\epsilon(u)\,\nabla u) + (c\,u, \rho_\epsilon(u)).$$

where

$$(\vec{b}\,u, \frac{1}{\epsilon}\rho'_\epsilon(u)\,\nabla u) \leq \nu\,(\nabla u, \frac{1}{\epsilon}\rho'_\epsilon(u)\,\nabla u) + \frac{\epsilon}{4\nu}\,meas(\{|u| \leq \epsilon\}).$$

Assume the inflow condition $\nu\,\frac{\partial}{\partial n}u - n\cdot\vec{b}\,u = 0$ on $\{s \in \partial\Omega : n\cdot b < 0\}$ and otherwise $\nu\,\frac{\partial}{\partial n}u = 0$. Note that for $u \in L^1(R^d)$

$$(u, \rho_\epsilon(u)) \to |u|_1 \quad \text{and} \quad (\psi, \rho_\epsilon(u)) \to (\psi, sign_0(u)) \ \text{ for } \ \psi \in L^1(\Omega)$$

as $\epsilon \to 0$. If $c(x) \leq \omega$, then it follows that

$$(\lambda-\omega)\,|u| \leq |\lambda\,u - \lambda\,Au|. \tag{1.33}$$

Since $H^1(\Omega)$ is dense in $L^1(\Omega)$, (1.33) holds for $u \in dom(A)$. For $\nu = 0$ case letting $\nu \to 0^+$ (1.33) holds for $dom(A) = \{u \in L^1(\Omega) : (\rho\,u)_x \in L^1(\Omega)\}$.

Example ($X = L^p(\Omega)$) Let $Au = \nu\,\Delta u + b\cdot\nabla u$ with homogeneous boundary condition $u = 0$ on $X = L^p(\Omega)$. Since

$$\langle \Delta u, u^*\rangle = \int_\Omega (\Delta u, |u|^{p-2}u) = -(p-1)\int_\Omega (\nabla u, |u|^{p-2}\nabla u)$$

and

$$(b\cdot\nabla u, |u|^{p-2}u)_{L^2} \le \frac{(p-1)\nu}{2}|(\nabla u, |u|^{p-2}\nabla u)_{L^2} + \frac{|b|_\infty^2}{2\nu(p-1)}\,(|u|^p, 1)_{L^2}$$

we have

$$\langle Au, u^*\rangle \le \omega |u|^2$$

for some $\omega > 0$.

Example (Fractional PDEs I)

In this section we consider the nonlocal diffusion equation of the form

$$u_t = Au = \int_{R^d} J(z)(u(x+z) - u(x))\,dz.$$

Or, equivalently

$$Au = \int_{(R^d)^+} J(z)(u(x+z) - 2u(x) + u(x-z))\,dz$$

for the symmetric kernel J in R^d. It will be shown that

$$(Au, \phi)_{L^2} = -\int_{R^d}\int_{(R^d)^+} J(z)(u(x+z) - u(x))(\phi(x+z) - \phi(x))\,dz\,dx$$

and thus A has a maximum extension.

Also, the nonlocal Fourier law is given by

$$Au = \nabla\cdot(\int_{R^d} J(z)\nabla u(x+z)\,dz).$$

Thus,

$$(Au, \phi)_{L^2} = \int_{R^d\times R^d} J(z)\nabla u(x+z)\cdot\nabla\phi(x)\,dz\,dx$$

Under the kernel J is completely monotone, one can prove that A has a maximal monotone extension.

1.4 Jump diffusion Model for American option

In this section we discuss the American option for the jump diffusion model

$$u_t + (x - \frac{\sigma^2}{2})u_x + \frac{\sigma^2 x^2}{2}u_{xx} + Bu + \lambda = 0, \quad u(T,x) = \psi,$$

$$(\lambda, u - \psi) = 0, \quad \lambda \le 0, \quad u \ge \psi$$

where the generator B for the jump process is given by

$$Bu = \int_{-\infty}^{\infty} k(s)(u(x+s) - u(x) + (e^s - 1)u_x)\, ds.$$

The CMGY model for the jump kernel k is given by

$$k(s) = \begin{cases} Ce^{-M|s|}|s|^{1+Y} = k^+(s) & s \geq 0 \\ \\ Ce^{-G|s|}|s|^{1+Y} = k^-(s) & s \leq 0 \end{cases}$$

Since

$$\int_{-\infty}^{\infty} k(s)(u(x+s) - u(x))\, ds = \int_0^{\infty} k^+(s)(u(x+s) - u(x))\, ds + \int_0^{\infty} k^-(s)(u(x-s) - u(x))\, ds$$

$$= \int_0^{\infty} \frac{k^+(s) + k^-(s)}{2}(u(x+s) - 2u(x) + u(x-s))\, ds + \int_0^{\infty} \frac{k^+(s) - k^-(s)}{2}(u(x+s) - u(x-s))\, ds.$$

Thus,

$$\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k(s)(u(x+s) - u(x)\, ds)\phi(x)\, dx$$

$$= \int_{-\infty}^{\infty}\int_0^{\infty} k_s(s)(u(x+s) - u(x))(\phi(x+s) - \phi(x))\, ds\, dx$$

$$+ \int_{-\infty}^{\infty}(\int_0^{\infty} k_u(s)(u(x+s) - u(s)))\phi(x)\, dx$$

where

$$k_s(s) = \frac{k^+(s) + k^-(s)}{2}, \quad k_u(s) = \frac{k^+(s) - k^-(s)}{2}$$

and hence

$$(Bu, \phi) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_s(s)(u(x+s) - u(x))(\phi(x+s) - \phi(s))\, ds\, dx$$

$$+ \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} k_u(s)(u(x+s) - u(s)))\phi(x)\, dx + \omega \int_{-\infty}^{\infty} u_x\phi\, dx.$$

where

$$\omega = \int_{-\infty}^{\infty}(e^s - 1)k(s)\, ds.$$

If we equip $V = H^1(R)$ by

$$|u|_V^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k_s(s)|u(x+s) - u(x)|^2\, ds\, dx + \frac{\sigma^2}{2}\int_{-\infty}^{\infty} |u_x|^2\, dx,$$

then $A + B \in \mathcal{L}(V, V^*)$ and $A + B$ generates the analytic semigroup on $X = L^2(R)$.

1.5 Numerical approximation of nonlocal operator

In this section we describe our higher order integration method for the convolution;

$$\int_0^\infty \frac{k^+(s)+k^-(s)}{2}(u(x+s)-2u(x)+u(x-s))\,ds+\int_0^\infty \frac{k^+(s)-k^-(s)}{2}(u(x+s)-u(x-s))\,ds.$$

For the symmetric part,

$$\int_{-\infty}^{\infty} s^2 k_s(s)\,\frac{u(x+s)-2u(x)+u(x-s)}{s^2}\,ds,$$

where we have

$$\frac{u(x+s)-2u(x)+u(x-s)}{s^2} \sim u_{xx}(x)+\frac{s^2}{12}u_{xxxx}(x)+O(s^4)$$

We apply the fourth order approximation of u_{xx} by

$$u_{xx}(x)\sim \frac{u(x+h)-2u(x)+u(x-h)}{h^2}-\frac{1}{12}\frac{u(x+2h)-4u(x)+6u(x)-4u(x-h)+u(x-2h)}{h^2}$$

and we apply the second order approximation of $u_{xxxx}(x)$ by

$$u_{xxxx}(x)\sim \frac{u(x+2h)-4u(x)+6u(x)-4u(x-h)+u(x-2h)}{h^4}.$$

Thus, one can approximate

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s)\,\frac{u(x+s)-2u(x)+u(x-s)}{s^2}\,ds$$

by

$$\rho_0\,(\frac{u_{k+1}-2u_k+u_{k-1}}{h^2}-\frac{1}{12}\frac{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}}{h^2})$$
$$+\frac{\rho_1}{12}\,\frac{u_{k+2}-4u_{k+1}+6u_k-4u_{k-1}+u_{k-2}}{h^2},$$

where

$$\rho_0=\int_{-\frac{h}{2}}^{\frac{h}{2}} s^2 k_s(s)\,ds \quad \text{and} \quad \rho_1=\frac{1}{h^2}\int_{-\frac{h}{2}}^{\frac{h}{2}} s^4 k_s(s)\,ds.$$

The remaining part of the convolution

$$\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j}+s)k_s(s)\,ds$$

can be approximated by three point quadrature rule based on

$$u(x_{k+j}+s)\sim u(x_{k+j})+u'(x_{k+j})s+\frac{s^2}{2}u''(x_{k+j})$$

with

$$u'(x_{k+j}) \sim \frac{u_{k+j+1} - u_{k+j-1}}{2h}$$

$$u''(x_{k+j}) \sim \frac{u_{k+j+1} - 2u_{k+j} + u_{k+j-1}}{h^2}.$$

That is,

$$\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} u(x_{k+j} + s) k_s(s)\, ds$$

$$\sim \rho_0^k u_{k+j} + \rho_1^k \frac{u_{k+j-1} - u_{k+j+1}}{2} + \rho_2^k \frac{u_{j+k+1} - 2u_{k+j} + u_{j+k-1}}{2}$$

where

$$\rho_0^k = \textstyle\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} k_s(s)\, ds$$

$$\rho_1^k = \tfrac{1}{h} \textstyle\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s - x_k) k_s(s)\, ds$$

$$\rho_2^k = \tfrac{1}{h^2} \textstyle\int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} (s - x_k)^2 k_s(s)\, ds.$$

For the skew-symmetric integral

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} k_u(s)(u(x+s) - u(x-s))\, ds \sim \rho_2\, u_x(x) + \frac{\rho_3}{6} h^2\, u_{xxx}(x)$$

where

$$\rho_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} 2s k_u(s)\, ds, \quad \rho_3 = \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} 2s^3 k_u(s)\, ds.$$

We may use the forth order difference approximation

$$u_x(x) \sim \frac{u(x+h) - u(x-h)}{2h} - \frac{u(x+2h) - 2u(x+h) + 2u(x-h) - u(x-2h)}{6h}$$

and the second order difference approximation

$$u_{xxx}(x) \sim \frac{u(x+2h) - 2u(x+h) + 2u(x-h) - u(x-2h)}{h^3}$$

and obtain

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} k_u(s)(u(x+s) - u(x-s))\, ds$$

$$\sim \rho_2 \,(\frac{u_{k+1} - u_{k-1}}{2h} - \frac{u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-1}}{6h}) + \frac{\rho_3}{6} \frac{u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-1}}{h}.$$

Example (Fractional PDEs II) Consider the fractional equation of the form

$$\int_{-t}^{0} g(\theta) u'(t+\theta)\, d\theta = Au, \quad u(0) = u_0,$$

where the kernel g satisfies

$$g > 0, \quad g \in L^1(-\infty, 0) \text{ and non-decreasing.}$$

For example, the case of the Caputo (fractional) derivative has

$$g(\theta) = \frac{1}{\Gamma(1-\alpha)} |\theta|^{-\alpha}.$$

Define $z(t,\theta) = u(t+\theta), \theta \in (-\infty, 0]$. Then, $\frac{d}{dt} z = \frac{\partial}{\partial \theta} z$. Thus, we define the linear operator $\mathcal{A}$ on $Z = C((-\infty, 0]; X)$ by

$$\mathcal{A} z = z'(\theta) \text{ with } dom(\mathcal{A}) = \{z' \in X : \int_{-\infty}^{0} g(\theta) z'(\theta)\, d\theta = Az(0)\}$$

Theorem 1.1 *Assume A is m-dissipative in a Banach space X. Then, $\mathcal{A}$ is dissipative and $R(\lambda\, I - \mathcal{A}) = Z$ for $\lambda > 0$. Thus, $\mathcal{A}$ generates the C_0-semigroup $T(t)$ on $Z = C((-\infty, 0]; X)$.*

Proof: First we show that $\mathcal{A}$ is dissipative. For $\phi \in dom\,(\mathcal{A})$ suppose $|\phi(0)| > |\phi(\theta)|$ for all $\theta < 0$. Define

$$g_\epsilon(\theta) = \frac{1}{\epsilon} \int_{\theta-\epsilon}^{\theta} g(\theta)\, d\theta.$$

For all $x^* \in F(\phi(0))$

$$\langle \int_{-\infty}^{0} g_\epsilon(\theta)(\phi') d\theta, x^* \rangle$$

$$= -\langle \int_{-\infty}^{0} \frac{g(\theta) - g(\theta - \epsilon)}{\epsilon} \langle \phi(\theta) - \phi(0), x^* \rangle \, d\theta > 0$$

since

$$\langle \phi(\theta) - \phi(0), x^* \rangle \leq (|\phi(\theta)| - |\phi(0)|) |\phi(0)| < 0, \quad \theta < 0.$$

Thus,

$$\lim_{\epsilon \to 0^+} \langle \int_{-\infty}^{0} g_\epsilon(\theta)(\phi') d\theta, x^* \rangle = \langle \int_{-\infty}^{0} g(\theta) \phi' d\theta, x^* \rangle > 0. \tag{1.34}$$

But, since there exists a $x^* \in F(\phi(0))$ such that

$$\langle Ax, x^* \rangle \leq 0$$

which contradicts to (1.34). Thus, there exists θ_0 such that $|\phi(\theta_0)| = |\phi|_Z$. Since $\langle \phi(\theta), x^* \rangle \leq |\phi(\theta)|$ for $x^* \in F(\phi(\theta_0))$, $\theta \to \langle \phi(\theta), x^* \rangle$ attains the maximum at θ_0 and thus $\langle \phi'(\theta_0), x^* \rangle = 0$ Hence,

$$|\lambda\, \phi - \phi'|_Z \geq \langle \lambda\, \phi(\theta_0) - \phi'(\theta_0), x^* \rangle = \lambda\, |\phi(\theta_0)| = \lambda\, |\phi|_Z. \tag{1.35}$$

For the range condition $\lambda \phi - \mathcal{A} \phi = f$ we note that

$$\phi(\theta) = e^{\lambda \theta} \phi(0) + \psi(\theta)$$

where

$$\psi(\theta)=\int_\theta^0 e^{\lambda(\theta-\xi)}f(\xi)\,d\xi.$$

Thus,

$$(\Delta(\lambda)\,I-A)\phi(0)=\int_{-\infty}^0 g(\theta)\psi'(\theta)\,d\theta)$$

where

$$\Delta(\lambda)=\lambda\int_{-\infty}^0 g(\theta)e^{\lambda\theta}\,d\theta>0$$

Thus,

$$\phi(0)=(\Delta(\lambda)\,I-A)^{-1}\int_{-\infty}^0 g'(\theta)\psi(\theta)\,d\theta.$$

Since $\mathcal{A}$ is dissipative and

$$\lambda\psi-\psi'=f\in Z,\quad \psi(0)=0,$$

thus $|\psi|_Z\leq \dfrac{1}{\lambda}|f|_Z$. Thus $\phi=(\lambda\,I-\mathcal{A})^{-1}f\in Z$. $\square$

Example (Renewable system) We discuss the renewable system of the form

$$\tfrac{dp_0}{dt}=-\textstyle\sum_i\lambda_i\,p_0(t)+\sum_i\int_0^L\mu_i(x)p_i(x,t)\,dx$$

$$(p_i)_t+(p_i)_x=-\mu_i(x)p,\quad p(0,t)=\lambda_i\,p_0(t)$$

for $(p_0,p_i,\;1\leq i\leq d)\in R\times L^1(0,T)^d$. Here, $p_0(t)\geq 0$ is the total utility and $\lambda_i\geq 0$ is the rate for the energy conversion to the i-th process p_i. The first equation is the energy balance law and s is the source = generation $-$consumption. The second equation is for the transport (via pipeline and storage) for the process p_i and $\mu_i\geq 0$ is the renewal rate and $\bar\mu\geq 0$ is the natural loos rate. $\{\lambda_i\geq 0\}$ represent the distribution of the utility to the i-th process.

Assume at the time $t=0$ we have the available utility $p_0(0)=1$ and $p_i=0$. Then we have the following conservation

$$p_0(t)+\int_0^t p_i(s)\,ds=1$$

if $t\leq L$. Let $X=R\times L^1(0,L)^d$. Let $A(\mu)$ defined by

$$Ax=(-\sum_i\lambda_i\,p_0+\sum_i\int_0^L\mu_i(x)\,dx,-(p_i)_x-\mu_i(x)p_i)$$

with domain

$$dom(A)=\{(p_0,p_i)\in R\times W^{11}(0,L)^d:p_i(0)=\lambda_i\,p_0\}$$

Let

$$\mathrm{sign}_\epsilon(s)=\begin{cases}\frac{s}{|s|} & |s|>\epsilon\\ \frac{s}{\epsilon} & |s|\leq\epsilon\end{cases}$$

Then,

$$(A(p_0,p),(\mathrm{sign}_0(p_0),\mathrm{sign}_\epsilon(p)) \le -(\sum_i \lambda_i)|p_0| + |\int_0^L \mu_i p_i\,dx|$$

$$\sum_i(\Psi_\epsilon(p_i(0)) - \Psi_\epsilon(p_i(L)) - \int_0^L \mu_i p_i \mathrm{sign}_\epsilon\,dx)$$

where

$$\Psi_\epsilon(s) = \begin{cases} |s| & |s| > \epsilon \\ \\ \frac{s^2}{2\epsilon} + \frac{\epsilon}{2} & |s| \le \epsilon \end{cases}$$

Since

$$\mathrm{sign}_\epsilon \to \mathrm{sign}_\epsilon, \quad \Psi_\epsilon \to |s|$$

by the Lebesgue dominated convergence theorem, we have

$$(A(p_0,p),(\mathrm{sign}_0(p_0),\mathrm{sign}_0(p)) \le 0.$$

The resolvent equation

$$A(p_0,p) = (s,f), \tag{1.36}$$

has a solution

$$p_i(x) = \lambda_i p_0 e^{-\int_0^x \mu_i} + \int_0^x e^{-\int_s^x \mu_i} f(s)\,ds$$

$$(\sum_i \lambda_i)(1 - e^{\int_0^L \mu_i})\,p_0 = s + \int_0^L \mu_i p_i(x)\,dx$$

Thus, A generates the contractive C_0 semigroup $S(t)$ on X. Moreover, it is cone preserving $S(t)\mathcal{C}^+ \subset \mathcal{C}^+$ since the resolvent is positive cone preserving.

Example (Bi-domain equation)

The electrical behavior of the cardiac tissue is described by a system consisting of PDEs coupled with ordinary differential equations which model the ionic currents associated with the reaction terms. The bi-domain model is a mathematical model for the electrical properties of cardiac muscle that takes into account the anisotropy of both the intracellular and extracellular spaces. It is formed of the bi-domain equations. The bi-domain model is now widely used to model defibrillation of the heart. In this paper we consider the feedback control for bi-domain model.

The weak form of the the bi-domain equation is given by

$$\begin{aligned} &(\frac{d}{dt}u,\phi) - (\mathcal{B}(\nabla u + \nabla u_e),\nabla\phi)_\Omega + (F(u,v),\phi) = 0 \\ &(\mathcal{B}\nabla u + (\mathcal{A}+\mathcal{B})\nabla u_e,\nabla\psi)_\Omega = \langle s,\psi\rangle, \end{aligned} \tag{1.37}$$

for all $(\phi,\psi) \in H^1(\Omega)\times H^1(\Omega)/R$, where $(u,u_e) \in H^1(\Omega)\times H^1(\Omega)/R$ is the solution pair and s is the control current. We consider the boundary current control:

$$\langle s,\psi\rangle = \int_\Gamma s(t,x)\psi(x)\,dx.$$

Here, $\mathcal{A}$ and $\mathcal{B}$ are elliptic operators of the form

$$\mathcal{B}\phi = \nabla\cdot(\bar\sigma_i\nabla\phi), \quad \mathcal{A}\phi = \nabla\cdot(\bar\sigma_e\nabla\phi),$$

where $\bar{\sigma}_i,\ \bar{\sigma}_e$ are respectively the intracellular and extracellular conductivity tensors. Note that one can write (1.37) as

$$\frac{d}{dt}u(t) - \mathcal{L}u(t) + F(u(t), v(t))) + \mathcal{C}s(t) = 0, \tag{1.38}$$

where

$$\mathcal{L} = (\mathcal{A}^{-1} + \mathcal{B}^{-1})^{-1} = \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}\mathcal{A}.$$

and

$$\mathcal{C}s = \mathcal{B}(\mathcal{A} + \mathcal{B})^{-1}s.$$

That is, $v = u + u_e$ satisfies

$$(\mathcal{A} + \mathcal{B})v = \mathcal{A}u + s$$

where

$$\langle s, \phi \rangle = \int_\Gamma s(t, x)\phi(x)\, dx$$

with $\langle s, 1 \rangle = 0$. Thus, $\mathcal{L}$ is an self adjoint elliptic operator on $L^2(\Omega)$. The boundary current control becomes the distributed control of the form $\mathcal{C}s(t)$.

Example (Second order equation) Let $V \subset H = H^* \subset V^*$ be the Gelfand triple. Let ρ be a bounded bilinear form on $H \times H$, μ and σ be bounded bilinear forms on $V \times V$. Assume ρ and σ are symmetric and coercive and $\mu(\phi, \phi) \geq 0$ for all $\phi \in V$. Consider the second order equation

$$\rho(u_{tt}, \phi) + \mu(u_t, \phi) + \sigma(u, \phi) = \langle f(t), \phi \rangle \text{ for all } \phi \in V. \tag{1.39}$$

Define linear operators M (mass), D (dampping), K and (stiffness) by

$$(M\phi, \psi)_H = \rho(\phi, \psi), \quad \phi,\ \psi \in H$$

$$\langle D\phi, \psi \rangle = \mu(\phi, \psi) \quad \phi,\ \psi \in V$$

$$\langle K\phi, \psi \rangle_{V^* \times V} = \sigma(\phi, \psi), \quad \phi,\ \psi \in V$$

We assume ρ is symmetric and H-coercive, σ is symmetric and V-coercive and $\mu(\phi, \phi) \geq 0$ for $\phi \in V$. Let $v = u_t$ and define A on $X = V \times H$ by

$$A(u, v) = (v, -M^{-1}(Ku + Dv))$$

with domain

$$dom\,(A) = \{(u, v) \in X : v \in V \text{ and } Ku + Dv \in H\}$$

The state space X is a Hilbert space with inner product

$$((u_1, v_1), (u_, v_2)) = \sigma(u_1, u_2) + \rho(v_1, v_2)$$

and

$$E(t) = |(u(t), v(t))|_X^2 = \sigma(u(t), u(t)) + \rho(v(t), v(t))$$

defines the energy of the state $x(t) = (u(t), v(t)) \in X$. First, we show that A is dissipative:

$$(A(u,v),(u,v))_X = \sigma(u,v) + \rho(-M^{-1}(Ku+Dv),v) = \sigma(u,v) - \sigma(u,v) - \mu(v,v) = -\mu(v,v) \leq 0$$

Next, we show that $R(\lambda I - A) = X$. That is, for $(f,g) \in X$ the exists a solution $(u,v) \in dom\,(A)$ satisfying

$$\lambda\, u - v = f, \quad \lambda M v + D v + K u = M g,$$

or equivalently $v = \lambda\, u - f$ and

$$\lambda^2 M u + \lambda D u + K u = M g + \lambda\, M f + D f \tag{1.40}$$

Define the bilinear form a on $V \times V$

$$a(\phi,\psi) = \lambda^2\, \rho(\phi,\psi) + \lambda\, \mu(\phi,\psi) + \sigma(\phi,\psi)$$

Then, a is bounded and V-coercive and if we let

$$F(\phi) = (M(g + \lambda\, f)\phi)_H + \mu(f,\phi)$$

then $F \in V^*$. It thus follows from the Lax-Milgram theory there exists a unique solution $u \in V$ to (1.40) and $Dv + Ku \in H$.

For example, consider the wave equation

$$\tfrac{1}{c^2(x)} u_{tt} + \kappa(x) u_t = \Delta u$$

$$[\tfrac{\partial u}{\partial n}] + \alpha u = \gamma\, u_t \text{ at } \Gamma$$

In this example we let $V = H^1(\Omega)/R$ and $H = L^2(\Omega)$ and define

$$\sigma(\phi,\psi) = \int_\Omega (\nabla\phi, \nabla\psi)\, dx + \int_\Gamma \alpha\phi\psi\, ds$$

$$\mu(\phi,\psi) = \int_{\partial\Omega} \kappa(x)\phi,\psi\, dx + \int_\Gamma \gamma)\phi,\psi\, ds$$

$$\rho(\phi,\psi) = \int_\Omega \frac{1}{c^2(x)}\, \phi\psi\, dx.$$

Example (Maxwell system for electro-magnetic equations)

$$\epsilon\, E_t = \nabla \times H, \quad \nabla \cdot E = \rho$$

$$\mu\, H_t = -\nabla \times E, \quad \nabla \cdot B = 0$$

with boundary condition

$$E \times n = 0$$

where E is Electric field, $B\mu\, H$ is Magnetic field and $D = \epsilon\, E$ is dielectric with $\epsilon,\ \mu$ is electric and magnetic permittivity, respectively. Let $X = L^2(\Omega)^d \times L^2(\Omega)^d$ with the norm defined by

$$|(E, H)|_X^2 = \int_\Omega (\epsilon\, |E|^2 + \mu\, |H|^2)\, dx.$$

The dissipativity follows from

$$\int_\Omega (E \cdot (\nabla \times H) - H \cdot (\nabla \times E))\, dx = \int_\Omega \nabla \cdot (E \times H)\, dx = \int_{\partial\Omega} n \cdot (E \times H)\, ds = 0$$

Let $\rho = 0$ and thus $\nabla \cdot E = 0$. The range condition is equivalent to

$$\epsilon\, E + \nabla \times \frac{1}{\mu}(\nabla \times E - g) = f$$

The weak form is given by

$$(\epsilon\, E, \psi) + (\frac{1}{\mu}\nabla \times E, \nabla \times \psi) = (f, \psi) + (g, \frac{1}{\mu}\nabla \times \psi). \tag{1.41}$$

for $\psi \in V = \{H^1(\Omega) : \nabla \cdot \psi = 0,\ \ n \times \psi = 0 \text{ at } \partial\Omega\}$. Since $|\nabla \times \psi|^2 = |\nabla\psi|^2$ for $\nabla \cdot \psi = 0$. the right hand side of (1.41) defines the bounded coercive quadratic form on $V \times V$, it follows from the Lax-Milgram equation that (1.41) has a unique solution in V.

1.6 Dual semigroup

Theorem (Dual semigroup) Let X be a reflexive Banach space. The adjoint $S^*(t)$ of the C_0 semigroup $S(t)$ on X forms the C_0 semigroup and the infinitesimal generator of $S^*(t)$ is A^*. Let X be a Hilbert space and $dom(A^*)$ be the Hilbert space with graph norm and X_{-1} be the strong dual space of $dom(A^*)$, then the extension $S(t)$ to X_{-1} defines the C_0 semigroup on X_{-1}.

Proof: (1) Since for $t,\ s \geq 0$

$$S^*(t + s) = (S(s)S(t))^* = S^*(t)S^*(s)$$

and

$$\langle x, S^*(t)y - y\rangle_{X\times X^*} = \langle S(t)x - x, y\rangle_{X\times X^*} \to 0.$$

for $x \in X$ and $y \in X^*$ Thus, $S^*(t)$ is weakly star continuous at $t = 0$ and let B is the generator of $S^*(t)$ as

$$Bx = w^* - \lim \frac{S^*(t)x - x}{t}.$$

Since

$$(\frac{S(t)x - x}{t}, y) = (x, \frac{S^*(t)y - y}{t}),$$

for all $x \in dom(A)$ and $y \in dom(B)$ we have

$$\langle Ax, y\rangle_{X\times X^*} = \langle x, By\rangle_{X\times X^*}$$

and thus $B = A^*$. Thus, A^* is the generator of a w^*- continuous semigroup on X^*.
(2) Since

$$S^*(t)y - y = A^* \int_0^t S^*(s)y\, ds$$

for all $y \in Y = \overline{dom\,(A^*)}$. Thus, $S^*(t)$ is strongly continuous at $t = 0$ on Y.
(3) If X is reflexive, $\overline{dom\,(A^*)} = X^*$. If not, there exists a nonzero $y_0 \in X$ such that $\langle y_0, x^* \rangle_{X \times X^*} = 0$ for all $x^* \in dom(A^*)$. Thus, for $x_0 = (\lambda\, I - A)^{-1} y_0$ $\langle \lambda\, x_0 - Ax_0, x^* \rangle = \langle x_0, \lambda\, x^* - A^* x^* \rangle = 0$. Letting $x^* = (\lambda\, I - A^*)^{-1} x_0^*$ for $x_0^* \in F(x_0)$, we have $x_0 = 0$ and thus $y_0 = 0$, which yields a contradiction.
(4) $X_1 = dom\,(A^*)$ is a closed subspace of X^* and is a invariant set of $S^*(t)$. Since A^* is closed, $S^*(t)$ is the C_0 semigroup on X_1 equipped with its graph norm. Thus,

$$(S^*(t))^* \text{ is the } C_0 \text{ semigroup on } X_{-1} = X_1^*$$

and defines the extension of $S(t)$ to X_{-1}. Since for $x \in X \subset X_{-1}$ and $x^* \in X^*$

$$\langle S(t)x, x^* \rangle = \langle x, S^*(t)x^* \rangle,$$

$S(t)$ is the restriction of $(S^*(t))^*$ onto X. □

1.7 Stability

Theorem (Datko 1970, Pazy 1972). A strongly continuous semigroup $S(t)$, $t \geq 0$ on a Banach space X is uniformly exponentially stable if and only if for $p \in [1, \infty)$ one has

$$\int_0^\infty |S(t)x|^p\, dt < \infty \text{ for all } x \in X.$$

Theorem. (Gearhart 1978, Pruss 1984, Greiner 1985) A strongly continuous semigroup on $S(t)$, $t \geq 0$ on a Hilbert space X is uniformly exponentially stable if and only if the half-plane $\{\lambda \in C : Re\lambda > 0\}$ is contained in the resolvent set $\rho(A)$ of the generator A with the resolvent satisfying

$$|(\lambda\, I - A)^{-1}|_\infty < \infty$$

1.8 Sectorial operator and Analytic semigroup

In this section we have the representation of the semigroup $S(t)$ in terms of the inverse Laplace transform. Taking the Laplace transform of

$$\frac{d}{dt}x(t) = Ax(t) + f(t)$$

we have

$$\hat{x} = (\lambda\, I - A)^{-1}(x(0) + \hat{f})$$

where for $\lambda > \omega$

$$\hat{x} = \int_0^\infty e^{-\lambda t} x(t)\, dt$$

is the Laplace transform of $x(t)$. We have the following the representation theory (inverse formula).

Theorem (Resolvent Calculus) For $x \in dom(A^2)$ and $\gamma > \omega$

$$S(t)x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda\, I - A)^{-1} x \, d\lambda. \tag{1.42}$$

Proof: Let A_μ be the Yosida approximation of A. Since $\mathrm{Re}\, \sigma(A_\mu) \leq \dfrac{\omega_0}{1 - \mu\omega_0} < \gamma$, we have

$$u_\mu(t) = S_\mu(t)x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda\, I - A_\mu)^{-1} x \, d\lambda.$$

Note that

$$\lambda(\lambda\, I - A)^{-1} = I + (\lambda\, I - A)^{-1} A. \tag{1.43}$$

Since

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda} \, d\lambda = 1$$

and

$$\int_{\gamma - i\infty}^{\gamma + i\infty} |\lambda - \omega|^{-2} \, d\lambda < \infty,$$

we have

$$|S_\mu(t)x| \leq M\, |A^2 x|,$$

uniformly in $\mu > 0$. Since

$$(\lambda\, I - A_\mu)^{-1} x - (\lambda\, I - A)^{-1} x = \frac{\mu}{1 + \lambda\mu} (\nu\, I - A)^{-1} (\lambda\, I - A)^{-1} A^2 x,$$

where $\nu = \dfrac{\lambda}{1 + \lambda\mu}$, $\{u_\mu(t)\}$ is Cauchy in $C(0, T; X)$ if $x \in dom(A^2)$. Letting $\mu \to 0^+$, we obtain (1.42). $\square$

Next we consider the sectorial operator. For $\delta > 0$ let

$$\Sigma_\omega^\delta = \{\lambda \in C : arg(\lambda - \omega) < \frac{\pi}{2} + \delta\}$$

be the sector in the complex plane C. A closed, densely defined, linear operator A on a Banach space X is a sectorial operator if

$$|(\lambda\, I - A)^{-1}| \leq \frac{M}{|\lambda - \omega|} \text{ for all } \lambda \in \Sigma_\omega^\delta.$$

For $0 < \theta < \delta$ let $\Gamma = \Gamma_{\omega,\theta}$ be the integration path defined by

$$\Gamma^\pm = \{z \in C : |z| \geq \delta,\ arg(z - \omega) = \pm(\tfrac{\pi}{2} + \theta)\},$$

$$\Gamma_0 = \{z \in C : |z| = \delta,\ |arg(z - \omega)| \leq \tfrac{\pi}{2} + \theta\}.$$

For $0 < \theta < \delta$ define a family $\{S(t),\ t \geq 0\}$ of bounded linear operators on X by

$$S(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda\, I - A)^{-1}x\, d\lambda. \tag{1.44}$$

Theorem (Analytic semigroup) If A is a sectorial operator on a Banach space X, then A generates an analytic (C_0) semigroup on X, i.e., for $x \in X$ $t \to S(t)x$ is an analytic function on $(0,\infty)$. We have the representation (1.44) for $x \in X$ and

$$|AS(t)x|_X \leq \frac{M_\theta}{t}|x|_X \ \ (\omega = 0).$$

Proof: Since

$$AS(t)x = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda(\lambda\, I - A)^{-1}x - x)\, d\lambda.$$

we have

$$|AS(t)x| \leq M\int_0^\infty e^{-sin\theta\, tz}\, dz|x| = \frac{M}{\sin\theta\, t}|x|.\square$$

The elliptic operator A defined by the Lax-Milgram theorem defines a sectorial operator on Hilbert space X.

Theorem (Sectorial operator) Let $V,\ H$ are Hilbert spaces and assume $H \subset V^*$. Let $\rho(u,v)$ is bounded bilinear form on $H \times H$ and

$$\rho(u,u) \geq |u|_H^2 \text{ for all } u \in H$$

Let $a(u,v)$ to be a bounded bilinear form on $V \times V$ with

$$\sigma(u,u) \geq \delta\, |u|_V^2 \text{ for all } u \in V.$$

Define the linear operator A by

$$\rho(Au,\phi) = a(u,\phi) \text{ for all } \phi \in V.$$

Then, for $Re\,\lambda > 0$we have

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V,V^*)} \leq \tfrac{1}{\delta}$$

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H)} \leq \tfrac{M}{|\lambda|}$$

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V^*,H)} \leq \tfrac{M}{\sqrt{|\lambda|}}$$

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H,V)} \leq \tfrac{M}{\sqrt{|\lambda|}}$$

Proof: Let $a(u,v)$ to be a bounded bilinear form on $V \times V$. Define $M \in \mathcal{L}(H,H)$ by

$$(Mu,v) = \rho(u,v) \text{ for all } u,\ v \in H$$

and $A_0 \in \mathcal{L}(V,V^*)$ by

$$\langle A_0u,v\rangle = \sigma(u,v) \text{ for } v \in V$$

Then, $A = M^{-1}A_0$ and for $f \in V^*$ and $Re\,\lambda > 0$, $(\lambda\, I - A)u = M^{-1}f$ is equivalent to

$$\lambda\rho(u,\phi) + a(u,\phi) = \langle f, \phi\rangle, \text{ for all } \phi \in V. \tag{1.45}$$

It follows from the Lax-Milgram theorem that (1.45) has a unique solution, given $f \in V^*$ and

$$Re\,\lambda\rho(u,u) + a(u,u) \le |f|_{V^*}|u|_V.$$

Thus,

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V^*,V)} \le \frac{1}{\delta}.$$

Also,

$$|\lambda|\, |u|_H^2 \le |f|_{V^*}|u|_V + M\,|u|_V^2 = M_1\,|f|_{V^*}^2$$

for $M_1 = 1 + \frac{M}{\delta^2}$ and thus

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(V^*,H)} \le \frac{\sqrt{M_1}}{|\lambda|^{1/2}}.$$

For $f \in H \subset V^*$

$$\delta\,|u|_V^2 \le Re\,\lambda\,\rho(u,u) + a(u,u) \le |f|_H|u|_H, \tag{1.46}$$

and

$$|\lambda|\rho(u,u) \le |f|_H|u|_H + M|u|_V^2 \le M_1|f|_H|u|_H$$

Thus,

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H)} \le \frac{M_1}{|\lambda|}.$$

Also, from (1.46)

$$\delta\,|u|_V^2 \le |f|_H|u|_H \le M_1\,|f|^2.$$

which implies

$$|(\lambda\, I - A)^{-1}|_{\mathcal{L}(H,V)} \le \frac{M_2}{|\lambda|^{1/2}}.$$

1.9 Approximation Theory

In this section we discuss the approximation theory for the linear C_0-semigroup. Equivalence Theorem (Lax-Richtmyer) states that for consistent numerical approximations, stability and convergence are equivalent. In terms of the linear semigroup theory we have

Theorem (Trotter-Kato theorem) Let X and X_n be Banach spaces and A and A_n be the infinitesimal generator of C_0 semigroups $S(t)$ on X and $S_n(t)$ on X_n of $G(M,\omega)$ class. Assume a family of uniformly bounded linear operators $P_n \in \mathcal{L}(X, X_n)$ and $E_n \in \mathcal{L}(X_n, X)$ satisfy

$$P_nE_n = I \quad |E_nP_nx - x|_X \to 0 \text{ for all } x \in X \tag{1.47}$$

Then, the followings are equivalent.
(1) there exist a $\lambda_0 > \omega$ such that for all $x \in X$

$$|E_n(\lambda_0\, I - A_n)^{-1}P_nx - (\lambda_0\, I - A)^{-1}x|_X \to 0 \quad \text{as } n \to \infty, \tag{1.48}$$

(2) For every $x \in X$ and $T \geq 0$

$$|E_n S_n(t) P_n x - S(t)x|_X \to \quad \text{as } n \to \infty.$$

uniformly on $t \in [0,T]$.

Proof: Since for $\lambda > \omega$

$$E_n(\lambda\, I - A)^{-1} P_n x - (\lambda\, I - A)^{-1} x = \int_0^\infty E_n S_n(t) P_n x - S(t)x\, dt$$

(1) follows from (2). Conversely, from the representation theory

$$E_n S_n(t) P_n x - S(t)x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} (E_n(\lambda\, I - A)^{-1} P_n x - (\lambda\, I - A)^{-1} x)\, d\lambda$$

where

$$(\lambda\, I - A)^{-1} - (\lambda_0\, I - A)^{-1} = (\lambda - \lambda_0)(\lambda\, I - A)^{-1}(\lambda_0\, I - A)^{-1}.$$

Thus, from the proof of Theorem (Resolvent Calculus) (1) holds for $x \in dom(A^2)$. But since $dom(A^2)$ is dense in X, (2) implies (1). $\square$

Remark (Stability) If A_n is uniformly dissipative:

$$|\lambda u_n - A_n u_n| \geq (\lambda - \omega)\, |u_n|$$

for all $u_n \in dom(A_n)$ and some $\omega \geq 0$, then A_n generates ω contractive semigroup $S_n(t)$ on X_n.

Remark (Consistency)

$$(\lambda I - A_n) u_n = P_n f$$

$$P_n(\lambda I - A) u = P_n f$$

we have

$$(\lambda\, I - A_n)(P_n u - u_n) + P_n A u - A_n P_n u = 0$$

Thus

$$|P_n u - u_n| \leq M\, |P_n A u - A_n P_n u|$$

The consistency(1.48) follows from

$$|P_n A u - A_n P_n u| \to 0$$

for all u in a dense subset of $dom(A)$.

Corollary Let the assumptions of Theorem hold. The statement (1) of Theorem is equivalent to (1.47) and the followings:

(C.1) there exists a subset D of $dom(A)$ such that $\overline{D} = X$ and $\overline{(\lambda_0\, I - A)D} = X$.

(C.2) for all $u \in D$ there exists a sequence $\bar{u}_n \in dom(A_n)$ such that $\lim E_n \bar{u}_n = u$ and $\lim E_n A_n \bar{u}_n = Au$.

Proof: Without loss of generality we can assume $\lambda_0 = 0$. First we assume that condition (1) hold. We set $D = dom(A)$ and thus $AD = X$. For $u \in dom(A)$ we set $\bar{u}_n = A_n^{-1}P_nAu$ and $u = A^{-1}x$. Then,

$$E_n\bar{u}_n - u = E_nA_n^{-1}P_nx - A^{-1}x \to 0$$

and

$$E_nA_n\bar{u}_n - Au = E_nA_nA_n^{-1}P_nx - AA^{-1}x = E_nP_nx - x \to 0$$

as $n \to \infty$. Hence conditions (C.1)–(C.2) hold.

Conversely, we assume conditions (C.1)–(C.2) hold. For $x \in AD$ we choose $u \in D$ such that $u = A^{-1}x$ and set $u_n = A_n^{-1}P_nx = A_n^{-1}P_nAu$. We then for u we choose $\bar{u}_n$ according to (C.2). Thus, we obtain

$$|\bar{u}_n - P_nu| = |P_n(E_n\bar{u}_n - u)| \le M\,|E_n\bar{u}_n - u| \to 0$$

as $n \to \infty$ and

$$|\bar{u}_n - u_n| \le |A_n^{-1}(A_n\bar{u}_n - P_nAu)| \le |A_n^{-1}P_n||E_nA_n\bar{u}_n - Au| \to 0$$

as $n\infty$. It thus follows that $|u_n - P_nu| \to 0$ as $n \to \infty$. Since

$$E_nA_n^{-1}P_n - A^{-1} = E_n(A_n^{-1}P_nA - P_n)A^{-1} + (E_nP_n - I)A^{-1},$$

we have

$$|E_nA_n^{-1}P_nx - A^{-1}x| \le |E_n(u_n - P_nu)| + |E_nP_nu - u| \le M\,|u_n - P_nu| + |E_nP_nu - u| \to 0$$

as $n \to \infty$ for all $x \in AD$. $\square$

Example 1 (Trotter-Kato theoarem) Consider the heat equation on $\Omega = (0,1)\times(0,1)$:

$$\frac{d}{dt}u(t) = \Delta u, \quad u(0,x) = u_0(x)$$

with boundary condition $u = 0$ at the boundary $\partial\Omega$. We use the central difference approximation on uniform grid points: $(i\,h, j\,h) \in \Omega$ with mesh size $h = \frac{1}{n}$:

$$\frac{d}{dt}u_{i,j}(t) = \Delta_h u = \frac{1}{h}\left(\frac{u_{i+1,j}-u_{i,j}}{h} - \frac{u_{i,j}-u_{i-1,j}}{h}\right) + \frac{1}{h}\left(\frac{u_{i,j+1}-u_{i,j}}{h} - \frac{u_{i,j}-u_{i,j-1}}{h}\right)$$

for $1 \le i,\ j \le n_1$, where $u_{i,0} = u_{i,n} = u1, j = u_{n,j} = 0$ at the boundary node. First, let $X = C(\Omega)$ and $X_n = R^{(n-1)^2}$ with sup norm. Let $E_nu_{i,j}$ = the piecewise linear interpolation and $(P_nu)_{i,j} = u(i\,h, j\,h)$ is the point-wise evaluation. We will prove that Δ_h is dissipative on X_n. Suppose $u_{ij} = |u_n|_\infty$. Then, since

$$\lambda\,u_{i,j} - (\Delta_h u)_{i,j} = f_{ij}$$

and

$$-(\Delta_h u)_{i,j} = \frac{1}{h^2}(4u_{i,j} - u_{i+1,j} - u_{i,j+1} - u_{i-1,j} - u_{i,j-1}) \ge 0$$

we have

$$0 \le u_{i,j} \le \frac{f_{i,j}}{\lambda}.$$

Thus, Δ_h is dissipative on X_n with sup norm. Next $X = L^2(\Omega)$ and X_n with ℓ^2 norm. Then,

$$(-\Delta_h u_n, u_n) = \sum_{i,j} |\frac{u_{i,j} - u_{i-1,j}}{h}|^2 + |\frac{u_{i,j} - u_{i,j-1}}{h}|^2 \ge 0$$

and thus Δ_h is dissipative on X_n with ℓ^2 norm.

Example 2 (Galerkin method) Let $V \subset H = H^* \subset V^*$ is the Gelfand triple. Consider the parabolic equation

$$\rho(\frac{d}{dt}u_n, \phi) = a(u_n, \phi) \tag{1.49}$$

for all $\phi \in V$, where the ρ is a symmetric mass form

$$\rho(\phi, \phi) \ge c\,|\phi|_H^2$$

and a is a bounded coercive bilinear form on $V \times V$ such that

$$a(\phi, \phi) \ge \delta\,|\phi|_V^2.$$

Define A by

$$\rho(Au, \phi) = a(u, \phi) \text{ for all } \phi \in V.$$

By the Lax-Milgram theorem

$$(\lambda I - A)u = f \in H$$

has a unique solution satisfying

$$\lambda\rho(u, \phi) - a(u, \phi) = (f, \phi)_H$$

for all $\phi \in V$. Let $dom(A) = (I - A)^{-1}H$. Assume

$$V_n = \{u = \sum a_k \phi_k^n, \;\; \phi_k^n \in V\} \text{ is dense in } V$$

Consider the Galerkin method, i.e. $u_n(t) \in V_n$ satisfies

$$\rho(\frac{d}{dt}u_n(t), \phi) = a(u_n, \phi).$$

Since for $u = (\lambda\, I - A)^{-1}f$ and $\bar{u}_n \in V_n$

$$\lambda\rho(u_n, \phi) + a(u_n, \phi) = (f, \phi) \text{ for } \phi \in V_n$$

$$\lambda\rho(\bar{u}_n, \phi) + a(\bar{u}_n, \phi) = \lambda\rho(\bar{u}_n - u, \phi) + a(\bar{u}_n - u, \phi) + (f, \phi) \text{ for } \phi \in V_n$$

$$\lambda\rho(u_n - \bar{u}_n, \phi) + a(u_n - \bar{u_n}, \phi) = \lambda\rho(\bar{u}_n - u, \phi) + a(\bar{u}_n - u, \phi).$$

Thus,

$$|u_n - \bar{u}_n| \le \frac{M}{\delta}|\bar{u_n} - u|_V.$$

Example 2 (Discontinuous Galerkin method) Consider the parabolic system for $u = \vec{u} \in L^2(\Omega)^d$

$$\frac{\partial}{\partial t} u = \nabla \cdot (a(x)\nabla u) + c(x)u$$

where $a \in R^d \times d$ is symmetric and

$$\underline{a}|\xi|^2 \leq (\xi, a(x), \xi)_{R^d} \leq \overline{a}|\xi|^2, \quad \xi \in R^d$$

for $0 < \underline{a} \leq \overline{a} < \infty$. The region Ω is dived into n non-overlapping sub-domains Ω_i with boundaries $\partial\Omega_i$ such that $\Omega = \cup \Omega_i$. At the interface $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ define

$$[[u]] = u|_{\partial\Omega_i} - u|_{\partial\Omega_j}$$

$$<< u >>= \tfrac{1}{2}(u|_{\partial\Omega_i} + u|_{\partial\Omega_j}).$$

The approximate solution $u_h(t)$ in

$$V_h = \{u_h \in L^2(\Omega) : u_h \text{ is linear on } \Omega_i\}.$$

Define the bilinear for on $V_h \times V_h$

$$a_h(u,v) = \sum_i \int_{\Omega_i} (a(x)\nabla u, \nabla v)\, dx - \sum_{i>j} \int_{\Gamma_{ij}} (<< n\cdot(a\nabla u) >> [[v]] \pm << n\cdot(a\nabla v) >> [[u]] + \frac{\beta}{h}[[u]][[v]]\, ds),$$

whee h is the meshsize and $\beta > 0$ is sufficiently large. If $+$ on the third term a_h is symmetric and for the case $-$ then a_h enjoys the coercivity

$$a_h(u,u) \geq \sum_i \int_{\Omega_i} (a(x)\nabla u, \nabla u)\, dx, \in u \in V_h,$$

regardless of $\beta > 0$.

Example 3 (Population dynaims) The transport equation

$$\tfrac{\partial p}{\partial t} + \tfrac{\partial p}{\partial x} + m(x)p(x,t) = 0$$

$$p(0,t) = \textstyle\int \beta(x)p(x,t)\, dx$$

Define the difference approximation

$$A_n p = (-\frac{p_i - p_{i-1}}{h} - m(x_i)p_i,\ 1 \leq i \leq n), \quad p_0 = \sum_i \beta_i\, p_i$$

Then,

$$(A_n p, sign_0(p)) \leq (\sum m_i - \beta_i)|p_i| \leq 0$$

Thus, A_n on $L^1(0,1)$ is dissipative.

$$(A, E_n\phi) - (P_n A_n, \phi).$$

Example 4 (Yee's scheme)

Consider the two dimensional Maxwell's equation. Consider the staggered grid; i.e. $E = (E^1_{i-\frac{1}{2},j}, E^2_{i,j-\frac{1}{2}})$ is defined at the the sides and $H = H_{i-\frac{1}{2},j-\frac{1}{2}}$ is defined at the center of the cell $\Omega_{i,j} = ((i-1)h, ih) \times ((j-1)h, jh)$.

$$\begin{aligned}
\epsilon_{i-\frac{1}{2},j}\,\tfrac{d}{dt}E^1_{i-\frac{1}{2},j} &= -\tfrac{H_{i-\frac{1}{2},j+\frac{1}{2}}-H_{i-\frac{1}{2},j-\frac{1}{2}}}{h}\\
\epsilon_{i,j+\frac{1}{2}}\,\tfrac{d}{dt}E^2_{i,j+\frac{1}{2}} &= \tfrac{H_{i+\frac{1}{2},j+\frac{1}{2}}-H_{i-\frac{1}{2},j-+\frac{1}{2}}}{h}\\
\mu_{i-\frac{1}{2},j-\frac{1}{2}}\,\tfrac{d}{dt}H_{i-\frac{1}{2},j-\frac{1}{2}} &= \tfrac{E^2_{i,j-\frac{1}{2}}-E^2_{i-1,j-\frac{1}{2}}}{h} - \tfrac{E^1_{i-\frac{1}{2},j}-E^1_{i-\frac{1}{2},j-1}}{h},
\end{aligned} \tag{1.50}$$

where $E^1_{i-\frac{1}{2},j} = 0,\ j = 0,\ j = N$ and $E^2_{i,j-\frac{1}{2},j} = 0,\ i = 0,\ j = N.$

Since

$$\sum_{i=1}^{N}\sum_{j=1}^{N} -\frac{H_{i-\frac{1}{2},j+\frac{1}{2}} - H_{i-\frac{1}{2},j-\frac{1}{2}}}{h}E^1_{i-\frac{1}{2},j} + \frac{H_{i+\frac{1}{2},j+\frac{1}{2}} - H_{i-\frac{1}{2},j-+\frac{1}{2}}h^2}{E}_{i,j+\frac{1}{2}}$$

$$+(\frac{E^2_{i,j-\frac{1}{2}} - E^2_{i-1,j-\frac{1}{2}}}{h} - \frac{E^1_{i-\frac{1}{2},j} - E^1_{i-\frac{1}{2},j-1}}{h})H_{i-\frac{1}{2},j-\frac{1}{2}} = 0$$

(1.50) is uniformly dissipative. The range condition $\lambda I - A_h = (f,g) \in X_h$ is equivalent to the minimization for E

$$\min \quad \frac{1}{2}(\epsilon_{i-\frac{1}{2},j}E^1_{i-\frac{1}{2},j} + \epsilon_{i,j+\frac{1}{2}}E^2_{i,j+\frac{1}{2}}) + \frac{1}{2}\frac{1}{\mu_{i,j}}(|\frac{E^2_{i,j-\frac{1}{2}} - E^2_{i-1,j-\frac{1}{2}}}{h}|^2 + |\frac{E^1_{i-\frac{1}{2},j} - E^1_{i-\frac{1}{2},j-1}}{h}|^2)$$

$$-(f^1_{i-\frac{1}{2},j} - \tfrac{1}{\mu_{i,j}}\tfrac{g_{i-\frac{1}{2},j+\frac{1}{2}}-g_{i-\frac{1}{2},j-\frac{1}{2}}}{h}, E^1_{i-\frac{1}{2},j} - (f^2_{i,j+\frac{1}{2}} + \tfrac{1}{\mu_{i,j}}\tfrac{H_{i+\frac{1}{2},j+\frac{1}{2}}-H_{i-\frac{1}{2},j-+\frac{1}{2}}}{h})E^2_{i,j+\frac{1}{2}}.$$

Example 5 Legende-Tau method

2 Dissipative Operators and Semigroup of Nonlinear Contractions

3 Evolution equations