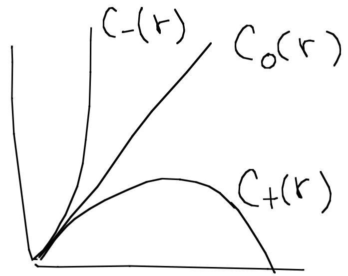
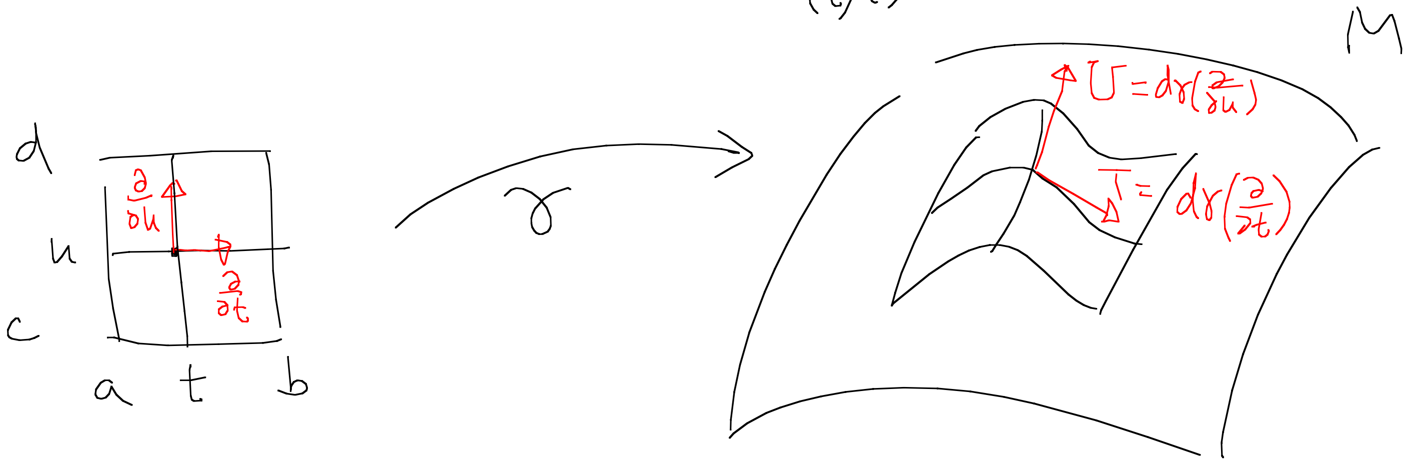


To generalize the above to arbitrary complete Riem. manifold, we need to study variations of geodesic.

Let $\gamma : [a,b]\times[c,d] \to M$ be a C^∞ map from the rectangle $[a,b]\times[c,d]$ to a complete Riem manifold M (of any dimension ≥ 2). Denote a point in $[a,b]\times[c,d]$ by (t,u). Then we can define 2 tangent vector fields along γ by

$$\begin{cases} T(t,u) = d\gamma\left(\left.\dfrac{\partial}{\partial t}\right|_{(t,u)}\right) \\ U(t,u) = d\gamma\left(\left.\dfrac{\partial}{\partial u}\right|_{(t,u)}\right) \end{cases}$$



$\forall$ fixed $u \in [c,d]$, a curve

$\gamma_u : [a,b] \to M : t \longmapsto \gamma(t,u)$ is defined.

Suppose $0 \in [c,d]$. Then γ_0 is called the <u>base curve</u> of γ.

If γ_u are geodesics $\forall u \in [c,d]$, we call γ a <u>one-parameter family of geodesics</u>.

In this case, the vector field $T = \gamma_u'$ and hence

$$D_T T = 0.$$

We also have $[T, U] = d\gamma\left(\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right]\right) = 0.$

Hence

$$\begin{cases} [T, U] = 0 \\ D_T T = 0 \end{cases} \quad \text{along } \gamma.$$

Then $D_T D_T U = D_T(D_U T)$

$$= D_T D_U T - D_U \cancel{D_T T}^{0} - D_{\cancel{[T,U]}^{0}} T$$

$$= -R_{TU} T$$

Therefore, along the base geodesic γ_0, we have

$$\boxed{D_{\gamma_0'} D_{\gamma_0'} U + R_{\gamma_0' U} \gamma_0' = 0} \quad \text{(Jac)}$$

or simply $\boxed{U'' + R_{\gamma_0' U} \gamma_0' = 0}$

where $U'' = D_{\gamma_0'} D_{\gamma_0'} U$ (similarly $U' = D_{\gamma_0'} U$)

Def: • Equation (Jac) is called the Jacobi equation along γ_0.

• Solutions of (Jac) are called Jacobi fields along γ_0.

Note: The vector field U constructed above is called a tranversal vector field (or variational vector field) of $\{\gamma_u\}$.

Hence, we have

Lemma 7: A transversal vector field of a 1-parameter family of geodesics is a Jacobi field.

eg: If M = 2 dim'l complete Riem. manifold.

Denote $C(r) = \{x \in M : d(x, o) = r\}$

$c(r) = \text{length } C(r)$, where $o \in M$ is fixed.

Let (ρ, θ) = polar coordinates on T_oM.

Let $\delta > 0$ small s.t. $\exp_o$ is a diffeomorphism on

$$B(\delta) = \{v \in T_oM : \rho(v) < \delta\}.$$

We can parametrize a circle of radius r in $B(\delta)$ (centered at o)

by

$$\begin{aligned} \tilde{\gamma} : [0, 2\pi] &\to B(\delta) \\ \theta &\mapsto (r, \theta) \end{aligned}$$

Then $C(r) = \exp_0(\tilde{\gamma})$ and

$$c(r) = \int_0^{2\pi} \left| (d\exp_0)_{(r,\theta)} \left(\frac{\partial}{\partial\theta}\right) \right| d\theta$$

The fact is:

$(d\exp_0)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ is a <u>tranversal</u> <u>vecta</u> <u>field</u> (of the family of radial geodesics) with <u>specific initial values</u>.

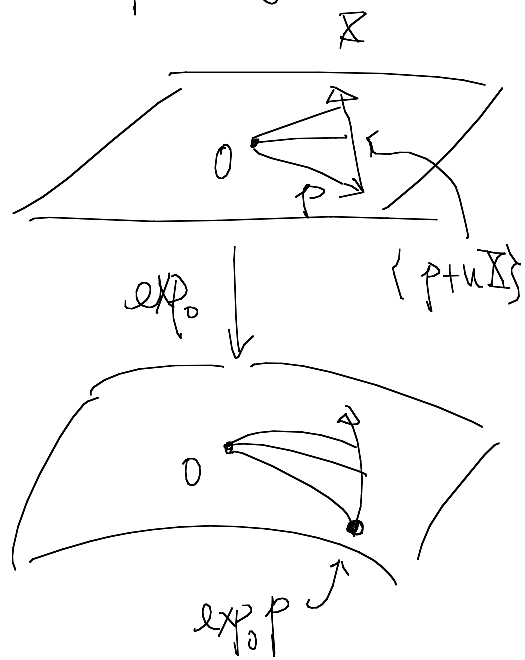
<u>General setting</u> (for this fact):

Let • M = complete Riem. manifold of dim $n \geq 2$

- $0 \in M$ fixed point.
- $p \in T_0 M$
- $X \in T_p(T_0 M) \cong T_0 M$

Define $\Gamma : [0, r] \times [0, 1] \to M$, where $r = |p|$ by

$$\Gamma(t, u) = \exp_0 \left[\frac{t}{r} (p + uX) \right]$$



Then $\forall u \in [0, 1]$, $\Gamma_u(t) = \Gamma(t, u)$ is a geodesic (with initial tangent vector $\frac{1}{r}(p + uX)$).

$\Rightarrow$ $\Gamma(t,u)$ is a 1-para. family of geodesics,

Let $V(t)$ = transversal vector field along Γ_0,

and $\delta > 0$ be s.t. $\exp_0$ is a diffeo. on

$$B(\delta) = \{ v \in T_oM : |v| < \delta \} \qquad (|v| = \rho(v))$$

in polar coordinate

Set $B_\delta = \{ x \in M : d(o,x) < \delta \}$.

Then $B_\delta = \exp_o(B(\delta))$.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_oM &

$\{\alpha^1, \cdots, \alpha^n\}$ be the dual basis of $\{e_1, \cdots, e_n\}$

Then $\{\alpha^1, \cdots, \alpha^n\}$ are coordinate functions on T_oM.

Define a coordinate system on B_δ by

$$x^i = \alpha^i \circ \exp_0^{-1} : B_\delta \to \mathbb{R}$$

Then we have

<u>Claim</u>: $\begin{cases} \left\langle \frac{\partial}{\partial x^i}\Big|_0, \frac{\partial}{\partial x^j}\Big|_0 \right\rangle = \delta_{ij}, & \forall i,j \\ D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = 0, & \forall i,j \end{cases}$

(Note: coordinate systems satisfying these conditions are called <u>normal coordinate systems</u>.)

Pf: The 1^{st} eq. is clearly follows from:

$$(d\exp_0)_0 = \mathrm{Id}.$$

$0 \in M$ $\qquad$ $0 \in T_0M$

To see the 2^{nd}, we define a bilinear form

$$\beta : T_0M \times T_0M \longrightarrow \mathbb{R}^n$$

by
$$\beta(e_i, e_j) = D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j}$$

Then $\forall\, v = \sum v^i e_i \in T_0M$,

$$\beta(v,v) = \sum_{i,j} v^i v^j D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = D_{\sum v^i \frac{\partial}{\partial x^i}\big|_{i=0}} \left(\sum v^j \frac{\partial}{\partial x^j} \right)$$

Note that $\sum v^j \frac{\partial}{\partial x^j}\Big|_0$ is the initial tangent vector of the geodesic $\exp_0(t \sum v^i e_i)$. Hence $\beta(v,v)=0$ by the geodesic eqt.

$\Rightarrow \beta \equiv 0$ on T_0M

i.e. $D_{\frac{\partial}{\partial x^i}\big|_0} \frac{\partial}{\partial x^j} = 0 \quad \forall i, j$

(This completes the proof of the claim) ※

Now assume $p = \sum p^i e_i$, $X = \sum X^i e_i$ (under $T_p(T_0M) \cong T_0M$)

For $\varepsilon > 0$ small, $\varepsilon p, \varepsilon X \in B(\delta)$.

Then in the above coordinate system $\{x^1, \cdots, x^n\}$,

$\left(= \exp_0\left[\frac{t}{r}(p+uX)\right] \right)$

the coordinate vector of $\Gamma(t,u) = \frac{t}{r}(\vec{p} + u\vec{X})$,

where $\vec{p} = (p^1, \dots, p^n)$ & $\vec{X} = (X^1, \dots, X^n)$,

for $(t,u) \in [0, \varepsilon r] \times [0, \varepsilon]$

And the base geodesic is $\Gamma_0(t) = \Gamma(t,0)$

$$(\text{in coordinate}) = \frac{t}{r}\vec{p}$$

$$\Rightarrow \quad V(t) = \frac{\partial}{\partial u}\Gamma(t,u)$$

$$(\text{in coordinate}) = \frac{t}{r}\vec{X}$$

i.e. $\quad V(t) = \frac{t}{r}\sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)}$

Therefore $U(0)=0$, and

$$U'(0) = D_{\Gamma_0'(0)} U = \frac{d}{dt}\Big|_{t=0} \left(\frac{t}{r} \sum X^i \frac{\partial}{\partial x^i}\Big|_{(t,0)} \right)$$

$$= \frac{1}{r} \sum X^i \frac{\partial}{\partial x^i}\Big|_0 + 0$$

In conclusion, the tranversal vector field $U(t)$ of $\Gamma(t,u) = \exp_{p_0}\left[\frac{t}{r}(p+uX)\right]$ satisfies

$$\begin{cases} U(0)=0 \\ U'(0) = \frac{1}{r}\vec{X} \text{ (in coordinate)}, \end{cases}$$

where $r=|p|$.

[~~Recall~~ Note: $V(t) = \frac{t}{r}\left(d\exp_o\right)_{\exp_o\left(\frac{t}{r}p\right)}(X)$ (check!)]

Applying the above to $M = \mathbb{R}^2, S^2$ & $\mathbb{H}^2$ with

$p = (r, \theta)$, $X = \left.\frac{\partial}{\partial\theta}\right|_{(r,\theta)}$.

Therefore $V(r) = \left(d\exp_o\right)_{(r,\theta)}\left(\frac{\partial}{\partial\theta}\right)$ (at $t = r$)

is a Jacobi field satisfying

$$\begin{cases} V(0) = 0 \\ |V'(0)| = \frac{1}{r}\left|\frac{\partial}{\partial\theta}\right| = 1. \end{cases}$$

((r,θ) = polar coordinates)

Let $W(t)$ = unit parallel vector field along Γ_0 s.t.

$$\langle W(t), \Gamma_0'(t) \rangle = 0.$$

On the other hand,

Gauss lemma $\Rightarrow$ $U(t) = (d\exp_o)_{(t,0)}\left(\frac{\partial}{\partial\theta}\right)$

is normal to $\Gamma_0'(t)$.

In our case of $\dim M = 2$,

$$U(t) = (d\exp_o)_{(t,0)}\left(\frac{\partial}{\partial\theta}\right) = f(t) W(t).$$

for some function $f \in C^\infty[0,r]$.

Then $D_{\Gamma_0'(t)} U(t) = f'(t) W(t)$

& $D_{\Gamma_0'(t)} D_{\Gamma_0'(t)} U(t) = f''(t) W(t)$

(since W is parallel)

Now (Jac) $\Rightarrow$

$$f''(t) W(t) + R_{\Gamma_0', fW} \Gamma_0' = 0$$

$$\Leftrightarrow \quad f''(t) + \langle R_{\Gamma_0' W} \Gamma_0', W \rangle f = 0$$

$$\Longrightarrow \quad f'' + K f = 0$$

where K = Gauss curvature at $\Gamma_0(t)$

(since $|\Gamma_0'(t)| = |W(t)| = 1$ & $\langle \Gamma_0', W \rangle = 0$)

We may also assume $\langle W, \frac{\partial}{\partial \theta} \rangle > 0$, we have

$$\begin{cases} f'' + Kf = 0 \\ f(0) = 0 \\ f'(0) = 1 \end{cases}$$

$\therefore$ The signature of K has implication on

$$c(r) = \int_0^{2\pi} \left| (d\exp_o)_{(r,\theta)} \left(\frac{\partial}{\partial\theta}\right) \right| d\theta = \int_0^{2\pi} f \, d\theta$$

In particular, if $K \equiv 0, +1, -1$ we have

$$f(r) = \begin{cases} r & , K \equiv 0 \\ \sin r & , K \equiv +1 \\ \sinh r & , K \equiv -1 \end{cases}$$

Prop: Let $K \geq +1$, then $c(r) \leq 2\pi \sin r$, for small r.

Pf: Consider a comparison function $h(t) = \sin t$

Then $\begin{cases} h'' + h = 0 \\ h(0) = 0 \\ h'(0) = 1. \end{cases}$

$$\Rightarrow \quad (hf' - fh')' = hf'' - fh''$$
$$= -Kfh + fh$$
$$= -(K-1)fh.$$

Since $f(0) = h(0) = 0$, $f'(0) = h'(0) = 1$, we have

$f \geq 0$, $h \geq 0$ for small $t > 0$.

$$\Rightarrow \quad (hf' - fh')' \leq 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad hf' - fh' \leq h(0)f'(0) - f(0)h'(0) = 0 \quad \text{for small } t>0$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' = \frac{hf' - fh'}{h^2} \leq 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad \frac{f(t)}{h(t)} \leq \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1$$

$$\Rightarrow \quad f(t) \leq h(t) = \sin t \quad \text{for small } t > 0$$

Hence $C(r) = \int_0^{2\pi} f(r)\, d\theta \leq 2\pi \sin r$ for small $r > 0$. ///

Prop: If $K \leq -1$, we have $C(r) \geq 2\pi \sinh(r)$.

(for small r at this moment)

Pf: Consider $h(t) = \sinh t$.

Then $\begin{cases} h''(t) - h(t) = 0 \\ h(0) = 0 \\ h'(0) = 1 \end{cases}$.

$$\Rightarrow \quad (hf' - fh')' = hf'' - fh''$$
$$= -Kfh - fh$$
$$= -(K+1) fh$$
$$\geq 0 \quad \text{for small } t > 0$$

$$\Rightarrow \quad hf' - fh' \geq h(0)f'(0) - f(0)h'(0) = 0$$

$$\Rightarrow \quad \left(\frac{f}{h}\right)' \geq 0$$

$$\Rightarrow \quad \frac{f(t)}{h(t)} \geq \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{h(\varepsilon)} = \frac{f'(0)}{h'(0)} = 1$$

$$\Rightarrow \quad f(t) \geq \sinh(t) \quad \text{for small } t > 0$$

$$\Rightarrow \quad C(r) \geq 2\pi \sinh(r) \quad \text{for small } r > 0.$$

#

Ch6 Jacobi field, Cartan-Hadamard Thm

§6.1 Jacobi field

Let γ = normalized geodesic (i.e. $|\gamma'|=1$)

Recall that the Jacobi eqt (for vector fields along γ)

is

$$U'' + R_{\gamma' U}\gamma' = 0 \qquad \text{(Jac)}$$

where $U'' = D_{\gamma'} D_{\gamma'} U \quad (U' = D_{\gamma'} U)$

Let $\{e_1(t), \cdots, e_n(t)\}$ be parallel vector fields along γ

s.t. $\forall t$

$$\begin{cases} e_1(t) = \gamma'(t) \\ \{e_i(t)\}_{i=1}^n \text{ is an athonormal basis of } T_{\gamma(t)}M. \end{cases}$$

Then $\forall$ vector field U along γ, we write

$$U(t) = \sum_{i=1}^n f^i(t)\, e_i(t), \quad \text{for some functions } f^i(t).$$

Similarly, the curvature can be written as

$$R_{e_i(t)e_j(t)}\, e_k(t) = \sum_{l=1}^n R_{ijk}^{\ \ \ l}(t)\, e_l(t),$$

where $R_{ijk}^{\ \ \ l}(t) = \langle R_{e_i(t)e_j(t)} e_k(t), e_l(t) \rangle$

Then the eqt (Jac) $\Rightarrow$

$$0 = V'' + R_{\gamma' V}\gamma'$$

$$= \left(\sum_i f^i e_i\right)'' + R_{e_1 V} e_1$$

$$= \sum_i (f^i)'' e_i + R_{e_1 (\sum_l f^l e_l)} e_1$$

$$= \sum_i (f^i)'' e_i + \sum_l f^l R_{e_1 e_l} e_1$$

$$= \sum_i (f^i)'' e_i + \sum_l f^l \sum_i R^i_{1l1} e_i$$

$$= \sum_i \left[(f^i)'' + \sum_l R^i_{1l1} f^l\right] e_i$$

$\therefore$ (Jac) $\Leftrightarrow$ $\boxed{(f^i)'' + \sum_l R^i_{1l1} f^l = 0 \quad \forall i = 1, \cdots, n}$

which is a 2^{nd} order linear ODE system.

Lemma 1

(1) Let γ be a geodesic. Then given any $v, w \in T_{\gamma(0)}M$, $\exists$ a unique Jacobi field $J(t)$ along γ s.t.

$$\begin{cases} J(0) = v \\ J'(0) = w. \end{cases}$$

(2) Unless $J \equiv 0$, the zero set of $J(t)$ along γ is discrete.

(Pf: ODE theory)

Lemma 2 : Let U be a vector field along a normalized geodesic γ. Then

U is a Jacobi field along γ

$\Leftrightarrow$ U is the transversal vector field of a one-parameter family of geodesics.

Pf: ($\Leftarrow$) Proved in previous chapter.

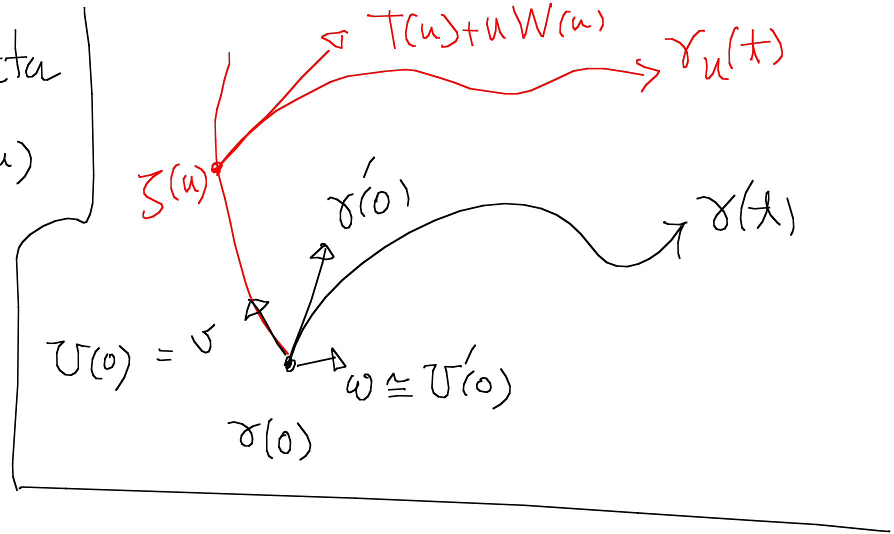
($\Rightarrow$) Let $v = U(0)$ & $w = U'(0)$

(by identifying $T_{\vec{p}}(T_{\gamma(0)}M) \cong T_{\gamma(0)}M$, $\forall \vec{p}$)

And let $\zeta : [0, \varepsilon] \to M$ be a geodesic

s.t. $\zeta(0) = \gamma(0)$ & $\zeta'(0) = v$

Define parallel vector fields $T(u)$ & $W(u)$ for $u \in [0, \varepsilon]$, along ζ such that



$$T(0) = \gamma'(0) \quad \& \quad W(0) = \omega$$

$\forall\, u \in [0, \varepsilon]$, define

$$\Gamma(t,u) = \gamma_u(t) = \exp_{\zeta(u)}\left[t\left(T(u) + uW(u)\right)\right].$$

Let V_1 = transversal vector field of γ_u along $\gamma = \gamma_0$.

Then V_1 is a Jacobi field.

Since $U_1(0) = \frac{\partial}{\partial u}\Big|_{u=0} \gamma_u(0)$

$$= \frac{\partial}{\partial u}\Big|_{u=0} \exp_{\zeta(u)}(0)$$

$$= \frac{\partial}{\partial u}\Big|_{u=0} \zeta(u) = \zeta'(0) = v.$$

Since $T_1 = d\Gamma\left(\frac{\partial}{\partial t}\right)$ is a vector field along Γ & when restricted to γ, we have

$$[T_1, U_1] = 0.$$

And hence

$$U_1'(0) = D_{\gamma'(0)} U_1 = D_{U_1(0)} T_1 \quad \left(\text{since } [T_1, U_1] = 0\right)$$

$$= D_v T_1 = D_{\zeta'(0)} T_1$$

Note that $T_1(\zeta(u)) = \left.\frac{\partial}{\partial t}\right|_{t=0} \exp_{\zeta(u)}\left[t(T(u)+uW(u))\right]$

$$= T(u) + uW(u)$$

$\therefore$ $U_1'(0) = D_{\zeta'(0)} T_1 = D_{\zeta'(0)}\left[T(u)+uW(u)\right]$

$= W(0)$ (since T, W are parallel along ζ)

$= w$

Altogether $U(0) = U_1(0)$ & $U'(0) = U_1'(0)$,

Uniqueness of Jacobi field $\Rightarrow U = U_1$

= transversal vector field $\#$

<u>Lemma 3</u>: Let U be a Jacobi field along a geodesic γ.

Then $\exists\, a, b \in \mathbb{R}$ such that

$$U = U^{\perp} + (at+b)\gamma',$$

where $U^{\perp}$ is a Jacobi field s.t. $\langle U^{\perp}, \gamma' \rangle = 0 \ \forall t$.

Pf: Consider

$$\frac{d^2}{dt^2}\langle U, \gamma' \rangle = \frac{d}{dt}\left(D_{\gamma'}\langle U, \gamma' \rangle \right)$$

$$= \frac{d}{dt}\left(\langle U', \gamma' \rangle + \langle U, \cancel{D_{\gamma'}\gamma'} \rangle \right)$$

$$= \langle U'', \gamma' \rangle + 0$$

$$= -\langle R_{\gamma' U}\gamma', \gamma' \rangle = 0$$

$$\Rightarrow \langle U, \gamma' \rangle = \tilde{a}\, t + \tilde{b} \quad \text{for some } \tilde{a}, \tilde{b} \in \mathbb{R}.$$

Let $U^{\perp} = U - \langle U, \frac{\gamma'}{|\gamma'|} \rangle \frac{\gamma'}{|\gamma'|}$

$$= U - \left(\frac{\tilde{a}}{|\gamma'|^2} t + \frac{\tilde{b}}{|\gamma'|^2} \right) \gamma'$$

Since $|\gamma'| \equiv \text{const.}$, $U^{\perp} = U - (at + b)\gamma'$

with $a = \frac{\tilde{a}}{|\gamma'|^2}$, $b = \frac{\tilde{b}}{|\gamma'|^2} \in \mathbb{R}$.

and satisfies $\langle U^{\perp}, \gamma' \rangle = 0$.

$$(V^\perp)'' = V'' - [(at+b)\gamma']''$$
$$= V'' = -R_{\gamma' V}\gamma'$$
$$= -R_{\gamma' V^\perp}\gamma' - (at+b)R_{\cancel{\gamma'}\gamma'}\gamma'$$
$$= -R_{\gamma' V^\perp}\gamma'$$
$\Rightarrow V^\perp$ is a Jacobi field. #

<u>Lemma 4</u> If V is a Jacobi field along a geodesic γ such that

$$\langle V(t_1), \gamma'(t_1)\rangle = \langle V(t_2), \gamma'(t_2)\rangle = 0$$

for 2 different t_1 & t_2. Then $\langle V(t), \gamma'(t)\rangle = 0$, $\forall t$.

(Pf: Since $\langle J(t), \gamma'(t) \rangle$ is linear in t)

In summary, we have the following facts of Jacobi fields:

(A) Let $\zeta : [0, \varepsilon] \to M$ be a curve in M,
$u \mapsto \zeta(u)$

$T(u)$, $W(u)$ parallel vector fields along ζ.

Then $\gamma_u(t) = \exp_{\zeta(u)}\left[t\left(T(u) + uW(u)\right)\right]$

determines a 1-para. family of geodesics $\{\gamma_u\}$

s.t. its transversal vector field $J(t)$ along γ_0

is a Jacobi field with $\begin{cases} U(0) = \xi'(0) \\ U'(0) = W(0) \end{cases}$.

(B) (If we take $\xi(u) \equiv x \in M$ (constant curve) in (A), then we have)

$\forall x \in M$; $T, \omega \in T_x M$. Then the 1-para. family of geodesics $\{\gamma_u\}$ defined by

$$\gamma_u(t) = \exp_x\left[t(T + u\omega)\right]$$

has a transversal vector field $U(t)$ s.t. $U(t)$ is a Jacobi field with

$$\begin{cases} U(0) = 0 \\ U'(0) = w. \end{cases}$$

(C) [Furthermore, adding condition $\langle T, w \rangle = 0$ to (B)]

Let $x \in M$; $T, w \in T_xM$ s.t. $\langle T, w \rangle = 0$.

Let $\gamma_u(t) = \exp_x[t(T + uw)]$,

Then the transversal vector field $U(t)$ of $\{\gamma_u\}$ is a <u>normal</u> Jacobi field with

$$\begin{cases} U(0) = 0 \\ U'(0) = w. \end{cases}$$

(normal Jacobi field $:=$ Jacobi field normal to the geodesic)

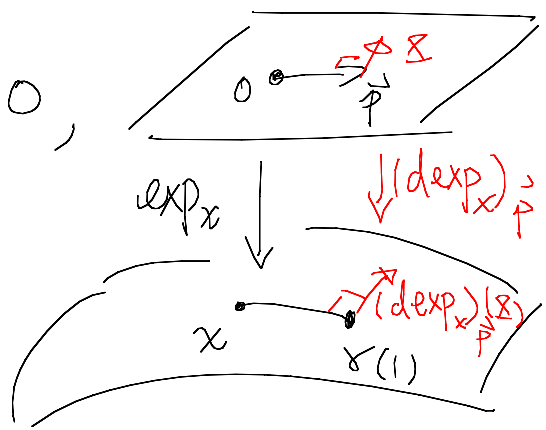
The proof of (C) needs (extension of) Gauss Lemma.

Lemma 5 (Gauss lemma)

Let M complete, $x \in M$, $\vec{p} \in T_x M$, $X \in T_{\vec{p}}(T_x M) \simeq T_x M$

If $\langle \vec{p}, X \rangle = 0$, then

$$\langle (d\exp_x)_{\vec{p}}(X), \gamma'(1) \rangle = 0,$$



where $\gamma : [0, 1] \to M$
$$t \mapsto \exp_x(t\vec{p}).$$

Pf: Let $\xi : [0, \varepsilon] \to T_xM$ be a curve in T_xM s.t.

$\xi(0) = \vec{p}$, $\xi'(0) = X$; and that

$$\xi([0,\varepsilon]) \subset S^{n-1}_{|\vec{p}|} \subset T_xM.$$

Such ξ exists since $X \perp \vec{p}$ (ie $X \in T_{\vec{p}} S^{n-1}_{|\vec{p}|}$)

Consider $\Gamma : [0,1] \times [0,\varepsilon] \to M$

$$(t, u) \mapsto \exp_x[t\xi(u)]$$

Let $T = d\Gamma\left(\frac{\partial}{\partial t}\right)$ & $U = d\Gamma\left(\frac{\partial}{\partial u}\right)$.

Then $\gamma(t) = \Gamma(t, 0)$,

$\gamma'(1) = T(\gamma(1))$

$$(d\exp_x)_{\vec{p}}(X) = U(\gamma(1)).$$

Since $|\xi(u)| = |\vec{p}|$

$$\Rightarrow \quad \langle T, T\rangle = |\vec{p}|^2 \quad (\text{geodesic has const. speed})$$

$$\therefore \quad T\langle U, T\rangle = \langle D_T U, T\rangle + \langle U, \cancel{D_T} T\rangle \quad (\gamma = \text{geodesic})$$

$$= \langle D_U T + \cancel{[T, U]}, T\rangle \quad \left([T,U] = d\Gamma\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}\right]\right)$$

$$= \langle D_U T, T\rangle = \frac{1}{2} U\langle T, T\rangle$$

$$= 0.$$

$$\Rightarrow \quad \langle U, T\rangle = \text{constant along } \gamma$$

$$= \lim_{t\to 0}\langle U(t), T(t)\rangle = \langle \cancel{U}^{\,0}(0), T(0)\rangle = 0 \quad \#$$

Pf of (C) : Let $\zeta : [0, \varepsilon] \to T_x M$

$u \mapsto t(T + u\omega)$

assumption

Then $\langle \zeta'(0), \zeta(0) \rangle = \langle t\omega, tT \rangle = t^2 \langle \omega, T \rangle = 0$,

and $(d\exp_x)_{(tT)} (\zeta'(0)) = U(t)$. ($U$ = transversal vector field of $\exp_x(t(T+u\omega))$)

Consider the curve $\gamma : [0,1] \to M$

$\tau \mapsto \exp_x(\tau(tT))$.

(Then $\gamma_0(t) = \exp_x(tT)$) (put $u=0$ in $\exp_x[t(T+u\omega)]$)

$$\Rightarrow \quad \gamma'(1) = \frac{d}{d\tau}\Big|_{\tau=1} [\exp_x(\tau t T)] = (d\exp_x)_{(tT)}(tT)$$

$$= t\,(d\exp_x)_{(tT)}(T)$$

$$= t\,\gamma_0'(t) \qquad (\text{" } ' \text{ " means derivative wrt } t)$$

Applying the Gauss Lemma to γ and $X = \zeta'(0)$,

$$\vec{p} = \zeta(0) = tT \qquad (\langle \vec{X}, \vec{p} \rangle = \langle \zeta'(0), \zeta(0) \rangle = 0),$$

we have

$$\langle U(t), \gamma_0'(t) \rangle = \langle (d\exp_x)_{(tT)}(\zeta'(0)),\ \frac{1}{t}\gamma'(1) \rangle$$

$$= \frac{1}{t} \langle (d\exp_x)_{(tT)}(X),\ \gamma'(1) \rangle = 0$$

$\Rightarrow U$ is normal. Other conclusions are clear from (B) ⊠