Ch2 Riemannian Metric, Connection & Parallel Transport.

Ref: 伍鴻熙, 沈純理, 虞言林 "黎曼几何初步", 北京大學出版社

2.1 Riemannian metric & connection

Def: Let M be a C^∞ manifold. A Riemannian metric g on M is given by an inner product g_x on each T_xM which depends smoothly on $x \in M$ in the sense that in any coordinates system U with coordinate functions $x^1, \cdots, x^n$,

$$g_{ij}(x) = g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \qquad (\forall i, j)$$

is a smooth function on the nbd.

(Caution: same notation, but not the $g_{ij}(x)$ in vector bundle)

Notation, most of the time we write

$$\langle\ ,\ \rangle_x \quad \text{for } g_x$$

$$(\text{and } \langle\ ,\ \rangle \quad \text{for } g.)$$

Note: • By definition, $(g_{ij}(x))$ is a symmetric positive definite $n\times n$ matrix $\forall x\in U$.

• g can be regarded as a $(0,2)$-tensor satisfying

$$\begin{cases} g(X,X)\geq 0 & \forall X\in\Gamma(TM) \\ g_x(X,X)=0 \Leftrightarrow X(x)=0 & \\ g(X,Y)=g(Y,X), & \forall X,Y\in\Gamma(TM) \end{cases}$$

Hence $\boxed{g=\sum_{i,j=1}^{n} g_{ij}(x)\,dx^i\otimes dx^j}$ in local coordinates

Def: A connection D (∇) on a C^∞ manifold M is a mapping

$$D: \Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

$$(V, X) \longmapsto D_V X,$$

such that

(C1) $D_{fV+gW} X = f D_V X + g D_W X$

(C2) $D_V(fX) = (Vf)X + f D_V X$

(C3) $D_V(X+Y) = D_V X + D_V Y$

where $V, W, X, Y \in \Gamma(TM)$; $f, g \in C^\infty(M)$.

(and $Vf = D_V f$ is the directional derivative of f in direction V)

<u>Note</u>: $D_V X$ is called the <u>covariant derivative</u> of X in the direction of V.

<u>Fact</u>: If $V, W \in \Gamma(TM)$ are vector fields s.t. $V(x) = W(x)$, then $(D_V X)(x) = (D_W X)(x)$, $\forall X \in \Gamma(TM)$.

(Pf: Ex.)

Using this fact, we have

<u>Def</u>: $\forall\, v \in T_xM$, one can define

$$D_v X \stackrel{\text{def}}{=} (D_V X)(x) \quad (\in T_xM)$$

where V is a vector field s.t. $V(x) = v$.

eg: Standard connection on $\mathbb{R}^n$

Recall the direction derivative of function

$$D_v f = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t|v|}$$

for a smooth function defined near $x \in \mathbb{R}^n$.

A smooth vector field X on $\mathbb{R}^n$ can be written as

$$X = \sum X^i(x) \frac{\partial}{\partial x^i}$$

(x^i = standard coordinates on $\mathbb{R}^n$, $\frac{\partial}{\partial x^i} = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th}$)

where $X^i(x)$ are smooth functions

Then $D_v X \overset{\text{def}}{=\!=} \sum D_v X^i(x) \frac{\partial}{\partial x^i}$, and

$$(D_V X)(x) \overset{\text{def}}{=\!=} D_{V(x)} X$$

define a connection on $\mathbb{R}^n$ (check: C1 – C3)

(By definition, we must have $D_v\left(\frac{\partial}{\partial x_j}\right)=0$, $\forall j=1,\cdots,n$)

Lemma: The set of connections on M is convex,

i.e. If $D^1, \cdots, D^k$ are connections on M

$f_1, \cdots, f_k$ are functions $\in C^\infty(M)$ with

$$\sum_{i=1}^{k} f_i = 1,$$

then $D = \sum_{i=1}^{k} f_i D^i$ is a connection on M.

$$\left(D_v X \overset{\text{def}}{=} \sum f_i D^i_v X \right)$$

Pf: C1 & C3 are clear & do not need $\sum f_i = 1$.

For C2, we have

$$D_V(f X) = \sum_i f_i D^i_V (f X)$$

$$= \sum_i f_i \left[(Vf) X + f D^i_V X \right]$$

$$= (Vf) X + f D_V X \quad \left(\text{since } \sum_i f_i = 1 \right)$$

※

Thm Let M be a C^∞ manifold. Then $\exists$ a connection on M.

Pf: Let $\{(U_i, \phi_i)\}$ be an atlas on M

Then $\{U_i\}$ is an open cover of M

$\Rightarrow$ $\exists$ partitions of unity $\{\varphi_i\}$ subordinate to $\{U_i\}$

(WLOG, we may assume $\{V_k\}_{k\in\Lambda'} = \{U_i\}_{i\in\Lambda}$)

On each U_i, the standard connection on $\mathbb{R}^n$ induces a connection $\overset{\circ}{D}{}^i$. Then $\sum \varphi_i D^i$ is a connection on M by the previous lemma. $\divideontimes$

<u>Remark</u>: Similar argument shows that there exists Riemannian metric on any manifold.

<u>Lemma</u>: Let $v \in T_xM$, and $\gamma : [0,\varepsilon) \to M$ be a curve such that $\gamma'(0) = v$. Suppose $X, Y \in \Gamma(TM)$

be 2 vector fields s.t. $X(\gamma(t)) = Y(\gamma(t))$, $\forall t \in [0, \varepsilon)$

Then $D_v X = D_v Y$.

(i.e. $D_{\gamma'(0)} X$ is determined by $X \circ \gamma$)

(Pf : Ex)

<u>Thm</u>: Let M = manifold

$g = \langle \, , \rangle$ = Riemannian metric on M

Then $\exists !$ connection D s.t.

$\left(\begin{smallmatrix}\text{compatible} \\ \text{with } g\end{smallmatrix}\right)$ (L1) $\quad X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$

$\left(\begin{smallmatrix}\text{torsion} \\ \text{free}\end{smallmatrix}\right)$ (L2) $\quad D_X Y - D_Y X - [X, Y] = 0$.

Pf: (Uniqueness)

In coordinates, any vector field can be written as

$$\underline{X} = \sum \underline{X}^i \frac{\partial}{\partial x^i}$$

$$\Rightarrow \qquad D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \quad \text{for some } \Gamma_{ij}^k \text{ (functions)}$$

Now for $\underline{X} = \underline{X}^j \frac{\partial}{\partial x^j}$, $V = V^i \frac{\partial}{\partial x^i}$, then

$$D_V \underline{X} = D_{V^i \frac{\partial}{\partial x^i}} \left(\underline{X}^j \frac{\partial}{\partial x^j} \right) = V^i D_{\frac{\partial}{\partial x^i}} \left(\underline{X}^j \frac{\partial}{\partial x^j} \right)$$

$$= V^i \left(\frac{\partial \underline{X}^j}{\partial x^i} \frac{\partial}{\partial x^j} + \underline{X}^j D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right)$$

$$= V^i\left(\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j} + X^j \Gamma_{ij}^k \frac{\partial}{\partial x^k}\right)$$

$$= V^i\left(\frac{\partial X^k}{\partial x^i} + \Gamma_{ij}^k X^j\right)\frac{\partial}{\partial x^k}$$

$\therefore \{\Gamma_{ij}^k\}$ determines $D_V X$.

Let $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle \quad \forall i, j$

$$\Rightarrow \quad \frac{\partial}{\partial x^i} g_{jk} = \frac{\partial}{\partial x^i}\left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right\rangle$$

$$= \left\langle D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right\rangle + \left\langle \frac{\partial}{\partial x^j}, D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right\rangle$$

$$= \left\langle \Gamma_{ij}^l \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right\rangle + \left\langle \frac{\partial}{\partial x^j}, \Gamma_{ik}^l \frac{\partial}{\partial x^l}\right\rangle$$

$$= g_{lk}\Gamma^{l}_{ij} + g_{jl}\Gamma^{l}_{ik}$$

$$\Rightarrow \quad \begin{cases} \dfrac{\partial g_{jk}}{\partial x^i} = g_{lk}\Gamma^{l}_{ij} + g_{jl}\Gamma^{l}_{ik} & \text{———} \quad (1) \\[2ex] \dfrac{\partial g_{ki}}{\partial x^j} = g_{li}\Gamma^{l}_{jk} + g_{kl}\Gamma^{l}_{ji} & \text{———} \quad (2) \\[2ex] \dfrac{\partial g_{ij}}{\partial x^k} = g_{lj}\Gamma^{l}_{ki} + g_{il}\Gamma^{l}_{kj} & \text{———} \quad (3) \end{cases}$$

Note that by (L2),

$$0 = D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \cancelto{0}{\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]}$$

$$= \left(\Gamma^{k}_{ij} - \Gamma^{k}_{ji}\right)\frac{\partial}{\partial x^k}$$

$$\Rightarrow \quad \boxed{\Gamma_{ij}^{k} = \Gamma_{ji}^{k}} \qquad \forall i, j, k$$

Then (1) + (2) − (3) ⇒

$$\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} = 2 g_{lk} \Gamma_{ij}^{l}$$

Denote the inverse matrix of (g_{ij}) by (g^{ij}).

Then $\qquad g^{sk} g_{kl} = \delta_l^s \quad \forall s, l$

$$\Rightarrow \quad \boxed{\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left[\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right]} \quad \text{— } (\Gamma)$$

∴ $\{\Gamma_{ij}^k\}$ & hence D satisfying L1 & L2 is uniquely determined by g.

<u>Existence</u>: Let $\{(U_\beta, \phi_\beta)\}$ = atlas of M. For $X = X^j \frac{\partial}{\partial x^j}$ & $V = V^i \frac{\partial}{\partial x^i}$ on U_β, we define

$$D_V X \overset{\text{def}}{=} V^i \left(\frac{\partial X^k}{\partial x^i} + \Gamma_{ij}^k X^j \right) \frac{\partial}{\partial x^k}$$

with Γ_{ij}^k defined by (Γ)

Then one can check that $D_V X$ doesn't depend on the coordinate (U_β, ϕ_β). Hence it defines a connection D on M. The properties L1 & L2 are then easy to check. ※

Note:

- The connection given by this theorem is called the Levi-Civita connection of g, (a Riemannian connection of g)

- The coefficients Γ_{ij}^{k} of D are called Christoffel symbols if D is Levi-Civita.

- The formula (Γ) is equivalent to

$$\langle D_X Y, Z\rangle = \frac{1}{2}\left\{ \begin{array}{l} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ + \langle Z,[X,Y]\rangle + \langle Y,[Z,X]\rangle - \langle X,[Y,Z]\rangle \end{array} \right\}$$

for $X, Y, Z \in \Gamma(TM)$

__eg__ On S^3, there exist $\hat{i}, \hat{j}, \hat{k}$ orthonormal vector fields such that $[\hat{i},\hat{j}]=\hat{k}$, $[\hat{j},\hat{k}]=\hat{i}$ & $[\hat{k},\hat{i}]=\hat{j}$.

$$\begin{aligned}\langle D_{\hat{i}}\hat{j}, \hat{k}\rangle &= \frac{1}{2}\Big\{ \hat{i}\,\cancelto{0}{\langle \hat{j},\hat{k}\rangle} + \hat{j}\,\cancelto{0}{\langle \hat{k},\hat{i}\rangle} - \hat{k}\,\cancelto{0}{\langle \hat{i},\hat{j}\rangle} \\ &\qquad + \langle \hat{k},[\hat{i},\hat{j}]\rangle + \langle \hat{j},[\hat{k},\hat{i}]\rangle - \langle \hat{i},[\hat{j},\hat{k}]\rangle \Big\} \\ &= \frac{1}{2}\Big\{\langle \hat{k},\hat{k}\rangle + \langle \hat{j},\hat{j}\rangle - \langle \hat{i},\hat{i}\rangle\Big\} = \frac{1}{2}\end{aligned}$$

Similarly, $\langle D_{\hat{i}}\hat{j}, \hat{i}\rangle = \langle D_{\hat{i}}\hat{j}, \hat{j}\rangle = 0$

Hence $D_{\hat{i}}\hat{j} = \frac{1}{2}\hat{k}$ (Similarly for others: Ex.)

Geometry meaning of Levi-Civita connection

<u>Def</u>: Let N be a (embedded) submanifold of M. Suppose g is a metric on M, then the <u>induce metric</u> $\bar{g}$ of g on N is defined by

$$\bar{g}(X,Y) = g(X,Y), \quad \forall X,Y \in TN \subset TM$$

(eg. If $N \subset M$ is open, then $\bar{g} = g|_N$)

<u>Def</u>: Let (M,g) be a Riemannian manifold, D = Levi-Civita connection of g.

Suppose $N \subset M$ is a submanifold, then one can define a connection on N by

$$\bar{D}_X Y = (D_X Y)^{\perp}$$

where $(\ )_x^{\perp} : T_x M \to T_x N$ is the orthogonal projection (wrt g_x on $T_x M$)

Facts • $\bar{D}$ is well-defined, ie. $\bar{D}$ satisfies C1 – C3.

• $\bar{D}$ is the Levi-Civita connection of the induce metric $\bar{g}$. (Pf: Ex)

Note: If $M = \mathbb{R}^n$, g = standard metric (= flat metric) then Levi-Civita connection D = usual directional derivative.

Hence, the facts above give a geometry interpretation of the Levi-Civita connection on submanifolds N of $\mathbb{R}^n$.

"Meaning" of L2: $D_X Y - D_Y X - [X, Y] = 0$

L2 doesn't use the metric g, and in local coordinates

$$L2 \iff \Gamma_{ij}^k = \Gamma_{ji}^k$$

Hence, connections satisfying (L2) are called symmetric

Moreover, $T(X, Y) = D_X Y - D_Y X - [X, Y]$

defines a $(1,2)$-tensor on M called the torsion tensor,

i.e. $T \in \Gamma(TM \otimes (\otimes^2 T^*M))$ (i.e. linear in X, Y (Ex))

Hence D is symmetric $\Leftrightarrow T \equiv 0$

$\Leftrightarrow$ D is <u>torsion free</u>.

2.2 <u>Parallel Transport</u>

Let D be a connection (not necessary Levi-Civita) on M;

$\gamma : [a,b] \to M$ be an <u>embedded</u> curve such that

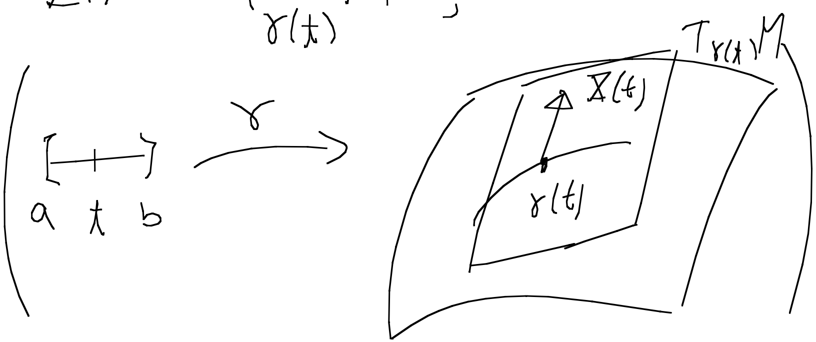
$\gamma([a,b])$ is contained in a coordinate neighborhood

with coordinate functions $\{x^i\}$.

Suppose X is a <u>vector field along γ</u>, i.e., X depends

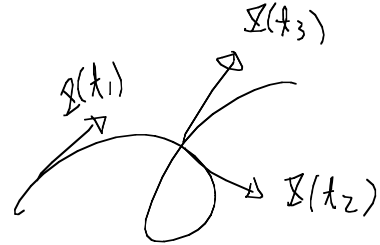
smoothly on t and $X(t) \in T_{\gamma(t)}M$, $\forall t \in [a,b]$

$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "X"] \arrow[r, "\gamma"'] \arrow[ur, phantom, "\circlearrowright", shift right=3ex] & M \end{tikzcd}$$



Since γ is embedded, X can be extended to a smooth vector field $\tilde{X}$ on M.

(Not true for immerse curve:



)

Now for any 2 extensions $\tilde{X}$ & $\tilde{Y}$, we must have

$$\tilde{X}(\gamma(t)) = \tilde{Y}(\gamma(t)) = X(\gamma(t))$$

$$\Rightarrow \quad D_{\gamma'(t)} \tilde{X} = D_{\gamma'(t)} \tilde{Y}$$

$\therefore$ $D_{\gamma'(t)} X$ is well-defined.

In local coordinates,

$$\gamma'(t) = \sum \gamma'^{i}(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

$$\mathbb{X}(t) = \sum \mathbb{X}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)}$$

for some functions $\gamma'^i(t)$ & $\mathbb{X}^i(t)$.

Recall that $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$ (for some Γ^k_{ij})

Therefore $D_{\gamma'(t)} \mathbb{X} = D_{\gamma'(t)} \left(\mathbb{X}^j \frac{\partial}{\partial x^j} \right)$

$$= \left(D_{\gamma'(t)} \mathbb{X}^j \right) \frac{\partial}{\partial x^j} + \mathbb{X}^j D_{\gamma'(t)} \frac{\partial}{\partial x^j}$$

$$= \frac{d\mathbb{X}^j}{dt} \frac{\partial}{\partial x^j} + \mathbb{X}^j \gamma'^i D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$$

$$= \left(\frac{d\mathbb{X}^k}{dt} + \mathbb{X}^j \gamma'^i \Gamma^k_{ij} \right) \frac{\partial}{\partial x^k}$$

$$D_{\gamma'(t)} X = 0 \iff \frac{dX^k}{dt} + \left(\Gamma^k_{ij} \gamma'^i\right) X^j = 0 \quad , \forall k = 1, \cdots, n$$

linear ODE system in $X^1, \cdots, X^n$.

<u>Linear</u> ODE theory $\Rightarrow$

$\forall\, v \in T_{\gamma(a)}M$, $\exists!$ soln. $X(t)$ to the IVP

$$\begin{cases} D_{\gamma'(t)} X = 0 \quad , \forall\, t \in [a, b] \\ X(a) = v \end{cases}$$

<u>Def</u>: A vector field X along γ is called <u>parallel</u> if $D_{\gamma'} X = 0$.

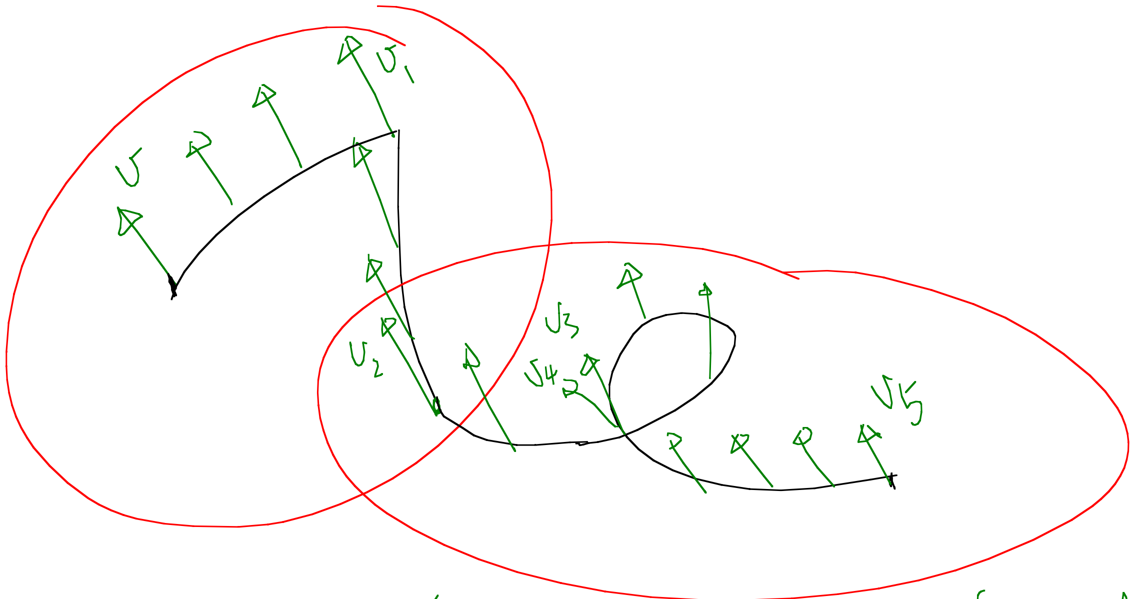
<u>Def</u>: A vector $w \in T_{\gamma(b)}M$ is called a <u>parallel transport</u>

of a vector $v \in T_{\gamma(a)}M$ along γ if $\exists$ a parallel vector field X along γ such that

$$X(a) = v \quad \& \quad X(b) = w$$

Note: parallel transport w of v (along γ) is uniquely determined by v. (for fixed γ)

Note: If γ is not embedded or not contained in a chart or γ is only piecewise smooth, we can use subdivision to define parallel transport of a vector $v \in T_{\gamma(a)}M$ along γ.



(V_3 may not equal to V_4 for curved space)

Hence we have

<u>Thm</u> $\forall$ immersed curve $\gamma : [a,b] \to M$ & $v \in T_{\gamma(a)}M$, $\exists!$ parallel vector field $X(t)$ along γ s.t. $X(a) = v$.

Hence $\exists!$ $w \in T_{\gamma(b)}M$ s.t. w is the parallel transport of v along γ.

This Thm $\Rightarrow$ one can define $\forall$ immersed curve $\gamma : [a,b] \to M$

a mapping

$$P^{\gamma} : T_{\gamma(a)}M \longrightarrow T_{\gamma(b)}M$$

$$v \longmapsto w = \text{parallel transport of } v \text{ along } \gamma.$$

<u>Thm</u>: $P^{\gamma} : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is an vector space isomorphism.

(Pf: Ex.)

- P^{γ} is called <u>parallel transport</u> <u>from $\gamma(a)$ to $\gamma(b)$ along γ</u>.