Exercise 1 Suggested Solution

(1) Let \( \{A_k\}_{k=1}^{\infty} \) be a sequence of measurable sets in \((X, \mathcal{M})\). Let

\[ A = \{ x \in X : x \in A_k \text{ for infinitely many } k \}, \]

and

\[ B = \{ x \in X : x \in A_k \text{ for all except finitely many } k \}. \]

Show that \( A \) and \( B \) are measurable.

**Solution**

\[ A = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k. \]

\[ B = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} A_k. \]

(2) Let \( \Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be continuous. Show that \( \Psi(f, g) \) are measurable for any measurable functions \( f, g \). This result contains Proposition 1.3 as a special case.

**Solution** Note that every open set \( G \subseteq \mathbb{R}^2 \) can be written as a countable union of set of the form \( V_1 \times V_2 \) where \( V_1, V_2 \) open in \( \mathbb{R} \). (Think of \( V_1 \times V_2 = (a, b) \times (c, d), a, b, c, d \in \mathbb{Q} \).)

Let \( G \subseteq \mathbb{R} \) be open. Then \( \Phi^{-1}(G) \) is open in \( \mathbb{R}^2 \), so

\[ \Phi^{-1}(G) = \bigcup_n (V_n^1 \times V_n^2), \]
Then
\[ h^{-1}(\Phi^{-1})(G) = \bigcup_n h^{-1}(V_1^1 \times V_2^1) = \bigcup_n f^{-1}(V_1^1) \cap g^{-1}(V_2^1) \]
is measurable since \( f \) and \( g \) are measurable. Hence \( h = (f, g) \).

(3) Show that \( f : X \to \mathbb{R} \) is measurable if and only if \( f^{-1}([a, b]) \) is measurable for all \( a, b \in \mathbb{R} \).

**Solution** By def \( f : X \to \mathbb{R} \) is measurable if \( f^{-1}(G) \) is measurable. \( \forall G \) open in \( \mathbb{R} \). Every open set \( G \) in \( \mathbb{R} \) can be written as a countable union of \((a, b), [−\infty, a), (b, \infty], a, b \in \mathbb{R}\). So \( ff \) is measurable iff \( f^{-1}(a, b), f^{-1}[−\infty, a), f^{-1}(b, \infty] \) are measurable.

\( \Rightarrow \) Use
\[
\begin{align*}
   f^{-1}(a, b) &= \bigcap_n f^{-1} \left( a - \frac{1}{n}, b + \frac{1}{n} \right) \\
   f^{-1}[−\infty, a) &= \bigcap_n f^{-1} \left( −\infty, a + \frac{1}{n} \right) \\
   f^{-1}(b, \infty] &= \bigcap_n f^{-1} \left( b - \frac{1}{n}, \infty \right)
\end{align*}
\]

\( \Leftarrow \) Use
\[
\begin{align*}
   f^{-1}(a, b) &= \bigcup_n f^{-1} \left[ a - \frac{1}{n}, b + \frac{1}{n} \right] \\
   f^{-1}[−\infty, a) &= \bigcap_n f^{-1} \left( −\infty, a - \frac{1}{n} \right) \\
   f^{-1}(b, \infty] &= \bigcap_n f^{-1} \left[ b + \frac{1}{n}, \infty \right].
\end{align*}
\]

(4) Let \( f : X \times [a, b] \to \mathbb{R} \) satisfy (a) for each \( x, y \mapsto f(x, y) \) is Riemann integrable, and (b) for each \( y, x \mapsto f(x, y) \) is measurable with respect to some
σ-algebra $\mathcal{M}$ on $X$. Show that the function

$$F(x) = \int_a^b f(x,y) \, dy$$

is measurable with respect to $\mathcal{M}$.

**Solution** For simplicity let $[a,b] = [0,1]$. For $n \geq 1$, equally divide $[0,1]$ into subintervals of length $1/n$ and let

$$F_n(x) = \sum_{k=1}^{n} f\left(x, \frac{k}{n}\right) \frac{1}{n}.$$

Clearly $F_n$ is measurable (with respect to $\mathcal{M}$). Now

$$F(x) = \lim_{n \to \infty} F_n(x),$$

so it is also measurable.

(5) Let $f$, $g$, $f_k, k \geq 1$, be measurable functions from $X$ to $\mathbb{R}$.

(a) Show that $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable sets.

(b) Show that $\{x : \lim_{k \to \infty} f_k(x) \text{ exists and is finite}\}$ is measurable.

**Solution**

(a) Suffice to show $\{x : F(x) > 0\}$ and $\{x : F(x) = 0\}$ are measurable. If $F$ is measurable, use

$$\{x : F(x) > 0\} = F^{-1}(0, \infty]$$

$$\{x : F(x) = 0\} = F^{-1}[0, \infty] \cap F^{-1}[-\infty, 0]$$

(b) Since $g(x) = \limsup_{k \to \infty} f_k(x)$ and $\liminf_{k \to \infty} f_k(x)$ are measurable.

$$\{x : \lim_{k \to \infty} f_k(x) \text{ exists }\} = \{x : \liminf_{k \to \infty} f_k(x) = \limsup_{k \to \infty} f_k(x)\}$$
On the other hand, the set \( \{ x : g(x) < +\infty \} \) is also measurable, so is their intersection.

(6) There are two conditions (i) and (ii) in the definition of a measure \( \mu \) on \((X, \mathcal{M})\). Show that (i) can be replaced by the “nontriviality condition”: There exists some \( E \in \mathcal{M} \) with \( \mu(E) < \infty \).

**Solution** If \( \mu \) is a measure satisfying the nontriviality condition and (ii), let \( A_1 = E, \ A_i = \phi \) for \( i \geq 2 \) in (ii),

\[ \infty > \mu(E) = \sum_{i=1}^{\infty} \mu(A_i) \geq \mu(A_1) + \mu(A_2) = \mu(E) + \mu(\phi) \]

so \( 0 \geq \mu(\phi) \geq 0 \). We have \( \mu \) is a measure satisfying (i) and (ii).

If \( \mu \) is a measure satisfying (i) and (ii), taking \( E = \phi \), we have the nontriviality condition.

(7) Let \( \{A_k\} \) be measurable and \( \sum_{k=1}^{\infty} \mu(A_k) < \infty \) and

\[ A = \{ x \in X : x \in A_k \text{ for infinitely many } k \} \]

We know that \( A \) is measurable from (1). Show that \( A \) is measurable.

**Solution** Since \( \sum_{k=1}^{\infty} \mu(A_k) < \infty \), we have \( \sum_{k=n}^{\infty} \mu(A_k) \to 0 \) as \( n \to \infty \). For any \( n \in \mathbb{N} \), we have

\[ A \subset \bigcup_{k \geq n} A_k \]

and so

\[ \mu(A) \leq \sum_{k=n}^{\infty} \mu(A_k) . \]

Taking \( n \to \infty \), we have \( \mu(A) = 0 \).

This result is called Borel-Cantelli lemma.
Let $B$ be the set defined in (1). Let $\mu$ be a measure on $(X, \mathcal{M})$. Show that

$$\mu(B) \leq \liminf_{k \to \infty} \mu(A_k).$$

**Solution** Using the characterization

$$B = \bigcup_{k=1}^{\infty} \bigcap_{j \geq k} A_j,$$

and the fact that $\{\bigcap_{j \geq k} A_j\}$ is ascending in $k$, we have

$$\mu(B) = \lim_{k \to \infty} \mu\left(\bigcap_{j \geq k} A_j\right) = \liminf_{k \to \infty} \mu\left(\bigcap_{j \geq k} A_j\right) \leq \liminf_{k \to \infty} \mu(A_k).$$

(9) Here we review Riemann integral. Let $f$ be a bounded function defined on $[a, b], a, b \in \mathbb{R}$. Given any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ on $[a, b]$ and tags $z_j \in [x_j, x_{j+1}]$, there corresponds a Riemann sum of $f$ given by $R(f, P, z) = \sum_{j=0}^{n-1} f(z_j)(x_{j+1} - x_j)$. The function $f$ is called Riemann integrable with integral $L$ if for every $\varepsilon > 0$ there exists some $\delta$ such that

$$|R(f, P, z) - L| < \varepsilon,$$

whenever $\|P\| < \delta$ and $z$ is any tag on $P$. (Here $\|P\| = \max_{j=0}^{n-1} |x_{j+1} - x_j|$ is the length of the partition.) Show that
1. For any partition $P$, define its Darboux upper and lower sums by

$$\overline{R}(f, P) = \sum_{j} \sup \{ f(x) : x \in [x_j, x_{j+1}] \} (x_{j+1} - x_j),$$

and

$$\underline{R}(f, P) = \sum_{j} \inf \{ f(x) : x \in [x_j, x_{j+1}] \} (x_{j+1} - x_j)$$

respectively. Show that for any sequence of partitions $\{P_n\}$ satisfying $\|P_n\| \to 0$ as $n \to \infty$, $\lim_{n \to \infty} \overline{R}(f, P_n)$ and $\lim_{n \to \infty} \underline{R}(f, P_n)$ exist.

2. $\{P_n\}$ as above. Show that $f$ is Riemann integrable if and only if

$$\lim_{n \to \infty} \overline{R}(f, P_n) = \lim_{n \to \infty} \underline{R}(f, P_n) = L.$$

3. A set $E$ in $[a, b]$ is called of measure zero if for every $\varepsilon > 0$, there exists a countable subintervals $J_n$ satisfying $\sum_n |J_n| < \varepsilon$ such that $E \subset \bigcup_n J_n$. Prove Lebesgue’s theorem which asserts that $f$ is Riemann integrable if and only if the set consisting of all discontinuity points of $f$ is a set of measure zero. Google for help if necessary.

Solution:

(a) It suffices to show: For every $\varepsilon > 0$, there exists some $\delta$ such that

$$0 \leq \overline{R}(f, P) - \underline{R}(f) < \varepsilon,$$

and

$$0 \leq \underline{R}(f) - \overline{R}(f, P) < \varepsilon,$$

for any partition $P$, $\|P\| < \delta$, where

$$\overline{R}(f) = \inf_{P} \overline{R}(f, P),$$

and

$$\underline{R}(f) = \inf_{P} \underline{R}(f, P).$$
and

\[ \overline{R}(f) = \sup_P R(f, P). \]

If it is true, then \( \lim_{n \to \infty} \overline{R}(f, P_n) \) and \( \lim_{n \to \infty} \overline{R}(f, P_n) \) exist and equal to \( \overline{R}(f) \) and \( R(f) \) respectively.

Given \( \varepsilon > 0 \), there exists a partition \( Q \) such that

\[ \overline{R}(f) + \varepsilon/2 > \overline{R}(f, Q). \]

Let \( m \) be the number of partition points of \( Q \) (excluding the endpoints). Consider any partition \( P \) and let \( R \) be the partition by putting together \( P \) and \( Q \). Note that the number of subintervals in \( P \) which contain some partition points of \( Q \) in its interior must be less than or equal to \( m \). Denote the indices of the collection of these subintervals in \( P \) by \( J \). We have

\[ 0 \leq \overline{R}(f, P) - \overline{R}(f, R) \leq \sum_{j \in J} 2M \Delta x_j \leq 2M \times m \|P\|, \]

where \( M = \sup_{[a, b]} |f| \), because the contributions of \( \overline{R}(f, P) \) and \( \overline{R}(f, Q) \) from the subintervals not in \( J \) cancel out. Hence, by the fact that \( R \) is a refinement of \( Q \),

\[ \overline{R}(f) + \varepsilon/2 > \overline{R}(f, Q) \geq \overline{R}(f, R) \geq \overline{R}(f, P) - 2Mm \|P\|, \]

i.e.,

\[ 0 \leq \overline{R}(f, P) - \overline{R}(f) < \varepsilon/2 + 2Mm \|P\|. \]

Now, we choose

\[ \delta < \frac{\varepsilon}{1 + 4Mm}, \]
Then for $P$, $\|P\| < \delta$,

\[ 0 \leq \overline{R}(f, P) - \underline{R}(f) < \varepsilon. \]

Similarly, one can prove the second inequality.

(b) With the result in part a, it suffices to prove the following result: Let $f$ be bounded on $[a, b]$. Then $f$ is Riemann integrable on $[a, b]$ if and only if $\overline{R}(f) = \underline{R}(f)$. When this holds, $L = \overline{R}(f) = \underline{R}(f)$.

According to the definition of integrability, when $f$ is integrable, there exists some $L \in \mathbb{R}$ so that for any given $\varepsilon > 0$, there is a $\delta > 0$ such that for all partitions $P$ with $\|P\| < \delta$,

\[ |R(f, P, z) - L| < \varepsilon/2, \]

holds for any tags $z$. Let $(P_1, z_1)$ be another tagged partition. By the triangle inequality we have

\[ |R(f, P, z) - R(f, P_1, z_1)| \leq |R(f, P, z) - L| + |R(f, P_1, z_1) - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \]

Since the tags are arbitrary, it implies

\[ \overline{R}(f, P) - \underline{R}(f, P) \leq \varepsilon. \]

As a result,

\[ 0 \leq \overline{R}(f) - \underline{R}(f) \leq \overline{R}(f, P) - \underline{R}(f, P) \leq \varepsilon. \]

Note that the first inequality comes from the definition of the upper/lower Riemann integrals. Since $\varepsilon > 0$ is arbitrary, $\overline{R}(f) = \underline{R}(f)$.

Conversely, using $\overline{R}(f) = \underline{R}(f)$ in part a, we know that for $\varepsilon > 0$, there exists
a $\delta$ such that

$$0 \leq \overline{R}(f, P) - \overline{R}(f, P) < \varepsilon,$$

for all partitions $P$, $\|P\| < \delta$. We have

$$R(f, P, z) - R(f) \leq \overline{R}(f, P) - \overline{R}(f)$$

$$\leq \overline{R}(f, P) - \overline{R}(f, P)$$

$$< \varepsilon,$$

and similarly,

$$\overline{R}(f) - R(f, P, z) \leq \overline{R}(f, P) - \overline{R}(f, P) < \varepsilon.$$

As $\overline{R}(f) = R(f)$, combining these two inequalities yields

$$|R(f, P, z) - R(f)| < \varepsilon,$$

for all $P$, $\|P\| < \delta$, so $f$ is integrable, where $L = \overline{R}(f)$.

(c) For any bounded $f$ on $[a, b]$ and $x \in [a, b]$, its oscillation at $x$ is defined by

$$\omega(f, x) = \inf_{\delta} \{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\}$$

$$= \lim_{\delta \to 0^+} \{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\}.$$

It is clear that $\omega(f, x) = 0$ if and only if $f$ is continuous at $x$. The set of discontinuity of $f$, $D$, can be written as $D = \bigcup_{k=1}^{\infty} O(k)$, where $O(k) = \{x \in [a, b] : \omega(f, x) \geq 1/k\}$. Suppose that $f$ is Riemann integrable on $[a, b]$. It suffices to show that each $O(k)$ is of measure zero. Given $\varepsilon > 0$, by Integrability of $f$, we can find a partition $P$ such that

$$\overline{R}(f, P) - \overline{R}(f, P) < \varepsilon/2k.$$
Let \( J \) be the index set of those subintervals of \( P \) which contains some elements of \( O(k) \) in their interiors. Then

\[
\frac{1}{k} \sum_{j \in J} |I_j| \leq \sum_{j \in J} (\sup_{I_j} f - \inf_{I_j} f) \Delta x_j \\
\leq \sum_{j=1}^{n} (\sup_{I_j} f - \inf_{I_j} f) \Delta x_j \\
= \mathcal{R}(f, P) - \mathcal{R}(f, P) \\
< \varepsilon/2k.
\]

Therefore

\[
\sum_{j \in J} |I_j| < \varepsilon/2.
\]

Now, the only possibility that an element of \( O(k) \) is not contained by one of these \( I_j \) is it being a partition point. Since there are finitely many partition points, say \( N \), we can find some open intervals \( I'_1, \ldots, I'_N \) containing these partition points which satisfy

\[
\sum |I'_i| < \varepsilon/2.
\]

So \( \{I_j\} \) and \( \{I'_i\} \) together form a covering of \( O(k) \) and its total length is strictly less than \( \varepsilon \). We conclude that \( O(k) \) is of measure zero.

Conversely, given \( \varepsilon > 0 \), fix a large \( k \) such that \( \frac{1}{k} < \varepsilon \). Now the set \( O(k) \) is of measure zero, we can find a sequence of open intervals \( \{I_j\} \) satisfying

\[
O(k) \subseteq \bigcup_{j=1}^{\infty} I_j,
\]

\[
\sum_{j=1}^{\infty} |I_{ij}| < \varepsilon.
\]
One can show that $O(k)$ is closed and bounded, hence it is compact. As a result, we can find $I_{i_1}, \ldots, I_{i_N}$ from $\{I_j\}$ so that

$$O(k) \subseteq I_{i_1} \cup \ldots \cup I_{i_N},$$

$$\sum_{j=1}^{N} |I_j| < \varepsilon.$$  

Without loss of generality we may assume that these open intervals are mutually disjoint since, whenever two intervals have nonempty intersection, we can put them together to form a larger open interval. Observe that $[a, b] \setminus (I_{i_1} \cup \ldots \cup I_{i_N})$ is a finite disjoint union of closed bounded intervals, call them $V_i’s$, $i \in A$. We will show that for each $i \in A$, one can find a partition on each $V_i = [v_{i-1}, v_i]$ such that the oscillation of $f$ on each subinterval in this partition is less than $1/k$.

Fix $i \in A$. For each $x \in V_i$, we have

$$\omega(f, x) < \frac{1}{k}.$$  

By the definition of $\omega(f, x)$, one can find some $\delta_x > 0$ such that

$$\sup\{f(y) : y \in B(x, \delta_x) \cap [a, b]\} - \inf\{f(z) : z \in B(x, \delta_x) \cap [a, b]\} < \frac{1}{k},$$

where $B(y, \beta) = (y - \beta, y + \beta)$. Note that $V_i \subseteq \bigcup_{x \in V_i} B(x, \delta_x)$. Since $V_i$ is closed and bounded, it is compact. Hence, there exist $x_{i_1}, \ldots, x_{i_M} \in V_i$ such that $V_i \subseteq \bigcup_{j=1}^{M} B(x_{i_j}, \delta_{x_{i_j}})$. By replacing the left end point of $B(x_{i_j}, \delta_{x_{i_j}})$ with $v_{i-1}$ if $x_{i_j} - \delta_{x_{i_j}} < v_{i-1}$, and replacing the right end point of $B(x_{i_j}, \delta_{x_{i_j}})$ with $v_i$ if $x_{i_j} + \delta_{x_{i_j}} > v_i$, one can list out the endpoints of $\{B(x_{i_j}, \delta_{i_j})\}_{j=1}^{M}$ and use them to form a partition $S_i$ of $V_i$. It can be easily seen that each subinterval in $S_i$ is covered by some $B(x_{i_j}, \delta_{x_{i_j}})$, which implies that the oscillation of $f$ in each subinterval is less than $1/k$. So, $S_i$ is the partition that we want.
The partitions $S_i$’s and the endpoints of $I_{i_1}, ..., I_{i_N}$ form a partition $P$ of $[a, b]$. We have

$$
\overline{R}(f, P) - \underline{R}(f, P) = \sum_{i_j} (M_j - m_j) \Delta x_j + \sum (M_j - m_j) \Delta x_j
$$

$$
\leq 2M \sum_{j=1}^{N} |I_{i_j}| + \frac{1}{k} \sum \Delta x_j
$$

$$
\leq 2M \varepsilon + \varepsilon (b - a)
$$

$$
= [2M + (b - a)] \varepsilon,
$$

where $M = \sup_{[a, b]} |f|$ and the second summation is over all subintervals in $V_i, i \in A$. Hence $f$ is integrable on $[a, b]$. 