**Theorem 1.** Let $\Omega$ be an open set in $\mathbb{C}$, and $f : \Omega \to \mathbb{C}$ be a complex-valued function on $\Omega$. Then the following are equivalent:

(a) $f$ is holomorphic on $\Omega$;

(b) $f'(z) := \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$ exists for every $z \in \Omega$;

(c) $f = u + iv$ where $u$ and $v$ are real-valued, continuously differentiable, and satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x};$$

(d) $f$ is continuous on $\Omega$, and for every closed disc $D$ contained inside $\Omega$, we have the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw \quad \text{for all } z \text{ in the interior of } D;$$

(e) For every open disc $D$ contained inside $\Omega$, one can represent $f$ as a convergent power series inside $D$, i.e. there exists coefficients $a_0, a_1, \ldots$, such that

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{for all } z \in D,$$

where $z_0$ is the center of $D$;

(f) For every $z_0 \in \Omega$, there exists a non-empty open disc $D$ centered at $z_0$, such that $f$ can be represented as a convergent power series inside $D$;

(g) (Morera’s theorem) $f$ is continuous on $\Omega$, and and for every closed triangle $T$ contained inside $\Omega$, we have

$$\int_{\partial T} f(z) dz = 0;$$

(h) For every open disc $D$ contained inside $\Omega$, there exists a holomorphic function $F$ on $D$ such that $F' = f$ on $D$.

**Remarks.**

1. The Cauchy-Riemann equations in (c) can be reformulated using the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

as

$$\frac{\partial f}{\partial \bar{z}} = 0.$$
See e.g. p.12 of [1].

2. The derivation of (e) from (d) can also be used to show that every function that is holomorphic on an annulus admits a Laurent series expansion. See e.g. Problem 3 of Chapter 3 of [1].

3. Other important consequences of the Cauchy integral formula in (d) include the Cauchy’s inequalities, Liouville’s theorem, and the fundamental theorem of algebra. See e.g. Section 4 of Chapter 2 of [1].

4. From the power series expansion of a holomorphic function into power series as in (e), one sees that the zeros of a non-constant holomorphic function are isolated. This leads one to study also singularities of holomorphic functions that are isolated, and those are classified into three kinds: removable singularities, poles and essential singularities. See e.g. Theorem 4.8 of Chapter 2, and Sections 1-3 of Chapter 3 of [1] (the latter also contains two important theorems, namely Riemann’s removable singularity theorem, and the theorem of Casorati-Weierstrass on essential singularities).

5. It follows from Morera’s theorem (g) that if $f_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions, and $f_n$ converges uniformly on every compact subset of $\Omega$ to a function $f : \Omega \to \mathbb{C}$, then $f$ is also holomorphic on $\Omega$. See e.g. Section 5.2 of Chapter 2 of [1].

6. Morera’s theorem (g) has, as an important consequence, the symmetry principle for holomorphic functions. See e.g. Section 5.4 of Chapter 2 of [1].

7. One can also replace the open discs $D$ in (h) above by any simply connected domain contained in $\Omega$. In fact, given a holomorphic function $f$ on a simply connected domain $D$, one can construct a desired $F$ (known as a primitive of $f$) by

$$F(z) = \int_{\gamma_z} f(w)dw,$$

where $\gamma_z$ is any (polygonal) path, entirely contained in $D$, that joins a fixed point $z_0 \in D$ to $z \in D$. In particular, if $f$ is holomorphic and nowhere vanishing on a simply connected domain $D$, then one can construct a logarithm of $f$, by declaring it to be a primitive of the (holomorphic) function $f'/f$ on $D$. See e.g. Sections 5 and 6 of Chapter 3 of [1].

8. The contour integral

$$\int_{\gamma} \frac{f'(z)}{f(z)}dz$$

also occurs in the discussion of the argument principle. If $f : \Omega \to \mathbb{C}$ is meromorphic on $\Omega$, and $\gamma$ is a positively oriented simple closed curve in $\Omega$ that avoids the poles and zeroes of $f$, then the above integral is $2\pi i$ times

$$(\text{number of zeroes of } f \text{ inside } \gamma - \text{number of poles of } f \text{ inside } \gamma).$$

This in turn leads to three important theorems: Rouche’s theorem, the open mapping theorem, and the maximum modulus principle. See e.g. Section 4 of Chapter 3 of [1].

REFERENCES