1. Osculating sphere \ (Thm 2.10 in the Textbook)

In \( \mathbb{R}^3 \),

Thm:

\[ C = \text{Frenet curve in } \mathbb{R}^3 \text{ with } T(s) \neq 0. \text{ (parametrized by arc-length)} \]

The sphere with center \( C(s_0) + \frac{1}{k(s_0)} e_2(s_0) - \frac{K'(s_0)}{T(s_0) K^2(s_0)} e_3(s_0) \) passing through \( C(s_0) \)

has a point of contact with the curve \( C(s) \) at the point \( s_0 \) of 3rd order. This sphere is uniquely determined by these properties and is called the osculating sphere.

PF:

Assume centre \( m(s_0) = c(s_0) + \alpha e_1(s_0) + \beta e_2(s_0) + r e_3(s_0) \), \( \alpha, \beta, r \) are to be determined.
Consider

\[ r(s) = \langle m - c(s), m - c(s) \rangle, \quad m = m(s_0) \]

\[ r' = -2 \langle m - c(s), c'(s) \rangle \]

\[ r'' = -2 \langle m - c(s), c''(s) \rangle + 2 \langle c'(s), c'(s) \rangle \]

\[ r''' = -2 \langle m - c(s), c'''(s) \rangle + 2 \langle c'(s), c''(s) \rangle + 4 \langle c''(s), c''(s) \rangle \]

\[ = -2 \langle m - c(s), c'''(s) \rangle + 6 \langle c'(s), c''(s) \rangle \]

Note that \( \langle c', c'' \rangle = 0 \), \( r''' = -2 \langle m - c(s), c'''(s) \rangle \).

The optimal contact of the sphere with \( c(s) \) means that as many derivatives of \( r(s) \) as possible vanish at \( s = s_0 \), i.e.

\[ r'(s_0) = 0 \iff \langle m - c(s_0), c'(s_0) \rangle = 0 \]

\[ \iff \langle m - c(s_0), e_1(s_0) \rangle = 0 \]

\[ \iff 2 = 0. \]
\[ T^{\prime}(S_0) = 0 \iff \langle m - c(S_0), c''(S_0) \rangle = \langle c(S_0), c'(S_0) \rangle = 0 \]

\[ e_2 = \frac{c''}{k} \]

\[ \langle \beta e_2 + re_3, ke_2 \rangle = 1 \]

\[ \beta = \frac{1}{k(S_0)} \]

\[ T^{\prime\prime}(S_0) = 0 \iff \langle m - c(S_0), c'''(S_0) \rangle = 0 \]

\[ \langle \frac{1}{k} e_2 + re_3, k' e_2 + k(-k e_1 + T e_3) \rangle = 0 \]

\[ \frac{k'}{k} + nkL = 0 \]

\[ t = -\frac{k'(s_0)}{k(s_0)/l(s_0)} \]

\[ S_0 \quad m(s_0) = c(s_0) + \frac{1}{k(s_0)} e_2(s_0) - \frac{k'(s_0)}{k'(s_0)/l(s_0)} e_3(s_0) \]

Centre = \( m(s_0) \), Radius = \( \sqrt{\frac{1}{k(s_0)} + \frac{k'(s_0)}{k'(s_0)/l(s_0)}} \) \( \Rightarrow \) Sphere \( S_{s_0} \)

pass through \( c(s_0) \), \( T(s_0) = 0 \) \( \Rightarrow \) \( c'(s_0) \) is inside the tangent plane of \( S_{s_0} \) at \( S_0 \), \( \exists \) curve \( S(s) \) on \( S_{s_0} \) s.t. \( S(s_0) = c(s_0) \), \( S'(s_0) = c'(s_0) \)
\[ r''(s_0) = 0 \implies \delta''(s_0) = C''(s_0) \]
\[ r'''(s_0) = 0 \implies \delta'''(s_0) = C'''(s_0). \]

So we say this sphere has a pt of contact with the curve \( c(s) \) at \( s_0 \) of 3rd order.

2. Slope lines (Thm 2.11 in the textbook)

For a Frenet curve in \( \mathbb{R}^3 \), the following conditions are equivalent:

(i) \( \exists v \in \mathbb{R}^3 \setminus \{0\} \) with the property that \( \langle e_1, v \rangle \) is constant.

(ii) \( \frac{T}{K} \) is constant.

We call such curve slope line.

\textbf{Pf:} \hspace{0.5cm} (i) \implies (ii)

We may write \( \langle e_1, v \rangle = |v| \cos \theta \) for some constant \( \theta \).
Differentiation ⇒ 0 = \langle e_1, v \rangle

= k \langle e_2, v \rangle

⇒ \langle e_2, v \rangle = 0 \quad \text{since} \; k \neq 0

⇒ v = \frac{1}{k} \langle v, e_1 \rangle e_1 + \langle v, e_3 \rangle e_3

= |v| \cos \theta e_1 + |v| \sin \theta e_3 \quad \text{(change } \theta \text{ to } -\theta \text{ if necessary)}

Differentiation ⇒

0 = |v| \cos \theta e_1' + |v| \sin \theta e_3'

⇒ 0 = \cos \theta ke_2 + \sin \theta (-\mathbf{1} \cdot e_2)

= (k \cos \theta - \mathbf{1} \sin \theta) \cdot e_2

⇒ k \cos \theta - \mathbf{1} \sin \theta = 0

⇒ \frac{\mathbf{1}}{k} = \cot \theta \quad \theta \neq 0 \quad \text{otherwise} \; k \neq 0 \; \text{elsewhere}

(iii) ⇒ (i) \; \text{if} \; \frac{\mathbf{1}}{k} = \text{constant}, \; \exists \; \theta \; \text{such that}

\frac{\mathbf{1}}{k} = \cot \theta.
Define \( V = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_3 \)

\[
V' = \cos \theta \mathbf{e}_1' + \sin \theta \mathbf{e}_3' = (k \cos \theta - 2 \sin \theta) \mathbf{e}_2
\]

\[
= 0
\]

\[
\langle e_1, v \rangle = \cos \theta = \text{constant}!
\]

\[\square\]

Remark: Helix is an example.

Please also read Thm 2.11 in the textbook since I provide a different proof.

3. Suggested Exercise 3 in HW.

The Frenet two-frame of a plane curve with given curvature function \( k(s) \) can be described by the exponential series for the matrix
\[
\begin{pmatrix}
0 & \int_0^s k(t) \, dt \\
-\int_0^s k(t) \, dt & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
e_1(s) \\
e_2(s)
\end{pmatrix} = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix}
0 & \int_0^s k(t) \, dt \\
-\int_0^s k(t) \, dt & 0
\end{pmatrix}^i \mathbf{e}_0
\]

**PF:** Given a curvature function \( k(s) \), assume

\[
e_1 = \begin{pmatrix} \cos(d(s)) \\ \sin(d(s)) \end{pmatrix}
\]

then \( e_2 = \begin{pmatrix} -\sin(d(s)) \\ \cos(d(s)) \end{pmatrix} \)

\[
K e_2 = e_1' = \alpha'( -\sin(d(s)), \cos(d(s)) ) = \alpha' e_2
\]

\[\Rightarrow\]

\( K = \alpha' \)

\[\Rightarrow\]

\( d(s) = \int_0^s k(t) \, dt \) if \( \alpha(0) = 0 \) i.e.

\[e_0 = (1, 0)\)

\[\Rightarrow\]

\( e_1 = (\cos(s^o_0 k), \sin(s^o_0 k)) \)

\( e_2 = (-\sin(s^o_0 k), \cos(s^o_0 k)) \)
\[
\begin{pmatrix}
\cos x & \sin x \\
-\sin x & \cos x
\end{pmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix}
0 & x^n \\
-x & 0
\end{pmatrix}
\]

(note that \( \begin{pmatrix}
0 & x^n \\
-x & 0
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} \))

\[
\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots
\]

\[
\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - \cdots
\]