Problems

1. (Kühnel Ch.2 Q.17) In the orthogonal (but not normal) three-frame $c', c'', c' \times c''$, prove that the Frenet equations of a space curve take the equivalent form

$$
\begin{pmatrix}
  c' \\
  c'' \\
  c' \times c''
\end{pmatrix}' =
\begin{pmatrix}
  0 & -\kappa^2 & 0 \\
  1 & -\kappa & -\tau \\
  0 & -\kappa & \kappa
\end{pmatrix}
\begin{pmatrix}
  c' \\
  c'' \\
  c' \times c''
\end{pmatrix}.
$$

Hence, the entries of the matrix depend in some sense rationally (i.e. without roots) on $\kappa^2 = \langle c'', c'' \rangle$ and $\tau$ (because of the relation $\kappa'/\kappa = \frac{1}{2}(\log \kappa^2)'$).

2. (Kühnel Ch.2 Q.14) Show that the osculating cubic parabola of a Frenet curve $c$ in $\mathbb{R}^3$, defined by

$$
s \mapsto c(0) + se_1(0) + \frac{s^2}{2}\kappa(0)e_2(0) + \frac{s^3}{6}\kappa(0)\tau(0)e_3(0),
$$

has at the point $s = 0$ the same curvature $\kappa(0)$ and torsion $\tau(0)$ as $c$ itself.

3. (Kühnel Ch.2 Q.15) In spherical coordinates $\varphi, \theta$, let a regular curve be given by the functions $(\varphi(s), \theta(s))$ inside the sphere with parametrization $(\cos \varphi \cos \theta, \sin \varphi \cos \theta, \sin \theta)$. For $s = 0$ the tangent to this curve is tangent to the equator $\vartheta = 0$, i.e. $\vartheta'(0) = 0$. Prove that the geodesic curvature is given by $\vartheta''(0) = \frac{d^2\vartheta}{ds^2}|_{s=0}$, and the curvature is consequently $\kappa(0) = \sqrt{1 + (\vartheta''(0))^2}$.

4. (Kühnel Ch.2 Q.20) Show the following: (i) $c$ is a helix if and only if the Darboux vector $\vec{D}$ is a constant vector; (ii) $c$ is a slope line if and only if $\vec{D}/\|\vec{D}\|$ is constant.

5. (Kühnel Ch.2 Q.23) Let $c$ be a Frenet curve in $\mathbb{R}^n$. Show that

$$
det(c', c'', \ldots, c^{(n)}) = \prod_{i=1}^{n-1} (\kappa_i)^{n-i}.
$$

6. (do Carmo P.47 Q.3) Compute the curvature of the ellipse

$$
x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi, \quad a \neq b,
$$

and show that it has exactly four vertices (i.e. points at which $\kappa' = 0$), namely, the points $(\pm a, 0)$ and $(0, \pm b)$. 

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Suggested Exercises

1. (Kühnel Ch.2 Q.16) Show that a slope line with $\tau \neq 0$ lies on a sphere if and only if an equation
   \[ \kappa^2(s) = (-A^2 s^2 + Bs + C)^{-1} \]
   is satisfied for some constants $A, B, C$, where $A = \frac{\tau}{\kappa}$. Prove that a spherical slope line through a point on the equator can never reach the north pole. It ends at a point where it cuts a small circle around the north pole orthogonally.

2. (Kühnel Ch.2 Q.21) The axis of the accompanying screw-motion at a point $c(0)$ is the line in the direction of the Darboux vector $\vec{D}(0) = \tau(0)e_1(0) + \kappa(0)e_3(0)$ through the point
   \[ P(0) = c(0) + \frac{\kappa(0)}{\kappa^2(0) + \tau^2(0)}e_2(0). \]
   Show that under these circumstances the tangent to the curve which passes through all of these points, namely,
   \[ P(s) = c(s) + \frac{\kappa}{\kappa^2 + \tau^2}e_2(s), \]
   is proportional to $\vec{D}(s)$ if and only if $\kappa/(\kappa^2 + \tau^2)$ is constant.