MATH 3060 Mathematical Analysis III

Tutorial 2 (September 21)

The following were discussed in the tutorial this week:

1. Convergence Criterion, Theorem 1.5 and 1.6.

2. Let \( f(x) = \cosh(x) := \frac{e^x + e^{-x}}{2} \) on \([-\pi, \pi]\). Show that
   \[
f(x) \sim \frac{\sinh \pi}{\pi} + \frac{2 \sinh \pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2 + 1} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} = \frac{\pi \coth \pi + 1}{2}.
   \]

3. Let
   \[
f(x) = \begin{cases} -\pi/2 - x/2 & \text{if } -\pi \leq x < 0 \\ 0 & \text{if } x = 0 \\ \pi/2 - x/2 & \text{if } 0 < x \leq \pi. \end{cases}
   \]
   Show that for any \( x \in [-\pi, \pi] \)
   \[
f(x) = \frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}.
   \]

4. Suppose \( f : [a, b] \to \mathbb{R} \) is Lipschitz continuous at \( x \), for all \( x \in [a, b] \). Is it true that \( f \) is uniformly Lipschitz on \([a, b]\)?
   \textbf{(Hint: Consider} \( f : [0, 1] \to \mathbb{R} \) defined by
   \[
f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \in (0, 1]; \\ 0 & \text{if } x = 0. \end{cases}
   \]

5. The notion of local Lipschitzity is introduced:

   \textbf{Definition.} Let \( f : [a, b] \to \mathbb{R} \) be a function and \( x \in [a, b] \). We say that \( f \) is locally Lipschitz at \( x \) if there is a neighbourhood \( U_x := (x - \delta_x, x + \delta_x) \) (\( \delta_x > 0 \)) and a constant \( L_x > 0 \) such that
   \[|f(u) - f(v)| \leq L_x |u - v| \quad \text{for all } x, y \in [a, b] \cap U_x. \]

6. Suppose \( f : [a, b] \to \mathbb{R} \) is locally Lipschitz at \( x \), for any \( x \in [a, b] \). Show that \( f \) is uniformly Lipschitz on \([a, b]\).
   \textbf{(Hint: You may use the following result:}
   \textbf{Theorem} \textbf{(Heine-Borel Theorem). Let} \( [a, b] \) \textbf{be a closed bounded interval. If} \{U_\alpha\}_{\alpha \in \mathcal{A}} \textbf{is a family of open intervals such that} [a, b] \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha, \textbf{then there exist} \alpha_1, \ldots, \alpha_k \in \mathcal{A} \textbf{such that} [a, b] \subseteq \bigcup_{i=1}^{k} U_{\alpha_i}. \textbf{Use the local Lipschitzity and Heine-Borel theorem to obtain open intervals} U_1, \ldots, U_k \textbf{such that} [a, b] \subseteq U_1 \cup \cdots \cup U_k. \textbf{Show that, by shrinking the intervals slightly, one can assume that} U_i \setminus U_j \neq \emptyset \textbf{if} i \neq j \textbf{and the separation between} U_i, U_j \textbf{is positive if} U_i \cap U_j = \emptyset. \)