In this chapter we study Fourier series. Basic definitions and examples are given in Section 1. In Section 2 we prove the fundamental Riemann-Lebesgue lemma and discuss the Fourier series from the mapping point of view. Pointwise and uniform convergence of the Fourier series of a function to the function itself under various regularity assumptions are studied in Section 3. In Section 1.5 we establish the $L^2$-convergence of the Fourier series without any additional regularity assumption. There are two applications. In Section 1.4 it is shown that every continuous function can be approximated by polynomials in a uniform manner. In Section 1.6 a proof of the classical isoperimetric problem for plane curves is presented. In the two appendices basic facts on series of functions and sets of measure zero are present.

1.1 Definition and Examples

The concept of series of functions and their pointwise and uniform convergence were discussed in Mathematical Analysis II. Power series and trigonometric series are the most important classes of series of functions. We learned power series in Mathematical Analysis II and now we discuss Fourier series. You are referred to Appendix I for basic definitions of series and series of functions.

First of all, a trigonometric series on $[-\pi, \pi]$ is a series of functions of the form

$$
\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx), \quad a_n, b_n \in \mathbb{R}.
$$
As \( \cos 0x = 1 \) and \( \sin 0x = 0 \), we always set \( b_0 = 0 \) and express the series as

\[
a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).
\]

It is called a **cosine series** if all \( b_n \) vanish and **sine series** if all \( a_n \) vanish. Trigonometric series form an important class of series of functions. In Mathematical Analysis II, we studied the convergence of the series of functions. We recall

- Uniform convergence implies pointwise convergence of a series of functions,
- Absolute convergence implies pointwise convergence of a series of functions,
- Weierstrass M-Test for uniform and absolute convergence (see Appendix I).

For instance, using the fact that \( |\cos nx|, |\sin nx| \leq 1 \), Weierstrass M-Test tells us that a trigonometric series is uniformly and absolutely convergent when its coefficients satisfy \( \sum |a_n|, \sum |b_n| < \infty \), and this is the case when \( |a_n|, |b_n| \leq Cn^{-s}, \forall n \geq 1 \), for some constant \( C \) and \( s > 1 \). Since the partial sums are continuous functions and uniform convergence preserves continuity, the infinite series

\[
\phi(x) \equiv a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]

is a continuous function on \([-2\pi, 2\pi]\). \( \phi \) is also of period \( 2\pi \). For, by pointwise convergence, we have

\[
\phi(x + 2\pi) = \lim_{n \to \infty} \sum_{k=0}^{n} (a_k \cos(kx + 2k\pi) + b_k \sin(kx + 2k\pi))
\]

\[
= \lim_{n \to \infty} \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx)
\]

\[
= \phi(x),
\]

hence it is \( 2\pi \)-periodic.

In the literature there are many interesting convergence results concerning trigonometric series. Nevertheless, we will not go into this direction. Here our attention is on a special class of trigonometric series called Fourier series. Each Fourier series is associated with an integrable, periodic function.

Given a \( 2\pi \)-periodic function which is Riemann integrable function \( f \) on \([-\pi, \pi]\), its
Fourier series or Fourier expansion is the trigonometric series given by

\[
\begin{align*}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy, \quad n \geq 1 \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \, dy, \quad n \geq 1 \quad \text{and} \\
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, dy.
\end{align*}
\] (1.1)

Note that \(a_0\) is the average of the function over the interval. From this definition we gather two basic information. First, the Fourier series of a function involves the integration of the function over an interval, hence any modification of the values of the function over a subinterval, not matter how small it is, may change the Fourier coefficients \(a_n\) and \(b_n\). This is unlike power series which only depend on the local properties (derivatives of all order at a designated point). We may say Fourier series depend on the global information but power series only depend on local information. Second, recalling from the theory of Riemann integral, we know that two integrable functions which are equal almost everywhere have the same integral. (We will see the converse is also true, namely, two functions with the same Fourier series are equal almost everywhere.) In Appendix II we recall the concept of a measure zero set and some of its basic properties. Therefore, the Fourier series of two such functions are the same. In particular, the Fourier series of a function is completely determined with its value on the open interval \((-\pi, \pi)\), regardless its values at the endpoints.

The motivation of the Fourier series comes from the belief that for a “nice function” of period \(2\pi\), its Fourier series converges to the function itself. In other words, we have

\[
f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \quad \forall x \in \mathbb{R}.
\] (1.2)

When this holds, the coefficients \(a_n, b_n\) are given by (1.1). To see this, we multiply (1.2) by \(\cos mx\) and then integrate over \([-\pi, \pi]\). Using the relations

\[
\begin{align*}
\int_{-\pi}^{\pi} \cos nx \cos mx \, dx &= \begin{cases} 
\pi, & n = m \\
0, & n \neq m
\end{cases}, \\
\int_{-\pi}^{\pi} \cos nx \sin mx \, dx &= 0 \quad (n, m \geq 1), \quad \text{and} \\
\int_{-\pi}^{\pi} \cos nx \, dx &= \begin{cases} 
2\pi, & n = 0 \\
0, & n \neq 0
\end{cases},
\end{align*}
\]

we formally arrive at the expression of \(a_n, n \geq 0\), in (1.2). Similarly, by multiplying (1.2) by \(\sin mx\) and then integrate over \([-\pi, \pi]\), one obtain the expression of \(b_n, n \geq 1\), in (1.2) after using

\[
\begin{align*}
\int_{-\pi}^{\pi} \sin nx \sin mx \, dx &= \begin{cases} 
\pi, & n = m \\
0, & n \neq m
\end{cases}.
\end{align*}
\]
Of course, (1.2) arises from the hypothesis that every sufficiently nice function of period 2π is equal to its Fourier expansion. The study of under which “nice conditions” this could happen is one of the main objects in the theory of Fourier series.

We can associate a Fourier series for any integrable function on \([-2\pi, 2\pi]\). Indeed, it suffices to extend the function as a function of period 2π. The extension is straightforward; simply let \(\tilde{f}(x) = f(x-(n+1)\pi)\) where \(n\) is the unique integer satisfying \(n\pi < x \leq (n+2)\pi\). It is clear that \(\tilde{f}\) is equal to \(f\) on \((-\pi, \pi]\). As the function is defined on \([-\pi, \pi]\), apparently an extension in strict sense is possible only if \(f(-\pi) = f(\pi)\). Since the function value at one point does not change the Fourier series, from now on it will be understood that the extension of a function to a 2π-periodic function refers to the extension for the restriction of this function on \((-\pi, \pi]\). Note that for the 2π-periodic extension of a continuous function on \([-\pi, \pi]\) has a jump discontinuity at ±π when \(f(\pi) \neq f(-\pi)\). It is is continuous on \(\mathbb{R}\) if and only if \(f(-\pi) = f(\pi)\). In the following we will not distinguish \(f\) with its extension \(\tilde{f}\).

We will use
\[
f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
to denote the fact that the right hand side of this expression is the Fourier series of \(f\).

**Example 1.1** We consider the function \(f_1(x) = x\). Its extension is a piecewise smooth function with jump discontinuities at \((2n + 1)\pi, n \in \mathbb{Z}\). As \(f_1\) is odd and \(\cos nx\) is even,
\[
\pi a_n = \int_{-\pi}^{\pi} x \cos nx \, dx = 0, \quad n \geq 0,
\]
and
\[
\pi b_n = \int_{-\pi}^{\pi} x \sin nx \, dx \\
= -x \cos nx \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos nx \, dx \\
= (-1)^{n+1} \frac{2\pi}{n}.
\]
Therefore,
\[
f_1(x) \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.
\]
Since \(f_1\) is an odd function, it is reasonable to see that no cosine functions are involved in its Fourier series. How about the convergence of this Fourier series? Although the terms decay like \(O(1/n)\) as \(n \to \infty\), its convergence is not clear at this moment. On the other hand, this Fourier series is equal to 0 at \(x = \pm \pi\) but \(f_1(\pm \pi) = \pi\). So, one thing is sure, namely, the Fourier series is not always equal to its function. It is worthwhile to observe
that the bad points $\pm \pi$ are precisely the discontinuity points of $f_1$.

**Notation** The big O and small $\circ$ notations are very convenient in analysis. We say a sequence $\{x_n\}$ satisfies $x_n = O(n^s)$ means that there exists a constant $C$ independent of $n$ such that $|x_n| \leq Cn^s$ as $n \to \infty$, in other words, the growth (resp. decay $s \geq 0$) of $\{x_n\}$ is not faster (resp. slower $s < 0$) the $s$-th power of $n$. On the other hand, $x_n = o(n^s)$ means $|x_n|n^{-s} \to 0$ as $n \to \infty$.

Example 1.2 Next consider the function $f_2(x) = x^2$. Unlike the previous example, its $2\pi$-periodic extension is continuous on $\mathbb{R}$. After performing integration by parts, the Fourier series of $f_2$ is seen to be

$$f_2(x) \equiv x^2 \sim \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx.$$ 

As $f_2$ is an even function, this is a cosine series. The rate of decay of the Fourier series is like $O(1/n^2)$. Using Weierstrass M-test, this series converges uniformly to a continuous function. In fact, due to the following result, it converges uniformly to $f_2$. Note that $f_2$ is smooth on $(n\pi, (n+1)\pi)$, $n \in \mathbb{Z}$.

**Convergence Criterion.** The Fourier series of a continuous, $2\pi$-periodic function which is $C^1$-piecewise on $[-\pi, \pi]$ converges to the function uniformly.

A function is called $C^1$-**piecewise** on some interval $I = [a, b]$ if there exists a partition of $I$ into subintervals $\{I_j\}_{j=1}^N$ and there are $C^1$-function $f_j$ defined on $I_j$ such that $f = f_j$ on each $(a_j, a_{j+1})$ where $I_j = [a_j, a_{j+1}]$. This convergence criterion is a special case of Theorem 1.7 in Section 3.

We list more examples of Fourier series of functions and leave them for you to verify.

(a) $f_3(x) \equiv |x| \sim \frac{\pi}{2} - 4 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x$,

(b) $f_4(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in (-\pi, 0) \end{cases}$ $\sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$,

(c) $f_5(x) = \begin{cases} x(\pi - x), & x \in [0, \pi] \\ x(\pi + x), & x \in (-\pi, 0) \end{cases}$ $\sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x$.

Let $\{c_n\}_{n=-\infty}^{\infty}$ be a bisequence of complex numbers. (A bisequence is a map from $\mathbb{Z}$ to $\mathbb{C}$.) A (complex) trigonometric series is the infinite series associated to the bisequence
\[ \{c_n e^{inx}\}_{-\infty}^{\infty} \text{ and is denoted by } \sum_{-\infty}^{\infty} c_n e^{inx}. \] To be in line with the real case, it is said to be convergent at \( x \) if
\[
\lim_{n \to \infty} \sum_{k=-n}^{n} c_n e^{inx}
\]
exists. Now, a complex Fourier series can be associated to a complex-valued function. Let \( f \) be a \( 2\pi \)-periodic complex-valued function which is integrable on \([-\pi, \pi]\). Its Fourier series is given by the series
\[
f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx},
\]
where the Fourier coefficients \( c_n \) are given by
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}.
\]
Here for a complex function \( f \), its integration over some \([a, b]\) is defined to be
\[
\int_{a}^{b} f(x) dx = \int_{a}^{b} f_1(x) dx + i \int_{a}^{b} f_2(x) dx,
\]
where \( f_1 \) and \( f_2 \) are respectively the real and imaginary parts of \( f \). It is called integrable if both real and imaginary parts are integrable. The same as in the real case, formally the expression of \( c_n \) is obtained as in the real case by first multiplying the relation
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}
\]
with \( e^{inx} \) and then integrating over \([-\pi, \pi]\) with the help from the relation
\[
\int_{-\pi}^{\pi} e^{inx} e^{-inx} dx = \begin{cases} 
2\pi, & n = m \\
0, & n \neq m.
\end{cases}
\]
When \( f \) is of real-valued, there are two Fourier series, that is, the real and the complex ones. To relate them it is enough to observe the Euler’s formula \( e^{i\theta} = \cos \theta + i \sin \theta \), so for \( n \geq 1 \)
\[
2\pi c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\
= \int_{-\pi}^{\pi} f(x) (\cos nx + i \sin nx) dx \\
= \int_{-\pi}^{\pi} f(x) \cos nx dx - i \int_{-\pi}^{\pi} f(x) \sin nx dx \\
= \pi (a_n - ib_n).
\]
1.1. DEFINITION AND EXAMPLES

we see that
\[ c_n = \frac{1}{2}(a_n - ib_n), \quad n \geq 1, \quad c_0 = a_0. \]
By a similar computation, we have
\[ c_n = \frac{1}{2}(a_{-n} + ib_{-n}), \quad n \leq -1. \]
It follows that \( c_{-n} = \overline{c_n} \) for all \( n \). In fact, the converse is true, that is, a complex Fourier series is the Fourier series of a real-valued function if and only if \( c_{-n} = \overline{c_n} \) holds for all \( n \).
Indeed, letting
\[ f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \]
be the Fourier series of \( f \), it is straightforward to verify that
\[ \overline{f(x)} \sim \sum_{n=-\infty}^{\infty} d_n e^{inx}, \quad d_n = \overline{c_{-n}}. \]
Hence when \( f \) is real-valued, \( \overline{f} = f \) so \( c_n = \overline{c_{-n}} \) holds. The complex form of Fourier series sometimes makes expressions and computations more elegant. We will use it whenever it makes things simpler.

We have been working on the Fourier series of \( 2\pi \)-periodic functions. For functions of \( 2T \)-period, their Fourier series are not the same. They can be found by a scaling argument. Let \( f \) be \( 2T \)-periodic. The function \( g(x) = f(Tx/\pi) \) is a \( 2\pi \)-periodic function. Thus,
\[ f \left( \frac{Tx}{\pi} \right) = g(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \]
where \( a_0, a_n, b_n, n \geq 1 \) are the Fourier coefficients of \( g \). By a change of variables, we can express everything inside the coefficients in terms of \( f, \cos n\pi x/T \) and \( \sin n\pi x/T \). The result is
\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi}{T} x + b_n \sin \frac{n\pi}{T} x \right), \]
where
\[ a_n = \frac{1}{T} \int_{-T}^{T} f(y) \cos \frac{n\pi}{T} y \, dy, \]
\[ b_n = \frac{1}{T} \int_{-T}^{T} f(y) \sin \frac{n\pi}{T} y \, dy, \quad n \geq 1, \quad \text{and} \]
\[ a_0 = \frac{1}{2T} \int_{-T}^{T} f(y) \, dy. \]
It reduces to (1.1) when \( T \) is equal to \( \pi \).
1.2 Riemann-Lebesgue Lemma

From the examples of Fourier series of functions in the previous section we see that the coefficients decay to 0 eventually. We will show that this is generally true. This is the content of the following result.

**Theorem 1.1 (Riemann-Lebesgue Lemma).** The Fourier coefficients of a $2\pi$-periodic function integrable on $[-\pi, \pi]$ converge to 0 as $n \to \infty$.

We point out this theorem still holds when $[-\pi, \pi]$ is replaced by any $[a, b]$. The proof is essentially the same.

We will use $R[-\pi, \pi]$ to denote the vector space of all integrable functions. To prepare for the proof we study how to approximate an integrable function by step functions. Let $a_0 = -\pi < a_1 < \cdots < a_N = \pi$ be a partition of $[-\pi, \pi]$. A step function $s$ satisfies $s(x) = s_j$, $\forall x \in (a_j, a_{j+1}]$, $\forall j \geq 0$. The value of $s$ at $-\pi$ is not important, but for definiteness let’s set $s(-\pi) = s_0$. We can express a step function in a better form by introducing the characteristic function $\chi_E$ for a set $E \subset \mathbb{R}$:

$$\chi_E = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

Then,

$$s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}, \quad I_j = (a_j, a_{j+1}], \quad j \geq 1, \quad I_0 = [a_0, a_1].$$

**Lemma 1.2.** For every step function $s$ in $R[-\pi, \pi]$, there exists some constant $C$ independent of $n$ such that

$$|a_n|, \quad |b_n| \leq \frac{C}{n}, \quad \forall n \geq 1,$$

where $a_n, b_n$ are the Fourier coefficients of $s$.

**Proof.** Let $s(x) = \sum_{j=0}^{N-1} s_j \chi_{I_j}$. We have

$$\pi a_n = \int_{-\pi}^{\pi} \sum_{j=0}^{N-1} s_j \chi_{I_j} \cos nx \, dx$$

$$= \sum_{j=0}^{N-1} s_j \int_{a_j}^{a_{j+1}} \cos nx \, dx$$

$$= \frac{1}{n} \sum_{j=0}^{N-1} s_j \left( \sin na_{j+1} - \sin na_j \right).$$
It follows that
\[ |a_n| \leq \frac{C}{n}, \quad \forall n \geq 1, \quad C = \frac{2}{\pi} \sum_{j=0}^{N-1} |s_j|. \]

Clearly a similar estimate holds for \( b_n \).

**Lemma 1.3.** Let \( f \in R[-\pi, \pi] \). Given \( \varepsilon > 0 \), there exists a step function \( s \) such that \( s \leq f \) on \( [-\pi, \pi] \) and
\[
\int_{-\pi}^{\pi} (f - s) < \varepsilon.
\]

**Proof.** As \( f \) is integrable, it can be approximated from below by its Darboux lower sums. In other words, for \( \varepsilon > 0 \), we can find a partition \( -\pi = a_0 < a_1 < \cdots < a_N = \pi \) such that
\[
\left| \int_{-\pi}^{\pi} f - \sum_{j=0}^{N-1} m_j(a_{j+1} - a_j) \right| < \varepsilon,
\]
where \( m_j = \inf \{ f(x) : x \in [a_j, a_{j+1}] \} \). It follows that
\[
\left| \int_{-\pi}^{\pi} (f - s) \right| < \varepsilon
\]
after setting
\[
s(x) = \sum_{j=0}^{N-1} m_j \chi_{I_j}, \quad I_j = (a_j, a_{j+1}], \quad j \geq 1, \quad I_0 = [a_0, a_1].
\]

Now we prove Theorem 1.1.

**Proof.** For \( \varepsilon > 0 \), we can find \( s \) as constructed in Lemma 1.3 such that \( 0 \leq f - s \) and
\[
\int_{-\pi}^{\pi} (f - s) < \frac{\varepsilon}{2}.
\]
Let \( a'_n \) be the \( n \)-th Fourier coefficient of \( s \). By Lemma 1.2,
\[
|a'_n| < \frac{\varepsilon}{2},
\]
for all \( n \geq n_0 = \lfloor 2C/\varepsilon \rfloor + 1 \).
\[
|\pi(a_n - a'_n)| = \left| \int_{-\pi}^{\pi} (f - s) \cos nx \, dx \right| \\
\leq \int_{-\pi}^{\pi} |f - s| \\
= \int_{-\pi}^{\pi} (f - s) \\
< \frac{\varepsilon}{2}.
\]
It follows that for all \( n \geq n_0 \),
\[
|a_n| \leq |a_n - a'_n| + |a'_n| < \frac{\varepsilon}{2\pi} + \frac{\varepsilon}{2} < \varepsilon.
\]

The same argument applies to \( b_n \) too. \( \square \)

It is useful to bring in a “mapping” point of view between functions and their Fourier series. Let \( R_{2\pi} \) be the collection of all \( 2\pi \)-periodic complex-valued functions integrable on \([-\pi, \pi]\) and \( C \) consisting of all complex-valued bisequences \( \{c_n\} \) satisfying \( c_n \to 0 \) as \( n \to \pm\infty \). The Fourier series sets up a mapping \( \Phi \) from \( R_{2\pi} \) to \( C \) by sending \( f \) to \( \{\hat{f}(n)\} \) where, to make things clear, we have let \( \hat{f}(n) = c_n \), the \( n \)-th Fourier coefficient of \( f \). When real-functions are considered, restricting to the subspace of \( C \) given by those satisfying \( c_{-n} = \overline{c_n} \), \( \Phi \) maps all real functions into this subspace. Perhaps the first question we ask is: Is \( \Phi \) one-to-one? Clearly the answer is no, for two functions which differ on a set of measure zero have the same Fourier coefficients. However, we have the following result, to be proved in Section 5.

**Uniqueness Theorem.** The Fourier series of two functions in \( R_{2\pi} \) coincide if and only if they are equal almost everywhere.

Thus \( \Phi \) is essentially one-to-one. One can show that it is not onto. Despite of this, we may still study how various structures on \( R_{2\pi} \) and \( C \) are associated under \( \Phi \). Observe that both \( R_{2\pi} \) and \( C \) form vector spaces over \( \mathbb{C} \). In fact, there are obvious and surprising ones. Some of them are listed below and more can be found in the exercise.

**Property 1.** \( \Phi \) is a linear map. Observe that both \( R_{2\pi} \) and \( C \) form vector spaces over \( \mathbb{R} \) or \( \mathbb{C} \). The linearity of \( \Phi \) is clear from its definition.

**Property 2.** When \( f \in R_{2\pi} \) is \( k \)-th differentiable and all derivatives up to \( k \)-th order belong to \( R_{2\pi} \), \( \hat{f}'(n) = (in)^k \hat{f}(n) \) for all \( n \in \mathbb{Z} \). See Proposition 1.4 below for a proof. This property shows that differentiation turns into the multiplication of a factor \( (in)^k \) under \( \Phi \). This is amazing!

**Property 3.** Every translation in \( \mathbb{R} \) induces a “translation operation” on functions defined on \( \mathbb{R} \). More specifically, for \( a \in \mathbb{R} \), set \( f_a(x) = f(x + a) \), \( x \in \mathbb{R} \). Clearly \( f_a \) belongs to \( R_{2\pi} \). We have \( \hat{f}_a(n) = e^{ina} \hat{f}(n) \). This property follows directly from the definition. It shows that a translation in \( R_{2\pi} \) turns into the multiplication of a factor \( e^{ina} \) under \( \Phi \).

**Proposition 1.4.** Let \( f \) be a \( 2\pi \)-periodic function which is differentiable on \([-\pi, \pi]\) with \( f' \in R_{2\pi} \). If
\[
f'(x) \sim c_0 + \sum_{n=1}^{\infty} (c_n \cos nx + d_n \sin nx),
\]
then \( c_n = nb_n \), and \( d_n = -na_n \). In complex notations, \( \hat{f}'(n) = in\hat{f}(n) \).
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Proof. We compute

\[
\pi c_n = \int_{-\pi}^{\pi} f'(y) \cos ny \, dy \\
= f(y) \cos ny|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(y)(-n \sin ny) \, dy \\
= n \int_{-\pi}^{\pi} f(y) \sin ny \, dy \\
= \pi n a_n.
\]

Similarly,

\[
\pi d_n = \int_{-\pi}^{\pi} f'(y) \sin ny \, dy \\
= f(y) \sin ny|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f(y)n \cos ny \, dy \\
= -n \int_{-\pi}^{\pi} f(y) \cos ny \, dy \\
= -\pi n a_n.
\]

Property 2 links the regularity of the function to the rate of decay of its Fourier coefficients. This is an extremely important property. When \( f \) is a 2\( \pi \)-periodic function whose derivatives up to \( k \)-th order belong to \( R_{2\pi} \), applying Riemann-Lebesgue lemma to \( f^{(k)} \) we know that \( f^{(k)}(n) = o(1) \) as \( n \to \infty \). By Property 2 it follows that \( \hat{f}(n) = o(n^{-k}) \), that is, the Fourier coefficients of \( f \) decay faster than \( n^{-k} \). Since \( \sum_{n=1}^{\infty} n^{-2} < \infty \), an application of Weierstrass M-test establishes the following result: The Fourier series of \( f \) converges uniformly provided \( f, f', \) and \( f'' \) belong to \( R_{2\pi} \). Although the Fourier series converges uniformly, at this point we cannot conclude that the limit function is \( f \). This fact will be proved in the next section.

1.3 Convergence of Fourier Series

In this section we study the convergence of the Fourier series of a function to itself. Recall that the series \( a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \), or \( \sum_{n=-\infty}^{\infty} c_n e^{inx} \), where \( a_n, b_n, c_n \) are the Fourier coefficients of a function \( f \) converges to \( f \) at \( x \) means that the \( n \)-th partial sum

\[
(S_n f)(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)
\]
or

\begin{align*}
(S_n f)(x) &= \sum_{k=-n}^{n} c_k e^{ikx} \\
&\text{converges to } f(x) \text{ as } n \to \infty.
\end{align*}

We start by expressing the partial sums in closed form. Indeed,

\begin{align*}
(S_n f)(x) &= a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) (\cos ky \cos kx + \sin ky \sin kx) \, dy \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \frac{1}{2} + \sum_{k=1}^{n} \cos k(y-x) \right) f(y) \, dy \\
&= \frac{1}{\pi} \int_{-\pi}^{x+\pi} \left( \frac{1}{2} + \sum_{k=1}^{n} \cos k(z) \right) f(x+z) \, dz \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \frac{1}{2} + \sum_{k=1}^{n} \cos k(z) \right) f(x+z) \, dz,
\end{align*}

where in the last step we have used the fact that the integrals over any two periods are the same. Using the elementary formula

\[\cos \theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\sin \left( n + \frac{1}{2} \right) \theta - \sin \frac{1}{2} \theta}{2 \sin \frac{1}{2} \theta}, \quad \theta \neq 0,\]

we obtain

\[ (S_n f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin \left( n + \frac{1}{2} \right) \frac{z}{2 \sin \frac{1}{2} z} f(x+z) \, dz. \]

We express it in the form

\[ (S_n f)(x) = \int_{-\pi}^{\pi} D_n(z) f(x+z) \, dz, \]

where \( D_n \) is the \textbf{Dirichlet kernel}

\begin{align*}
D_n(z) &= \left\{ \begin{array}{ll}
\sin \left( n + \frac{1}{2} \right) \frac{z}{2 \sin \frac{1}{2} z}, & z \neq 0 \\
\frac{2n+1}{2\pi}, & z = 0.
\end{array} \right.
\end{align*}

Using \( \sin \theta/\theta \to 1 \) as \( \theta \to 0 \), we see that \( D_n \) is continuous on \([-\pi, \pi]\).

Taking \( f \equiv 1 \), \( S_n f = 1 \) for all \( n \). Hence

\[ 1 = \int_{-\pi}^{\pi} D_n(z) \, dz. \]
Using it we can write
\[(S_n f)(x) - f(x) = \int_{-\pi}^{\pi} D_n(z)(f(x + z) - f(x)) \, dz. \tag{1.3}\]

In order to show \(S_n f(x) \to f(x)\), it suffices to show the right hand side of (1.3) tends to 0 as \(n \to \infty\).

Thus, the Dirichlet kernel plays a crucial role in the study of the convergence of Fourier series. We list some of its properties as follows.

**Property I.** \(D_n(z)\) is an even, continuous, 2\(\pi\)-periodic function vanishing at \(z = 2k\pi/(2n + 1), -n \leq k \leq n, \) on \([-\pi, \pi]\).

**Property II.** \(D_n\) attains its maximum value \((2n + 1)/2\) at 0.

**Property III.**
\[\int_{-\pi}^{\pi} D_n(z) \, dz = 1\]

**Property IV.** For every \(\delta > 0\),
\[\int_{0}^{\delta} |D_n(z)| \, dz \to \infty, \quad \text{as } n \to \infty.\]

Only the last property needs a proof. Indeed, for each \(n\) we can fix an \(N\) such that \(\pi N \leq (2n + 1)\delta/2 \leq (N + 1)\pi\), so \(N \to \infty\) as \(n \to \infty\). We compute
\[
\int_{0}^{\delta} |D_n(z)| \, dz = \int_{0}^{\delta} \frac{|\sin(n + \frac{1}{2})z|}{2\pi|\sin \frac{\pi}{2}|} \, dz \\
\geq \frac{1}{\pi} \int_{0}^{\delta} \frac{|\sin t|}{t} \, dt \\
\geq \frac{1}{\pi} \int_{0}^{N\pi} \frac{|\sin t|}{t} \, dt \\
= \frac{1}{\pi} \sum_{k=1}^{N} \int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t} \, dt \\
= \frac{1}{\pi} \sum_{k=1}^{N} \int_{0}^{\pi} \frac{|\sin s|}{s + (k - 1)\pi} \, ds \\
\geq \frac{1}{\pi} \sum_{k=1}^{N} \int_{0}^{\pi} \frac{|\sin s|}{\pi k} \, ds \\
= c_0 \sum_{k=1}^{N} \frac{1}{k}, \quad c_0 = \frac{1}{\pi^2} \int_{0}^{\pi} |\sin s| \, ds > 0, \\
\to \infty,
\]
as \( N \to \infty \).

To elucidate the effect of the kernel, we fix a small \( \delta > 0 \) and split the integral into two parts:

\[
\int_{-\pi}^{\pi} \chi_A(z)D_n(z)(f(x+z) - f(x)) \, dz,
\]
and

\[
\int_{-\pi}^{\pi} \chi_B(z)D_n(z)(f(x+z) - f(x)) \, dz,
\]
where \( A = (-\delta, \delta) \) and \( B = [-\pi, \pi] \setminus A \). The second integral can be written as

\[
\int_{-\pi}^{\pi} \chi_B(z)(f(x+z) - f(x)) \frac{1}{2\pi \sin \frac{z}{2}} \sin \left( n + \frac{1}{2} \right) z \, dz.
\]

As \( |\sin z/2| \) has a positive lower bound on \( B \), the function

\[
\chi_B(z)(f(x+z) - f(x)) \frac{1}{2\pi \sin \frac{z}{2}}
\]
belongs to \( R[-\pi, \pi] \) and the second integral tends to 0 as \( n \to \infty \) in view of Riemann-Lebesgue lemma. The trouble lies on the first integral. It can be estimated by

\[
\int_{-\delta}^{\delta} |D_n(z)||f(x+z) - f(x)| \, dz.
\]

In view of Property IV, No matter how small \( \delta \) is, this term may go to \( \infty \) so it is not clear how to estimate this integral.

The difficulty can be resolved by imposing a further regularity assumption on the function. First a definition. For a function \( f \) defined on \([a, b]\) is called **Lipschitz continuous** at \( x \in [a, b] \) if there exist \( L \) and \( \delta \) such that

\[
|f(y) - f(x)| \leq L |y - x|, \quad \forall y \in [a, b], \quad |y - x| \leq \delta.
\] (1.4)

Here both \( L \) and \( \delta \) depend on \( x \). We point out that if \( f \in C[a, b] \) is Lipschitz continuous at \( x \), there exists some \( L' \) such that

\[
|f(y) - f(x)| \leq L'|y - x|, \quad \forall y \in [a, b].
\]

In fact, this comes from (1.4) if \( |y - x| \leq \delta \). For \( y \) satisfying \( |y - x| > \delta \), we have

\[
|f(y) - f(x)| \leq \frac{|f(y)| + |f(x)|}{|y - x|},
\]
hence we could take

\[
L' = \max \left\{ L, \frac{2M}{\delta} \right\},
\]
where \( M = \sup \{|f(y)| : y \in [a, b]\} \).
Theorem 1.5. Let \( f \) be a \( 2\pi \)-periodic function integrable on \([-\pi, \pi]\). Suppose that \( f \) is Lipschitz continuous at \( x \). Then \( \{S_n f(x)\} \) converges to \( f(x) \) as \( n \to \infty \).

Proof. Let \( \Phi_\delta \) be a cut-off function satisfying (a) \( \Phi_\delta \in C(\mathbb{R}) \), \( \Phi_\delta \equiv 0 \) outside \((-\delta, \delta)\), (b) \( \Phi_\delta \geq 0 \) and (c) \( \Phi_\delta = 1 \) on \((-\delta/2, \delta/2)\). We write

\[
(S_n f)(x) - f(x) = \int_{-\pi}^{\pi} D_n(z)(f(x + z) - f(x)) \, dz
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(n + \frac{1}{2})z \, (f(x + z) - f(x)) \, dz
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_\delta(z) \sin(n + \frac{1}{2})z \, (f(x + z) - f(x)) \, dz
\]

\[
+ \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - \Phi_\delta(z)) \frac{\sin(n + \frac{1}{2})z}{\sin \frac{z}{2}} \, (f(x + z) - f(x)) \, dz
\]

\[
\equiv I + II.
\]

By our assumption on \( f \), there exists \( \delta_0 > 0 \) such that

\[
|f(x + z) - f(x)| \leq L |z|, \quad \forall |z| < \delta_0.
\]

Using \( \sin \theta / \theta \to 1 \) as \( \theta \to 0 \), there exists \( \delta_1 \) such that \( 2|\sin z/2| \geq |z/2| \) for all \( z, |z| < \delta_1 \). For \( z, |z| < \delta \equiv \min \{\delta_0, \delta_1\} \), we have \( |f(x + z) - f(x)|/|\sin z/2| \leq 4L \) and

\[
|I| \leq \frac{1}{2\pi} \int_{-\delta}^{\delta} \Phi_\delta(z) \frac{|\sin(n + \frac{1}{2})z|}{|\sin \frac{z}{2}|} \, |f(x + z) - f(x)| \, dz
\]

\[
\leq \frac{1}{2\pi} \int_{-\delta}^{\delta} 4L \, dz
\]

\[
= \frac{4\delta L}{\pi}.
\]

For \( \epsilon > 0 \), we fix \( \delta \) so that

\[
\frac{4\delta L}{\pi} < \frac{\epsilon}{2}.
\]

After fixing \( \delta \), we turn to the second integral

\[
II = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \Phi_\delta(z))(f(x + z) - f(x)) \, \frac{\sin \left( n + \frac{1}{2} \right) z}{\sin \frac{z}{2}} \, dz
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \Phi_\delta(z))(f(x + z) - f(x)) \left( \cos \frac{z}{2} \sin nz + \sin \frac{z}{2} \cos nz \right) \, dz
\]

\[
\equiv \int_{-\pi}^{\pi} F_1(x, z) \, \sin nz \, dz + \int_{-\pi}^{\pi} F_2(x, z) \, \cos nz \, dz.
\]
As \(1 - \Phi_\delta(z) = 0\), for \(z \in (-\delta/2, \delta/2)\), \(|\sin z/2|\) has a positive lower bound on \((-\pi, -\delta/2) \cup (\delta/2, \pi)\), and so \(F_1\) and \(F_2\) are integrable on \([-\pi, \pi]\). By Riemann-Lebesgue lemma, for \(\varepsilon > 0\), there is some \(n_0\) such that

\[
\left| \int_{-\pi}^{\pi} F_1 \sin nz\,dz \right|, \left| \int_{-\pi}^{\pi} F_2 \cos nz\,dz \right| < \frac{\varepsilon}{4}, \quad \forall n \geq n_0. \tag{1.7}
\]

Putting (1.5), (1.6) and (1.7) together,

\[
|S_n f(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon, \quad \forall n \geq n_0.
\]

We have shown that \(S_n f(x)\) tends to \(f(x)\) when \(f\) is Lipschitz continuous at \(x\).

We leave some remarks concerning this proof. First, the cut-off function \(\Phi_\delta\) can be replaced by \(\chi_{[-\delta, \delta]}\) without affecting the proof. Second, the regularity condition Lipschitz continuity is used to kill off the growth of the kernel at \(x\). Third, this method used in this proof is a standard one. It will appear in many other places. For instance, a careful examination of it reveals a convergence result for functions with jump discontinuity after using the evenness of the Dirichlet kernel.

**Theorem 1.6.** Let \(f\) be a \(2\pi\)-periodic function integrable on \([-\pi, \pi]\). Suppose at some \(x \in [-\pi, \pi]\), \(\lim_{y \to x^+} f(y)\) and \(\lim_{y \to x^-} f(y)\) exist and there are \(\delta > 0\) and constant \(L\) such that

\[
|f(y) - f(x^+)| \leq L(y - x), \quad \forall y, \quad 0 < y - x < \delta,
\]

and

\[
|f(y) - f(x^-)| \leq L(x - y), \quad \forall y, \quad 0 < x - y < \delta.
\]

Then \(\{S_n f(x)\}\) converges to \((f(x^+) + f(x^-))/2\) as \(n \to \infty\).

Here \(f(x^+)\) and \(f(x^-)\) stand for \(\lim_{y \to x^+} f(y)\) and \(\lim_{y \to x^-} f(y)\) respectively. We leave the proof of this theorem as an exercise.

A function \(f\) defined on \([a, b]\) is called to satisfy a **Lipschitz condition** if there exists an \(L\) such that

\[
|f(x) - f(y)| \leq L|x - y|, \quad \forall x, y \in [a, b].
\]

When \(f\) satisfies a Lipschitz condition, it is Lipschitz continuous everywhere. The Lipschitz condition is some kind of “uniformly Lipschitz” condition. Every continuously differentiable function on \([a, b]\) satisfies a Lipschitz condition. In fact, by the fundamental theorem of calculus, for \(x, y \in [a, b]\),

\[
|f(y) - f(x)| = \left| \int_x^y f'(t)\,dt \right| \leq M|y - x|,
\]

where \(M = \sup\{|f'(t)| : t \in [a, b]\}\). Similarly, every piecewise \(C^1\)-function satisfies a Lipschitz condition.
1.3. CONVERGENCE OF FOURIER SERIES

Now, we have a theorem on the uniform convergence of the Fourier series of a function to the function itself.

**Theorem 1.7.** Let \( f \) a \( 2\pi \)-periodic function satisfying a Lipschitz condition. Its Fourier series converges to \( f \) uniformly as \( n \to \infty \).

**Proof.** Observe that when \( f \) is Lipschitz continuous on \([-\pi, \pi]\), \( \delta_0 \) and \( \delta_1 \) can be chosen independent of \( x \) and (1.5), (1.6) hold uniformly in \( x \). In fact, \( \delta_0 \) only depends on \( L \), the constant appearing in the Lipschitz condition. Thus the theorem follows if \( n_0 \) in (1.7) can be chosen uniformly in \( x \). This is the content of the lemma below. We apply it by taking \( f(x, y) \) to be \( F_1(x, z) \) or \( F_2(x, z) \).

**Lemma 1.8.** Let \( f(x, y) \) be periodic in \( y \) and \( f \in C([-\pi, \pi] \times [-\pi, \pi]) \). For any fixed \( x \),

\[
c(n, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x, y) e^{-iny} \, dy \to 0
\]

uniformly in \( x \) as \( n \to \infty \).

**Proof.** We need to show that for every \( \varepsilon > 0 \), there exists some \( n_0 \) independent of \( x \) such that

\[
|c(n, x)| < \varepsilon, \quad \forall n \geq n_0.
\]

Observe that

\[
2\pi c(n, x) = \int_{-\pi}^{\pi} f(x, y) e^{-iny} \, dy \\
= \int_{-\pi}^{\pi} f\left(x, z + \frac{\pi}{n}\right) e^{-inz + \frac{\pi}{n}} \, dz \\
= -\int_{-\pi}^{\pi} f\left(x, z + \frac{\pi}{n}\right) e^{-inz} \, dz \quad \text{(f is 2\pi-periodic)}.
\]

We have

\[
c(n, x) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(f(x, y) - f\left(x, y + \frac{\pi}{n}\right)\right) e^{-iny} \, dy.
\]

As \( f \in C([-\pi, \pi] \times [-\pi, \pi]) \), it is uniformly continuous in \([-\pi, \pi] \times [-\pi, \pi] \). For \( \varepsilon > 0 \), there exists a \( \delta \) such that

\[
|f(x, y) - f(x', y')| < \varepsilon \quad \text{if} \quad |x - x'|, \ |y - y'| < \delta.
\]

We take \( n_0 \) so large that \( \pi/n_0 < \delta \). Then, using \( |e^{-iny}| = 1 \),

\[
|c(n, x)| \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} \left|f(x, y) - f\left(x, y + \frac{\pi}{n}\right)\right| \, dy \\
\leq \frac{\varepsilon}{4\pi} \int_{-\pi}^{\pi} \, dy = \frac{\varepsilon}{2} \\
< \varepsilon, \quad \forall n \geq n_0.
\]
Example 1.3. We return to the functions discussed in Examples 1.1 and 1.2. Indeed, $f_1$ is smooth except at $n\pi$. According to Theorem 1.5, the series
\[
2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx
\]
converges to $x$ for every $x \in (-\pi, \pi)$. On the other hand, we observed before that the series tend to 0 at $x = \pm \pi$. As $f_1(\pi_+)=\pi$ and $f(\pi_-)=\pi$, we have $f_1(\pi_+)+f(\pi_-)=0$, which is in consistency with Theorem 1.5. In the second example, $f_2$ is continuous, $2\pi$-periodic. By Theorem 1.7, its Fourier series
\[
\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx
\]
converges to $x^2$ uniformly on $[-\pi, \pi]$.

So far we have been working on the Fourier series of $2\pi$-periodic functions. It is clear that the same results apply to the Fourier series of $2T$-periodic functions for arbitrary positive $T$.

We have shown the convergence of the Fourier series under some additional regularity assumptions on the function. But the basic question remains, that is, is the Fourier series of a continuous, $2\pi$-periodic function converges to itself? It turns out the answer is negative. A not-so-explicit example can be found in Stein-Shakarchi and an explicit but complicated one was given by Fejér (see Zygmund “Trigonometric Series”). You may google for more. In fact, using the uniform boundedness principle in functional analysis, one can even show that “most” continuous functions have divergent Fourier series. The situation is very much like in the case of the real number system where transcendental numbers are uncountable while algebraic numbers are countable despite the fact that it is difficult to establish a specific number is transcendental.

We present another convergence result where is concerned with pointwise convergence. It replaces regularity by monotonicity in the function under consideration. Theorem 1.9 and Proposition 1.10 are for optional reading.

Theorem 1.9. Let $f$ be a $2\pi$-periodic function integrable on $[-\pi, \pi]$. Suppose that it is piecewise continuous and increasing near some point $x$. Its Fourier series converges to $(f(x^+) + f(x^-))/2$ at $x$.

Proof. In the following proof we will take $x = 0$ for simplicity. We first write, using the
evenness of $D_n$,

$$(S_n f)(0) = \int_{-\pi}^{\pi} D_n(z) f(z) \, dz
= \int_{0}^{\pi} (f(z) + f(-z)) D_n(z) \, dz.$$ 

So,

$$(S_n f)(0) - \frac{1}{2}(f(0^+) + f(0^-)) = \int_{0}^{\pi} (f(z) - f(0^+) + f(-z) - f(0^-)) D_n(z) \, dz.$$ 

We will show that

$$\hat{\pi}_0 \left| f(z) - f(0^+) \right| \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}} \, dz \rightarrow 0 \quad (1.8)$$

and

$$\hat{\pi}_0 \left| f(z) - f(0^-) \right| \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}} \, dz \rightarrow 0 \quad (1.9)$$

as $n \to \infty$. Indeed, for a small $h > 0$, we consider

$$\int_{0}^{h} (f(z) - f(0^+)) \sin \frac{2n+1}{2} z \sin \frac{z}{2} \, dz = \int_{0}^{h} (f(z) - f(0^+)) \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}} \, dz$$

Using L’Hospital’s rule,

$$\frac{1}{\sin \frac{z}{2}} - \frac{1}{\frac{z}{2}} = \frac{z - 2 \sin \frac{z}{2}}{z \sin \frac{z}{2}} \to 0 \quad \text{as} \quad z \to 0.$$ 

Therefore, for $\varepsilon > 0$, we can find $h_1$ such that

$$\int_{0}^{h_1} \left| f(z) - f(0^+) \right| \left| \frac{1}{\sin \frac{z}{2}} - \frac{1}{\frac{z}{2}} \right| \left| \frac{2n+1}{2} z \right| \, dz$$

where $h_1$ is independent of $n$. Next, by the second mean-value theorem for integral (see below),

$$\int_{0}^{h} (f(z) - f(0^+)) \frac{\sin \frac{2n+1}{2} z}{\frac{z}{2}} \, dz = (f(h) - f(0^+)) \int_{k}^{h} \frac{\sin \frac{2n+1}{2} z}{\frac{z}{2}} \, dz$$

for some $k \in (0, h)$. As

$$\left| \int_{k}^{h} \frac{\sin \frac{2n+1}{2} z}{\frac{z}{2}} \, dz \right| = \left| 2 \int_{k}^{\ell} \frac{\sin t}{t} \, dt \right|, \quad \ell = \frac{2n+1}{2}$$

we get

$$\leq 2 \left| \int_{0}^{\ell} \frac{\sin t}{t} \, dt \right| + 2 \left| \int_{0}^{\ell} \frac{\sin t}{t} \, dt \right|$$

$$\leq 4 \sup \left| \int_{0}^{T} \frac{\sin t}{t} \, dt \right| \equiv 4L,$$
and we can find \( h_2 \leq h_1 \) such that

\[
4L \left| f(h) - f(0^+) \right| < \frac{\varepsilon}{3}, \quad \forall 0 < h \leq h_2,
\]

we have

\[
\left| \int_0^{h_2} \left( f(z) - f(0^+) \right) \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}} \, dz \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\]

Now, by Riemann-Lebesgue lemma, there exists some \( n_0 \) such that

\[
\left| \int_{h_2}^{n} \left( f(z) - f(0^+) \right) \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}} \, dz \right| < \frac{\varepsilon}{3}, \quad \forall n \geq n_0.
\]

Putting things together,

\[
\left| \int_0^{n} \left( f(z) - f(0^+) \right) \frac{\sin \frac{2n+1}{2} z}{\sin \frac{z}{2}} \, dz \right| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \quad \forall n \geq n_0.
\]

We have shown that (1.8) holds. To prove (1.9), it suffices to apply (1.8) to the function \( g(z) = f(-z) \).

\[
\text{Proposition 1.10 (Second Mean-Value Theorem).} \quad \text{Let } f \in R[a, b] \text{ and } g \text{ be monotone on } [a, b] \text{ and satisfy } g(a) = 0. \text{ There exists some } c \in (a, b) \text{ such that}
\]

\[
\int_a^b f(x)g(x) \, dx = g(b) \int_c^b f(x) \, dx.
\]

\[
\text{Proof.} \quad \text{Without loss of generality, we assume } g \text{ is increasing. Let}
\]

\[
a = x_0 < x_1 < \cdots < x_n = b
\]

be a partition \( P \) on \([a, b] \).

\[
\int_a^b f g = \sum_{j=1}^n g(x_j) \int_{x_{j-1}}^{x_j} f + \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)(g(x) - g(x_j)) \, dy.
\]

In case \( \|P\| \to 0 \), it is not hard to show that the second integral tends to zero, so

\[
\int_a^b f g = \lim_{\|P\| \to 0} \sum_{j=1}^n g(x_j) \int_{x_{j-1}}^{x_j} f.
\]

Letting \( F(x) = \int_x^b f \) and using \( F(x_a) = F(b) = 0 \), we have

\[
\sum_{j=1}^n g(x_j) \int_{x_{j-1}}^{x_j} f = \sum_{j=1}^n g(x_j)(F(x_j) - F(x_{j-1})) = g(x_1)F(x_0) + \sum_{j=1}^{n-1} (g(x_{j+1}) - g(x_j))F(x_j).
\]
Let $m = \inf_{[a,b]} F$ and $M = \sup_{[a,b]} F$. As $g$ is increasing,

$$mg(b) \leq g(x_1)F(x_0) + \sum_{j=1}^{n-1} (g(x_{j+1}) - g(x_j))F(x_j) \leq Mg(b).$$

Letting $\|P\| \to 0$, we conclude that

$$mg(b) \leq \int_a^b fg \leq Mg(b).$$

As $c \mapsto \int_c^b f$ is continuous and bounded between $m$ and $M$, there is some $c$ such that

$$\frac{1}{g(b)} \int_a^b fg = \int_c^b f.$$

\[\square\]

### 1.4 Weierstrass Approximation Theorem

As an application of Theorem 1.7, we prove a theorem of Weierstrass concerning the approximation of continuous functions by polynomials. First we consider how to approximate a continuous function by continuous, piecewise linear functions. A continuous function defined on $[a,b]$ is **piecewise linear** if there exists a partition $a = a_0 < a_1 < \cdots < a_n = b$ such that $f$ is linear on each subinterval $[a_j, a_{j+1}]$.

**Proposition 1.11.** Let $f$ be a continuous function on $[a,b]$. For every $\varepsilon > 0$, there exists a continuous, piecewise linear function $g$ such that $\|f - g\|_\infty < \varepsilon$.

Recall that $\|f - g\|_\infty = \sup\{ |f(x) - g(x)| : x \in [a,b] \}$.

**Proof.** As $f$ is uniformly continuous on $[a,b]$, for every $\varepsilon > 0$, there exists some $\delta$ such that $|f(x) - f(y)| < \varepsilon/2$ for $x, y \in [a, b], |x - y| < \delta$. We partition $[a,b]$ into subintervals $I_j = [a_j, a_{j+1}]$ whose length is less than $\delta$ and define $g$ to be the piecewise linear function satisfying $g(a_j) = f(a_j)$ for all $j$. For $x \in [a_j, a_{j+1}]$, $g$ is given by

$$g(x) = \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j} (x - a_j) + f(a_j).$$

We have

$$|f(x) - g(x)| = |f(x) - \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j} (x - a_j) + f(a_j)|$$

$$\leq |f(x) - f(a_j)| + \frac{|f(a_{j+1}) - f(a_j)|}{a_{j+1} - a_j} |x - a_j|$$

$$\leq |f(x) - f(a_j)| + |f(a_{j+1}) - f(a_j)|$$

$$< \varepsilon,$$
Next we study how to approximate a continuous function by trigonometric polynomials (or, equivalently, finite Fourier series).

**Proposition 1.12.** Let \( f \) be a continuous function on \([0, \pi]\). For \( \varepsilon > 0 \), there exists a trigonometric polynomial \( h \) such that \( \|f - h\|_{\infty} < \varepsilon \).

**Proof.** First we extend \( f \) to \([-\pi, \pi]\) by setting \( f(x) = f(-x) \) (using the same notation) to obtain a continuous function on \([-\pi, \pi]\) with \( f(-\pi) = f(\pi) \). By the previous proposition, we can find a continuous, piecewise linear function \( g \) such that \( \|f - g\|_{\infty} < \varepsilon/2 \). Since \( g(-\pi) = f(-\pi) = f(\pi) = g(\pi) \), \( g \) can be extended as a Lipschitz continuous, \( 2\pi \)-periodic function. By Theorem 1.7, there exists some \( N \) such that \( \|g - S_N g\|_{\infty} < \varepsilon/2 \). Therefore, \( \|f - S_N g\|_{\infty} \leq \|f - g\|_{\infty} + \|g - S_N g\|_{\infty} < \varepsilon/2 + \varepsilon/2 = \varepsilon \). The proposition follows after noting that every finite Fourier series is a trigonometric polynomial (see Exercise 1).

**Theorem 1.13 (Weierstrass Approximation Theorem).** Let \( f \in C[a, b] \). Given \( \varepsilon > 0 \), there exists a polynomial \( p \) such that \( \|f - p\|_{\infty} < \varepsilon \).

**Proof.** Consider \([a, b] = [0, \pi]\) first. Extend \( f \) to \([-\pi, \pi]\) as before and, for \( \varepsilon > 0 \), fix a trigonometric polynomial \( h \) such that \( \|f - h\|_{\infty} < \varepsilon/2 \). This is possible due to the previous proposition. Now, we express \( h \) as a finite Fourier series \( a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \). Using the fact that
\[
\cos \theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}, \quad \text{and} \quad \sin \theta = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \theta^{2n-1}}{(2n-1)!},
\]
where the convergence is uniform on \([-\pi, \pi]\), each \( \cos nx \) and \( \sin nx \), \( n = 1, \ldots, N \), can be approximated by polynomials. Putting all these polynomials together we obtain a polynomial \( p(x) \) satisfying \( \|h - p\|_{\infty} < \varepsilon/2 \). It follows that \( \|f - p\|_{\infty} \leq \|f - h\|_{\infty} + \|h - p\|_{\infty} < \varepsilon/2 + \varepsilon/2 = \varepsilon \).

When \( f \) is continuous on \([a, b]\), the function \( \varphi(t) = f\left(\frac{b-a}{\pi}t + a\right) \) is continuous on \([0, \pi]\). From the last paragraph, we can find a polynomial \( p(t) \) such that \( \|\varphi - p\|_{\infty} < \varepsilon \) on \([0, \pi]\). But then the polynomial \( q(x) = p\left(\frac{\pi}{b-a}(x - a)\right) \) satisfies \( \|f - q\|_{\infty} = \|\varphi - p\|_{\infty} < \varepsilon \) on \([a, b]\).

### 1.5 Mean Convergence of Fourier Series

In Section 2 we studied the uniform convergence of Fourier series. Since the limit of a uniformly convergent series of continuous functions is again continuous, we do not expect results like Theorem 1.6 applies to functions with jumps. In this section we will measure the distance between functions by a norm weaker than the uniform norm. Under the new
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$L^2$-distance, you will see that every integrable function is equal to its Fourier expansion almost everywhere.

Recall that there is an inner product defined on the $n$-dimensional Euclidean space called the Euclidean metric

$$\langle x, y \rangle_2 = \sum_{j=1}^{n} x_j y_j, \quad x, y \in \mathbb{R}^n.$$  

With this inner product, one can define the concept of orthogonality and angle between two vectors. Likewise, we can also introduce a similar product on the space of integrable functions. Specifically, for $f, g \in R[-\pi, \pi]$, the $L^2$-product is given by

$$\langle f, g \rangle_2 = \int_{-\pi}^{\pi} f(x)g(x) \, dx.$$  

The $L^2$-product behaves like the Euclidean metric on $\mathbb{R}^n$ except at one aspect, namely, the condition $\langle f, f \rangle_2 = 0$ does not imply $f \equiv 0$. This is easy to see. In fact, when $f$ is equal to zero except at finitely many points, then $\langle f, f \rangle_2 = 0$. From Appendix II, $\langle f, f \rangle_2 = 0$ if and only if $f$ is equal to zero except on a set of measure zero. This minor difference with the Euclidean inner product will not affect our discussion much. Parallel to the Euclidean case, we define the $L^2$-norm of an integrable function $f$ to be

$$\|f\|_2 = \sqrt{\langle f, f \rangle_2},$$  

and the $L^2$-distance between two integrable functions $f$ and $g$ by $\|f - g\|_2$. (When $f, g$ are complex-valued, one should define the $L^2$-product to be

$$\langle f, g \rangle_2 = \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx,$$

so that $\langle f, f \rangle_2 \geq 0$. We will be restricted to real functions in this section.) One can verify that the triangle inequality

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2$$

holds. We can also talk about $f_n \to f$ in $L^2$-sense, i.e., $\|f_n - f\|_2 \to 0$, or equivalently,

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} |f - f_n|^2 = 0, \quad \text{as } n \to \infty.$$  

This is a convergence in an average sense. It is not hard to see that when $\{f_n\}$ tends to $f$ uniformly, $\{f_n\}$ must tend to $f$ in $L^2$-sense. A moment’s reflection will show that the converse is not always true. Hence convergence in $L^2$-sense is weaker than uniform convergence. We will discuss various metrics and norms in Chapter 2.

Our aim in this section is to show that the Fourier series of every integrable function converges to the function in the $L^2$-sense.
Just like the canonical basis \( \{ e_1, \ldots, e_n \} \) in \( \mathbb{R}^n \), the functions
\[
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \right\}_{n=1}^{\infty}
\]
forms an “orthonormal basis” in \( R[-\pi, \pi] \), see Section 1.1. In the following we denote by
\[
E_n = \bigg\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx \bigg\rangle_{j=1}^{n}
\]
the \((2n+1)\)-dimensional vector space spanned by the first \(2n+1\) trigonometric functions.

We start with a general result. Let \( \{ \phi_n \}_{n=1}^{\infty} \) be an orthonormal set (or orthonormal family) in \( R[-\pi, \pi] \), i.e.,
\[
\int_\pi^{\pi} \phi_n \phi_m = \delta_{nm}, \quad \forall n, m \geq 1.
\]
Let
\[
S_n = \langle \phi_1, \ldots, \phi_n \rangle
\]
be the \(n\)-dimensional subspace spanned by \( \phi_1, \ldots, \phi_n \). For a general \( f \in R[-\pi, \pi] \), we consider the minimization problem
\[
\inf \{ \| f - g \|_2 : g \in S_n \}.
\]
From a geometric point of view, this infimum gives the \(L^2\)-distance from \( f \) to the finite dimensional subspace \( S_n \).

**Proposition 1.14.** The unique minimizer of (1.10) is attained at the function \( g = \sum_{j=1}^{n} \alpha_j \phi_j, \) where \( \alpha_j = \langle f, \phi_j \rangle \).

**Proof.** To minimize \( \| f - g \|_2 \) is the same as to minimize \( \| f - g \|_2^2 \). Every \( g \) in \( S_n \) can be written as \( g = \sum_{j=1}^{n} \beta_j \phi_j, \beta_j \in \mathbb{R} \). Let
\[
\Phi(\beta_1, \ldots, \beta_n) = \int_{-\pi}^{\pi} |f - g|^2
\]
\[
= \int_{-\pi}^{\pi} \left( f - \sum_{j=1}^{n} \beta_j \phi_j \right)^2
\]
\[
= \int_{-\pi}^{\pi} f^2 - 2 \sum_{j=1}^{n} \beta_j \alpha_j + \sum_{j=1}^{n} \beta_j^2.
\]
be a function from \( \mathbb{R}^n \) to \( \mathbb{R} \). We use elementary inequality \( 2ab \leq a^2 + b^2 \) in a tricky way,
\[
2 \sum_{j=1}^{n} \beta_j \alpha_j = 2 \sum_{j=1}^{n} \frac{\beta_j}{\sqrt{2}} \alpha_j
\]
\[
\leq \sum_{j=1}^{n} \frac{\beta_j^2}{2} + 2 \sum_{j=1}^{n} \alpha_j^2.
\]
Therefore,
\[ \Phi(\beta) \geq \int_{-\pi}^{\pi} f^2 - \frac{1}{2} \sum_{j=1}^{n} \beta_j^2 - 2 \sum_{j=1}^{n} \alpha_j^2 + \sum_{j=1}^{n} \beta_j^2 \]
\[ = \int_{-\pi}^{\pi} f^2 - 2 \sum_{j=1}^{n} \alpha_j^2 + \frac{1}{2} |\beta|^2 \]
\[ \to \infty, \]
as \( |\beta| \to \infty \). It implies that \( \Phi \) must attain a minimum at some finite point \( \gamma \). At this point \( \gamma \), \( \nabla \Phi(\gamma) = (0, \ldots, 0) \). We compute
\[ \frac{\partial \Phi}{\partial \beta_i} = -2\alpha_i + 2\beta_i. \]
Hence, \( \beta = \alpha \). As there is only one critical point, it must be the minimum of \( \Phi \).

Given an orthonormal set \( \{\phi_n\}_{n=1}^{\infty} \), one may define the “Fourier series” of an \( L^2 \)-function \( f \) with respect to the orthonormal set \( \{\phi_n\} \) to be the series \( \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n \) and set \( P_n f = \sum_{k=1}^{n} \langle f, \phi_k \rangle \phi_k \). This proposition asserts that the distance between \( f \) and \( S_n \) is realized at \( \|f - S_n f\|_2 \). The function \( P_n f \) is sometimes called the orthogonal projection of \( f \) on \( S_n \). As a special case, taking \( \{\phi_n\} = \{1/\sqrt{2\pi}, \cos nx/\sqrt{\pi}, \sin nx/\sqrt{\pi}\} \) and \( S_{2n+1} = E_n \), a direct computation shows that \( P_{2n+1} f = S_n f \), where \( S_n f \) is the \( n \)-th partial sum of the Fourier series of \( f \). Thus we can rewrite Proposition 1.14 in this special case as

**Corollary 1.15.** For \( f \in R_{2\pi} \), for each \( n \geq 1 \),
\[ \|f - S_n f\|_2 \leq \|f - g\|_2 \]
for all \( g \) of the form
\[ g = c_0 + \sum_{k=1}^{n} (c_j \cos kx + d_j \sin kx), \quad c_0, c_k, d_k \in \mathbb{R}. \]

Here is the main result of this section.

**Theorem 1.16.** For every \( f \in R_{2\pi} \),
\[ \lim_{n \to \infty} \|S_n f - f\|_2 = 0. \]

**Proof.** Let \( f \in R[-\pi, \pi] \). For \( \varepsilon > 0 \), we can find a \( 2\pi \)-periodic, Lipschitz continuous function \( g \) such that
\[ \|f - g\|_2 < \frac{\varepsilon}{2}. \]
Indeed, $g$ can be obtained by first approximating $f$ by a step function and then modifying the step function at its jumps. By Theorem 1.7, we can fix an $N$ so that in sup-norm

$$\|g - S_N g\|_\infty < \frac{\varepsilon}{\sqrt{2\pi}}.$$ 

Thus

$$\|g - S_N g\|_2 = \sqrt{\int_{-\pi}^{\pi} (g - S_N g)^2} \leq \|g - S_N g\|_\infty \sqrt{2\pi} < \frac{\varepsilon}{2}.$$ 

It follows from Corollary 1.15 that

$$\|f - S_N f\|_2 \leq \|f - S_N g\|_2 \leq \|f - g\|_2 + \|g - S_N g\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

As $S_N \subset S_n$ for all $n \geq N$, by Corollary 1.15 again, we have

$$\|f - S_n f\|_2 \leq \|f - S_N f\|_2 < \varepsilon.$$ 

We have the following result concerning the uniqueness of the Fourier expansion.

**Corollary 1.17.** (a) Suppose that $f_1$ and $f_2$ in $R_{2\pi}$ have the same Fourier series. Then $f_1$ and $f_2$ are equal almost everywhere.

(b) Suppose that $f_1$ and $f_2$ in $C_{2\pi}$ have the same Fourier series. Then $f_1 \equiv f_2$.

**Proof.** Let $f = f_2 - f_1$. The Fourier coefficients of $f$ all vanish, hence $S_n f = 0$, for all $n$. By Theorem 1.16, $\|f\|_2 = 0$. From Appendix II we know that $f^2$, hence $f$, must vanish almost everywhere. In other words, $f_2$ is equal to $f_1$ almost everywhere. (a) holds. To prove (b), letting $f$ be continuous and assuming $f(x)$ is not equal to zero at some $x$, by continuity it is non-zero for all points near $x$. Hence we may assume $x$ belongs to $(-\pi, \pi)$ and $|f(y)| > 0$ for all $y \in (x - \delta, x + \delta)$ for some $\delta > 0$. But then $\|f\|_2$ would be greater or equal to the integral of $|f|$ over $(x - \delta, x + \delta)$, which is positive. This contradiction shows that $f \equiv 0$. 

Another interesting consequence of Theorem 1.16 is the Parseval’s identity. In fact, this identity is equivalent to Theorem 1.16.

**Corollary 1.18 (Parseval’s Identity).** For every $f \in R_{2\pi}$,

$$\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

where $a_n$ and $b_n$ are the Fourier coefficients of $f$. 

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Proof. Making use of the relations such as \( \langle f, \cos nx/\sqrt{\pi} \rangle = \sqrt{\pi}a_n, n \geq 1 \), we have \( \langle f, S_n f \rangle = \| S_n f \|_2^2 = 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \). By Theorem 1.15,

\[
0 = \lim_{n \to \infty} \| f - S_n f \|_2^2 = \lim_{n \to \infty} \left( \| f \|_2^2 - 2\langle f, S_n f \rangle + \| S_n f \|_2^2 \right) = \| f \|_2^2 - 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \].
\]

The norm of \( f \) can be regarded as the length of the “vector” \( f \). Parseval’s Identity shows that the square of the length of \( f \) is equal to the sum of the length of the orthogonal projection of \( f \) onto each one-dimensional subspace spanned by the sine and cosine functions. This is an infinite dimensional version of the ancient Pythagoras theorem. It is curious to see what really comes out when you plug in some specific functions. For instance, we take \( f(x) = x \) and recall that its Fourier series is given by \( \sum 2(-1)^{n+1}/n \sin nx \). Therefore, \( a_n = 0, n \geq 0 \) and \( b_n = 2(-1)^{n+1}/n \) and Parseval’s identity yields Euler’s summation formula

\[
\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .
\]

You could find more interesting identities by applying the same idea to other functions.

The following result will be used in the next section.

**Corollary 1.19 (Wirtinger’s Inequality).** For every \( f \in R_{2\pi} \) satisfying \( f' \in R_{2\pi} \),

\[
\int_{-\pi}^{\pi} (f(x) - \overline{f})^2 \, dx \leq \int_{-\pi}^{\pi} f'^2(x) \, dx ,
\]

and equality holds if and only if \( f(x) = a_0 + a_1 \cos x + b_1 \sin x \).

Here

\[
\overline{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f
\]

is the average or mean of \( f \) over \([ -\pi, \pi ]\).

**Proof.** Noting that

\[
\overline{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cos 0 \, dx = a_0 ,
\]

\[
f(x) - \overline{f} = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) ,
\]

and equality holds if and only if \( f(x) = a_0 + a_1 \cos x + b_1 \sin x \).
by Theorem 1.6. By Parseval’s identity,

\[
\int_{-\pi}^{\pi} (f(x) - \overline{f})^2\,dx = \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2),
\]

and

\[
\int_{-\pi}^{\pi} (f(x) - \overline{f})^2\,dx = \int_{-\pi}^{\pi} f'(x)^2\,dx = \pi \sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2).
\]

Therefore, we have

\[
\int_{-\pi}^{\pi} f'(x)^2\,dx - \int_{-\pi}^{\pi} (f(x) - \overline{f})^2\,dx = \pi \sum_{n=1}^{\infty} (n^2 - 1)(a_n^2 + b_n^2),
\]

and the result follows.

This inequality is also known as Poincaré’s Inequality.

1.6 The Isoperimetric Problem

The classical isoperimetric problem known to the ancient Greeks asserts that only the circle maximizes the enclosed area among all simple, closed curves of the same perimeter. In this section we will present a proof of this inequality by Fourier series. To formulate this geometric problem in analytic terms, we need to recall some facts from advanced calculus.

Indeed, a parametric \( C^1 \)-curve is a map \( \gamma \) from some interval \([a,b]\) to \( \mathbb{R}^2 \) such that \( x \) and \( y \) belong to \( C^1[a,b] \) where \( \gamma(t) = (x(t), y(t)) \) and \( x'(t)^2 + y'(t)^2 > 0 \) for all \( t \in [a,b] \).

In the following a curve is always refereed to a parametric \( C^1 \)-curve. For such a curve, its length is defined to be

\[
L[\gamma] = \int_{a}^{b} \sqrt{x'(t)^2 + y'(t)^2}\,dt, \quad \gamma = (x, y).
\]

A curve is closed if \( \gamma(a) = \gamma(b) \) and simple if \( \gamma(t) \neq \gamma(s), \forall t \neq s \) in \([a,b] \). The length of a closed curve is called the perimeter of the curve.

When a closed, simple curve is given, the area it encloses is also fixed. Hence one should be able to express this enclosed area by a formula involving \( \gamma \) only. Indeed, this can be accomplished by the Green’s theorem. Recalling that the Green’s theorem states that for every pair of \( C^1 \)-functions \( P \) and \( Q \) defined on the curve \( \gamma \) and the region enclosed by the curve, we have

\[
\int_{\gamma} P\,dx + Q\,dy = \iint_{D} \left( \frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y) \right)\,dxdy,
\]
where the left hand side is the line integral along $\gamma$ and $D$ is the domain enclosed by $\gamma$ (see Fritzpatrick, p.543). Taking $P \equiv 0$ and $Q = x$, we obtain

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \iint_D 1 = \text{area of } D,$$

so

$$A[\gamma] = \iint_D 1dxdy = \int xdy = \int_a^b x(t)y'(t)dt .$$

The classical isoperimetric problem is: Among all simple, closed curves with a fixed perimeter, find the one whose enclosed area is the largest. We will see that the circle is the only solution to this problem.

To proceed further, let us recall the concept of reparametrization. Indeed, a curve $\gamma_1$ on $[a_1, b_1]$ is called a reparametrization of the curve $\gamma$ on $[a, b]$ if there exists a $C^1$-map $\xi$ from $[a_1, b_1]$ to $[a, b]$ with non-vanishing derivative so that $\gamma_1(t) = \gamma(\xi(t)), \forall t \in [a_1, b_1]$. It is known that the length remains invariant under reparametrizations.

Another useful concept is the parametrization by arc-length. A curve $\gamma = (x, y)$ on $[a, b]$ is called in arc-length parametrization if $x'^2(t) + y'^2(t) = 1, \forall t \in [a, b]$. We know that every curve can be reparametrized in arc-length parametrization. Let $\gamma(t) = (x(t), y(t)), t \in [a, b]$, be a parametrization of a curve. We define a function $\varphi$ by setting

$$\varphi(\tau) = \int_a^\tau \left( x'^2(t) + y'^2(t) \right)^{1/2} dt,$$

it is readily checked that $\varphi$ is a $C^1$-map from $[a, b]$ to $[0, L]$ with positive derivative, and $\gamma_1(s) = \gamma(\xi(s)), \xi = \varphi^{-1}$, is an arc-length reparametrization of $\gamma$ on $[0, L]$ where $L$ is the length of $\gamma$.

We now apply the Wirtinger’s Inequality to give a proof of the classical isoperimetric problem.

Let $\gamma : [a, b] \to \mathbb{R}^2$ be a closed, simple $C^1$-curve bounding a region $D$. Without loss of generality we may assume that it is parametrized by arc-length. Assuming the perimeter of $\gamma$ is equal to $2\pi$, we want to find the region that encloses the maximal area. The perimeter is given by

$$L[\gamma] = \int_0^{2\pi} \sqrt{x'^2(s) + y'^2(s)}ds = 2\pi ,$$

and the area is given by

$$A[\gamma] = \int_0^{2\pi} x(s)y'(s)ds .$$
Extending $\gamma_1$ and $\gamma_2$ as $2\pi$-periodic functions, we compute

\[
2A[\gamma] = \int_{-\pi}^{\pi} 2x(s)y'(s)ds \\
= \int_{-\pi}^{\pi} 2(x(s) - x) y'(s)ds \\
\leq \int_{-\pi}^{\pi} (x(s) - x)^2 ds + \int_{-\pi}^{\pi} y'^2(s)ds \quad \text{(by } 2ab \leq a^2 + b^2) \\
\leq \int_{-\pi}^{\pi} x'^2(s)ds + \int_{-\pi}^{\pi} y'^2(s)ds \quad \text{(by Wirtinger's Inequality)} \\
= \int_{-\pi}^{\pi} (x'^2(s) + y'^2(s))ds \\
= 2\pi, \quad \text{(use } x'^2(s) + y'^2(s) = 1) \\
\]

whence $A[\gamma] \leq \pi$. We have shown that the enclosed area of a simple, closed $C^1$-curve with perimeter $2\pi$ cannot exceed $\pi$. As $\pi$ is the area of the unit circle, the unit circle solves the isoperimetric problem.

Now the uniqueness case. We need to examine the equality signs in our derivation. We observe that the second equality holds if and only if $a_n = b_n = 0$ for all $n \geq 2$ in the Fourier series of $x(s)$. So, $x(s) = a_0 + a_1 \cos s + b_1 \sin s$, or

\[x(s) = a_0 + r \cos(s - x_0),\]

where

\[r = \sqrt{a_1^2 + b_1^2}, \quad \cos x_0 = \frac{a_1}{r}.
\]

(Note that $(a_1, b_1) \neq (0, 0)$. For if $a_1 = b_1 = 0$, $x(s)$ is constant and $x'^2 + y'^2 = 1$ implies $y'^2(s) = \pm s + b$, and $y$ can never be periodic.) Now we determine $y$. From the above calculation, when the first equality holds ($2ab = a^2 + b^2$ means $a - b = 0$),

\[x - \bar{x} - y' = 0.\]

So $y'(s) = x(s) - \bar{x} = r \cos(s - x_0)$, which gives

\[y(s) = r \sin(s - x_0) + c_0, \quad c_0 \text{ constant.}\]

It follows that $\gamma$ describes a circle of radius $r$ centered at $(a_0, c_0)$. Using the fact that the perimeter is $2\pi$, we conclude that $r = 1$, so the maximum must be a unit circle.

Summarizing, we have the following solution to the classical isoperimetric problem.

**Theorem 1.20.** Among all closed, simple $C^1$-curves of the same perimeter, only the circle encloses the largest area.
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The same proof also produces a dual statement, namely, among all regions which enclose the same area, only the circle has the shortest perimeter.

Appendix I Series of Functions

This appendix serves to refresh your memory after the long, free summer.

A sequence is a mapping $\varphi$ from $\mathbb{N}$ to $\mathbb{R}$. For $\varphi(n) = a_n$, we usually denote the sequence by $\{a_n\}$ rather than $\varphi$. This is a convention. We say the sequence is convergent if there exists a real number $a$ satisfying, for every $\varepsilon > 0$, there exists some $n_0$ such that $|a_n - a| < \varepsilon$ for all $n, n \geq n_0$. When this happens, we write $a = \lim_{n \to \infty} a_n$.

An (infinite) series is always associated with a sequence. Given a sequence $\{x_n\}$, set $s_n = \sum_{k=1}^{n} x_k$ and form another sequence $\{s_n\}$. This sequence is the infinite series associated to $\{x_n\}$ and is usually denoted by $\sum_{k=1}^{\infty} x_k$. The sequence $\{s_n\}$ is also called the sequence of $n$-th partial sums of the infinite series. By definition, the infinite series is convergent if $\{s_n\}$ is convergent. When this happens, we denote the limit of $\{s_n\}$ by $\sum_{k=1}^{\infty} x_k$, in other words, we have

$$\lim_{n \to \infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k.$$ 

So the notation $\sum_{k=1}^{\infty} x_k$ has two meanings, first, it is the notation for an infinite series and, second, the limit of its partial sums (whenever it exists).

When the target $\mathbb{R}$ is replaced by $\mathbb{C}$, we obtain a sequence or a series of complex numbers, and the above definitions apply to them after replacing the absolute value by the complex absolute value or modulus.

Let $\{f_n\}$ be a sequence of real- or complex-valued functions defined on some non-empty $E$ on $\mathbb{R}$. It is called convergent pointwisely to some function $f$ defined on the same $E$ if for every $x \in E$, $\{f_n(x)\}$ converges to $f(x)$ as $n \to \infty$. Keep in mind that $\{f_n(x)\}$ is a sequence of real or complex numbers, so its convergence has a valid meaning. A more important concept is the uniform convergence. The sequence $\{f_n\}$ is uniformly convergent to $f$ if, for every $\varepsilon > 0$, there exists some $n_0$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq n_0$. Equivalently, uniform convergence holds if, for every $\varepsilon > 0$, there exists some $n_1$ such that $\|f_n - f\|_\infty < \varepsilon$ for all $n \geq n_1$. Here $\|f\|_\infty$ denotes the sup-norm of $f$ on $E$.

An (infinite) series of functions is the infinite series given by $\sum_{k=1}^{\infty} f_k(x)$ where $f_k$ are defined on $E$. Its convergence and uniform convergence can be defined via its partial sums $s_n(x) = \sum_{k=1}^{n} f_k(x)$ as in the case of sequences of numbers.

Among several criteria for uniform convergence, the following test is very useful.
Weierstrass M-Test. Let \( \{f_k\} \) be a sequence of functions defined on some \( E \subset \mathbb{R} \). Suppose that there exists a sequence of non-negative numbers, \( \{x_k\} \), such that

1. \(|f_k(x)| \leq x_k\) for all \( k \geq 1 \), and
2. \( \sum_{k=1}^{\infty} x_k \) is convergent.

Then \( \sum_{k=1}^{\infty} f_k \) converges uniformly and absolutely on \( E \).

Appendix II  Sets of Measure Zero

Let \( E \) be a subset of \( \mathbb{R} \). It is called of measure zero, or sometimes called a null set, if for every \( \varepsilon > 0 \), there exists a (finite or infinite) sequence of intervals \( \{I_k\} \) satisfying (1) \( E \subset \bigcup_{k=1}^{\infty} I_k \) and (2) \( \sum_{k=1}^{\infty} |I_k| < \varepsilon \). (When the intervals are finite, the upper limit of the summation should be changed accordingly.) Here \( I_k \) could be an open, closed or any other interval and its length \(|I_k|\) is defined to be the \( b-a \) where \( a \leq b \) are the endpoints of \( I_k \).

The empty set is a set of measure zero from this definition. Every finite set is also null. For, let \( E = \{x_1, \cdots, x_N\} \) be the set. For \( \varepsilon > 0 \), the intervals \( I_k = (x_1 - \varepsilon/(4N), x_k + \varepsilon/(4N)) \) clearly satisfy (1) and (2) in the definition.

Next we claim that every countable set is also of measure zero. Let \( E = \{x_1, x_2, \cdots\} \) be a countable set. We choose \( I_k = \left(x_k - \frac{\varepsilon}{2k+2}, x_k + \frac{\varepsilon}{2k+2}\right) \).

Clearly, \( E \subset \bigcup_{k=1}^{\infty} I_k \). On the other hand,

\[
\sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} \frac{\varepsilon}{2k+1} = \frac{\varepsilon}{2} < \varepsilon.
\]

We conclude that every countable set is a null set.

There are uncountable sets of measure zero. For instance, the Cantor set which plays an important role in analysis, is of measure zero. Here we will not go into this.

The same trick in the above proof can be applied to the following situation.

**Proposition A.1.** The union of countably many null sets is a null set.
1.6. THE ISOPERIMETRIC PROBLEM

Proof. Let $E_k, k \geq 1$, be sets of measure zero. For $\varepsilon > 0$, there are intervals satisfying \( \{I^k_j\}, E_k \subset \cup_j I^k_j \), and \( \sum_j |I^k_j| < \varepsilon/2^k \). It follows that $E \equiv \cup_k E_k \subset \cup_j \cup_k I^k_j$, and

$$\sum_k \sum_j |I^k_j| < \sum_k \frac{\varepsilon}{2^k} = \varepsilon.$$  

The concept of a null set comes up naturally in the theory of Riemann integration. A theorem of Lebesgue asserts that a bounded function is Riemann integrable if and only if its discontinuity set is null. (This result can be found in an appendix of Bartle-Sherbert and also in my 2060 notes. It will be proved again in Real Analysis. Presently you may simply take it for granted.) Let us prove the following result.

**Proposition A.2.** Let $f$ be a non-negative integrable function on $[a, b]$. Then $\int_a^b f = 0$ if and only if $f$ is equal to 0 except on a null set. Consequently, two integrable functions $f, g$ satisfying

$$\int_a^b |f - g| = 0,$$

if and only if $f$ is equal to $g$ except on a null set.

Proof. We set, for each $k \geq 1$, $A_k = \{x \in [a, b] : f(x) > 1/k\}$. It is clear that

$$\{x : f(x) > 0\} = \bigcup_{k=1}^\infty A_k.$$

By Proposition A.1., it suffices to show that each $A_k$ is null. Thus let us consider $A_{k_0}$ for a fixed $k_0$. Recall from the definition of the Riemann integral, for every $\varepsilon > 0$, there exists a partition $a = x_1 < x_2 < \cdots < x_n = b$ such that

$$0 \leq \sum_{k=1}^{n-1} f(z_k)|I_k| = \left| \sum_{k=1}^{n-1} f(z_k)|I_k| - \int_a^b f \right| < \frac{\varepsilon}{k_0},$$

where $I_k = [x_k, x_{k+1}]$ and $z_k$ is an arbitrary tag in $[x_j, x_{j+1}]$. Let \( \{k_1, \ldots, k_m\} \) be the index set for which $I_{k_j}$ contains a point $z_{k_j}$ from $A_{k_0}$. Choosing the tag point to be $z_{k_j}$, we have $f(z_{k_j}) = 1/k_0$. Therefore,

$$\frac{1}{k_0} \sum_{k_j} |I_{k_j}| = \sum_{k_j} f(z_{k_j})|I_{k_j}| \leq \sum_{k=1}^{n-1} f(z_k)|I_k| < \frac{\varepsilon}{k_0},$$

so

$$\sum_{k_j} |I_{k_j}| < \varepsilon.$$

We have shown that $A_{k_0}$ is of measure zero.
Conversely, let $D$ be the set consisting of all discontinuity points of $f$. We claim that $A \subset D$. Assuming this, the aforementioned Lebesgue’s theorem tells us that $D$ is of measure zero, so is $A$. To prove the claim, let us assume that there is a some $x_0 \in A$ at which $f$ is continuous. We have $f(x_0) > 0$. Without loss of generality we may also assume $x_0$ is not one of the endpoints of the interval. By the definition of continuity, there exists some small $\delta > 0$ such that $|f(x) - f(x_0)| < f(x_0)/2$ for $x \in (x_0 - \delta, x_0 + \delta)$. Consequently, $f(x) \geq f(x_0) - f(x_0)/2 = f(x_0)/2$ for $x$ in this interval. We have

\[
0 = \int_a^b f = \int_a^{x_0-\delta} f + \int_{x_0-\delta}^{x_0+\delta} f + \int_{x_0+\delta}^b f \\
\geq \int_{x_0-\delta}^{x_0+\delta} f \\
\geq \frac{f(x_0)}{2} \times 2\delta > 0,
\]

contradiction holds. \hfill \Box

A property holds almost everywhere if it holds except on a null set. For instance, this proposition asserts that the integral of a non-negative function is equal to zero if and only if it vanishes almost everywhere.

**Comments on Chapter 1.** Historically, the relation (1.2) comes from a study on the one-dimensional wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

where $u(x, t)$ denote the displacement of a string at the position-time $(x, t)$. Around 1750, D’Alembert and Euler found that a general solution of this equation is given by

\[
f(x - ct) + g(x + ct)
\]

where $f$ and $g$ are two arbitrary twice differentiable functions. However, D. Bernoulli found that the solution could be represented by a trigonometric series. These two different ways of representing the solutions led to a dispute among the mathematicians at that time, and it was not settled until Fourier gave many convincing examples of representing functions by trigonometric series in 1822. His motivation came from heat conduction. After that, trigonometric series have been studied extensively and people call it Fourier series in honor of the contribution of Fourier. Nowadays, the study of Fourier series has matured into a branch of mathematics called harmonic analysis. It has equal importance in theoretical and applied mathematics, as well as other branches of natural sciences and engineering.

In some books the Fourier series of a function is written in the form

\[ \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \]

instead of

\[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \]

so that the formula for \( a_0 \) is the same as the other \( a_n \)'s (see (1.1)). However, our notation has the advantage that \( a_0 \) has a simple meaning, i.e., it is the average of the function over a period.

Concerning the convergence of a Fourier series to its function, we point out that an example of a continuous function whose Fourier series diverges at some point can be found in Stein-Sharachi. More examples are available by googling. The classical book by A. Zygmund, “Trigonometric Series” (1959) reprinted in 1993, contains most results before 1960. After 1960, one could not miss to mention Carleson’s sensational work in 1966. His result in particular implies that the Fourier series of every function in \( R_{2\pi} \) converges to the function itself almost everywhere.

There are several standard proofs of the Weierstrass approximation theorem, among them Rudin’s proof in “Principles” by expanding an integral kernel and Bernstein’s proof based on binomial expansion are both worth reading. Recently the original proof of Weierstrass by the heat kernel is available on the web. It is nice to take a look too. In Chapter 3 we will reproduce Rudin’s proof and then discuss Stone-Weierstrass theorem, a far reaching generalization of Weierstrass approximation theorem.

The elegant proof of the Isoperimetric Inequality by Fourier series presented here is due to Hurwitz (1859-1919). You may google under “proofs of the Isoperimetric Inequality” to find several different proofs in the same spirit. The Isoperimetric Inequality has a higher dimensional version which asserts that the ball has the largest volume among all domains having the same surface area. However, the proof is much more complicated.

The aim of this chapter is to give an introduction to Fourier series. It will serve the purpose if your interest is aroused and now you consider to take our course on Fourier analysis in the future. (Watch out! This course will not be offered in 2016/17.) Not expecting a thorough study, I name Stein-Shakarchi as the only reference.