Solutions to Assignment 1

1. Show that every trigonometric series can be written in the form

\[ a_0 + \sum_{n=1}^{\infty} r_n \cos(nx - \theta_n), \quad r_n \geq 0, \quad \theta_n \in [0, 2\pi). \]

**Solution** Choose \( \theta_n \in [0, 2\pi) \) to satisfy

\[ \cos \theta_n = \frac{a_n}{r_n}, \quad \sin \theta_n = \frac{b_n}{r_n}, \]

where

\[ r_n = \sqrt{a_n^2 + b_n^2}. \]

2. Consider the trigonometric series

\[ a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \]

Suppose that it is uniformly and absolutely convergent on some \( (a, b) \) in \([-\pi, \pi]\). Show that it is uniformly and absolutely convergent on \([-\pi, \pi]\). Hint: Use (1).

**Solution** Let

\[ g(x) = |a_0| + \sum_{n=1}^{\infty} r_n |\cos(nx - \theta_n)|. \]

By assumption, \( g \) is a continuous function on \([-\pi, \pi]\). Using \(|\cos \theta| \geq \cos^2 \theta \geq 2^{-1}(1 + \cos 2\theta)\), we have

\[ \int_a^b g(x)dx \geq \sum_{n} \int_a^b r_n |\cos(nx - \theta_n)|dx \]

\[ \geq \frac{1}{2} \sum_{n} r_n \left( b - a + \frac{\sin(2nx - 2\theta_n)}{2n}\right) \]

\[ \geq \frac{b - a}{4} \sum_{n} r_n, \quad \forall n \geq n_1, \]

if we choose \( n_1 \) such that \( 1/n \leq (b - a)/2 \). The desired result comes from the M-test.

3. Consider the trigonometric series as in (2). Suppose that it is uniformly convergent on some \( (a, b) \) in \([-\pi, \pi]\). Show that

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0. \]

**Hint:** Use (1).

**Solution** The proof is in the same spirit of (2). By assumption, for \( \varepsilon > 0 \), there is some \( n_1 \) such that

\[ \left| \sum_{n=n_1}^{m} r_n \cos(nx - \theta_n) \right| < \varepsilon, \]

if we choose \( m \) large enough.
for all \(n, m \geq n_1\). Taking \(m = n + 1\), we have \(\left| r_n \cos(nx - \theta_n) \right| < \varepsilon\). By integrating over \([a, b]\), we have

\[
\varepsilon(b - a) > \int_a^b |r_n \cos(nx - \theta)| \, dx
\]

\[
\geq \int_a^b r_n \cos^2 nx \, dx
\]

\[
= \frac{1}{2} r_n \left( b - a + \frac{\sin(2nx - 2\theta_n)}{2n} \right)
\]

\[
\geq \frac{b - a}{4} r_n,
\]

for all large \(n\). We conclude that \(r_n \to 0\) as \(n \to \infty\).

4. A finite Fourier series is of the form \(a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)\). A trigonometric polynomial is of the form \(p(\cos x, \sin x)\) where \(p(x, y)\) is a polynomial of two variables \(x, y\).

(a) Write down the general expressions for trigonometric polynomial of degree one, two and three.

(b) Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.

Solution

(a) A trigonometric polynomial of degree \(N\) is given by

\[
f(x) = p(\cos x, \sin x) = \sum_{n=0}^{N} \sum_{k=0}^{n} c_{k,n-k}(\cos x)^k(\sin x)^{n-k},
\]

where \(p(x, y) = \sum_{n=0}^{N} \sum_{k=0}^{n} c_{k,n-k} x^k y^{n-k}\) is a general polynomial of degree \(N\) in two variables \(x, y\).

(b) Suppose \(f(x)\) is a trigonometric polynomial of order \(N\), say

\[
f(x) = p(\cos x, \sin x) = \sum_{n=0}^{N} \sum_{k=0}^{n} c_{k,n-k}(\cos x)^k(\sin x)^{n-k}.
\]

By Euler’s formula, \(\cos x = \frac{1}{2}(e^{ix} + e^{-ix})\), \(\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})\), one has

\[
f(x) = p\left(\frac{e^{ix} + e^{-ix}}{2}, \frac{e^{ix} - e^{-ix}}{2i}\right)
\]

\[
= \sum_{n=0}^{N} \sum_{k=0}^{n} c_{k,n-k} \left(\frac{e^{ix} + e^{-ix}}{2}\right)^k \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^{n-k}
\]

\[
= \sum_{n=0}^{N} \sum_{k=0}^{n} c_{k,n-k} 2^n \left[ \sum_{r=0}^{k} e^{2irx} \right] \left[ \sum_{s=0}^{n-k} e^{2isx} \right]
\]

\[
= \sum_{n=0}^{N} \sum_{m=-n}^{n} d_t e^{inx}
\]

\[
= \sum_{n=-N}^{N} c_n e^{inx},
\]

(b) Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.
by collecting terms in the last two steps. Hence \( f(x) \) is a finite Fourier series of order \( N \).

On the other hand, suppose \( f(x) \) is a finite Fourier series given by

\[
f(x) = \sum_{n=-N}^{N} c_n e^{inx}.
\]

Applying Euler’s formula and expanding the binomials, we have

\[
f(x) = c_0 + \sum_{n=1}^{N} \left( c_n (e^{ix})^n + c_{-n} (e^{-ix})^n \right)
\]

\[
= c_0 + \sum_{n=1}^{N} \left( c_n (\cos x + i \sin x)^n + c_{-n} (\cos x - i \sin x)^n \right)
\]

\[
= c_0 + \sum_{n=1}^{N} \left( c_n \sum_{k=0}^{n} (\cos x)^k (i \sin x)^{n-k} + c_{-n} \sum_{k=0}^{n} (\cos x)^k (-i \sin x)^{n-k} \right)
\]

\[
= c_0 + \sum_{n=1}^{N} \sum_{k=0}^{n} i^{n-k} (c_n + (-1)^{n-k} c_{-n}) (\cos x)^k (\sin x)^{n-k}
\]

where \( p(x, y) = c_0 + \sum_{n=1}^{N} \sum_{k=0}^{n} i^{n-k} (c_n + (-1)^{n-k} c_{-n}) x^k y^{n-k} \) is a polynomial of degree \( N \) in two variables \( x, y \). Hence \( f(x) \) is a trigonometric polynomial of degree \( N \).

5. Let \( f \) be a \( 2\pi \)-periodic function which is integrable over \([-\pi, \pi]\). Show that it is integrable over any finite interval and

\[
\int_I f(x)dx = \int_J f(x)dx,
\]

where \( I \) and \( J \) are intervals of length \( 2\pi \).

**Solution** It is clear that \( f \) is also integrable on \([n\pi, (n+2)\pi]\), \( n \in \mathbb{Z} \), so it is integrable on any finite interval. To show the integral identity it suffices to take \( J = [-\pi, \pi] \) and \( I = [a, a+2\pi] \) for some real number \( a \). Since the length of \( I \) is \( 2\pi \), there exists some \( n \) such that \( n\pi \in I \) but \((n+2)\pi \) does not belong to the interior of \( I \). We have

\[
\int_a^{a+2\pi} f(x)dx = \int_a^{n\pi} f(x)dx + \int_{n\pi}^{a+2\pi} f(x)dx.
\]

Using

\[
\int_a^{n\pi} f(x)dx = \int_{a+2\pi}^{(n+2)\pi} f(x)dx
\]

(by a change of variables), we get

\[
\int_a^{a+2\pi} f(x)dx = \int_{a+2\pi}^{(n+2)\pi} f(x)dx + \int_{n\pi}^{a+2\pi} f(x)dx = \int_{n\pi}^{(n+2)\pi} f(x)dx.
\]

Now, using a change of variables again we get

\[
\int_{n\pi}^{(n+2)\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx.
\]
6. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.

**Solution** Write

\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \]

Suppose \( f(x) \) is an even function. Then, for \( n \geq 1 \), we have

\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx f(x) \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} \sin nx f(x) \, dx + \int_{0}^{\pi} \sin nx f(x) \, dx \right]. \]

By a change of variable and using \( f(-x) = f(x) \) since \( f(x) \) is an even function,

\[ \int_{-\pi}^{0} \sin nx f(x) \, dx = \int_{0}^{\pi} \sin(-nx) f(-x) \, dx = -\int_{0}^{\pi} \sin nx f(x) \, dx, \]

one has

\[ b_n = \frac{1}{\pi} \left[ -\int_{0}^{\pi} \sin nx f(x) \, dx + \int_{0}^{\pi} \sin nx f(x) \, dx \right] = 0. \]

Hence the Fourier series of every even function \( f \) is a cosine series.

Now suppose \( f(x) \) is an odd function. Then, for \( n \geq 1 \), we have

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos nx f(x) \, dx = \frac{1}{\pi} \left[ \int_{-\pi}^{0} \cos nx f(x) \, dx + \int_{0}^{\pi} \cos nx f(x) \, dx \right]. \]

By a change of variable and using \( f(-x) = -f(x) \) since \( f(x) \) is an odd function,

\[ \int_{-\pi}^{0} \cos nx f(x) \, dx = \int_{0}^{\pi} \cos(-nx) f(-x) \, dx = -\int_{0}^{\pi} \cos nx f(x) \, dx, \]

one has

\[ a_n = \frac{1}{\pi} \left[ -\int_{0}^{\pi} \cos nx f(x) \, dx + \int_{0}^{\pi} \cos nx f(x) \, dx \right] = 0. \]

Furthermore, by a change of variable and using \( f(-x) = -f(x) \),

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \left[ \int_{-\pi}^{0} f(x) \, dx + \int_{0}^{\pi} f(x) \, dx \right] \]

\[ = \frac{1}{2\pi} \left[ \int_{0}^{\pi} f(x) \, dx + \int_{0}^{\pi} f(x) \, dx \right] = 0. \]

Hence the Fourier series of every odd function \( f \) is a sine series.

7. Here all functions are defined on \([-\pi, \pi]\). Verify their Fourier expansion and determine their convergence and uniform convergence (if possible).

(a)

\[ x^2 \sim \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx, \]

(b)

\[ |x| \sim \frac{\pi}{2} - 4 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x, \]
(c) \[ f(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0] \end{cases} \approx \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x, \]

(d) \[ g(x) = \begin{cases} x(\pi - x), & x \in [0, \pi) \\ x(\pi + x), & x \in (-\pi, 0) \sim 4 \pi \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x. \]

Solution

(a) Consider the function \( f_1(x) = x^2 \). As \( f_1(x) \) is even, its Fourier series is a cosine series and hence \( b_n = 0 \).
\[
 a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \frac{x^3}{3} \bigg|_{-\pi}^{\pi} = \frac{\pi^2}{3},
\]
and by integration by parts,
\[
 a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx \\
 = \frac{1}{n\pi} x^2 \sin nx \bigg|_{-\pi}^{\pi} - \frac{2}{n\pi} \int_{-\pi}^{\pi} x \sin nx dx \\
 = \frac{2}{n^2 \pi} x \cos nx \bigg|_{-\pi}^{\pi} - \frac{2}{n^2 \pi} \int_{-\pi}^{\pi} \cos nx dx \\
 = 4 \left( -\frac{1}{n} \right)^n \frac{1}{n^2}.
\]
For \( n \geq 1 \),
\[
 |a_n| = \left| -4 \left( -\frac{1}{n} \right)^{n+1} \frac{1}{n^2} \right| \leq \frac{4}{n^2}.
\]
We conclude that the Fourier series converges uniformly by the Weierstrass M-test.

(b) Consider the function \( f_2(x) = |x| \). As \( f_2(x) \) is even, its Fourier series is a cosine series and hence \( b_n = 0 \).
\[
 a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{2\pi} \frac{x^2}{2} \bigg|_{-\pi}^{\pi} = \frac{\pi}{2},
\]
and by integration by parts,
\[
 a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx dx \\
 = \frac{2}{n\pi} x \sin nx \bigg|_{0}^{\pi} - \frac{2}{n\pi} \int_{0}^{\pi} \sin nx dx \\
 = -\frac{2}{n^2 \pi} \cos nx \bigg|_{0}^{\pi} \\
 = -2 \frac{(-1)^n - 1}{n^2 \pi}.
\]
For \( n \geq 1 \),
\[
 |a_n| = 2 \frac{((-1)^n - 1)}{n^2 \pi} \leq \frac{4}{\pi n^2}.
\]
We conclude that the Fourier series converges uniformly by the Weierstrass M-test.
(c) As \( f(x) \) is odd, its Fourier series is a sine series and hence \( a_n = 0 \).

\[
\begin{align*}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin nx \, dx \\
&= \frac{2}{n\pi} \cos nx \bigg|_0^\pi \\
&= 2 \frac{(-1)^n - 1}{n\pi}.
\end{align*}
\]

Now we consider the convergence of the series \( \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x \). Fix \( x \in (-\pi, 0) \cup (0, \pi) \). Using the elementary formula

\[
\sum_{n=1}^{N} \sin(2n-1)x = \frac{\sin^2(N+1)x}{\sin x},
\]

one has that the partial sums \( |\sum_{n=1}^{N} \sin(2n-1)x| = |\frac{\sin^2(N+1)x}{\sin x}| \leq \frac{1}{|\sin x|} \) are uniformly bounded. This also holds for \( x = 0 \), in which case \( |\sum_{n=1}^{N} \sin(2n-1)0| = 0 \). Furthermore, the coefficients \( \frac{1}{2n-1} \) decreases to 0. We conclude that the Fourier series converges pointwisely by Dirichlet’s test.

(d) As \( g(x) \) is odd, its Fourier series is a sine series and hence \( a_n = 0 \). By integration by parts,

\[
\begin{align*}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) \sin nx \, dx \\
&= -\frac{2}{n\pi} x(\pi - x) \cos nx \bigg|_0^\pi + \frac{2}{n\pi} \int_0^{\pi} (\pi - 2x) \cos nx \, dx \\
&= \frac{2}{n^2\pi} (\pi - 2x) \sin nx \bigg|_0^\pi + \frac{4}{n^2\pi} \int_0^{\pi} \sin nx \, dx \\
&= -\frac{4}{n^3\pi} \cos nx \bigg|_0^\pi \\
&= -\frac{4}{n^3\pi} [(-1)^n - 1].
\end{align*}
\]

As

\[|b_n| \leq \frac{8}{\pi n^3},\]

we conclude that the Fourier series converges uniformly by the Weierstrass M-test.

8. Show that

\[
x^2 \sim \frac{4\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n},
\]

for \( x \in [0, 2\pi] \). Compare it with 3(a).

**Solution**  It shows that a function may have two different Fourier expansion over a subinterval. Here we have two on \([0, \pi]\).

Consider the function \( f(x) = x^2 \).

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{2\pi} \left. \frac{x^3}{3} \right|_0^\pi = \frac{4\pi^2}{3},
\]
and by integration by parts,
\[ a_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx \, dx \]
\[ = \frac{1}{n\pi} x^2 \sin nx \bigg|_0^{2\pi} - \frac{1}{n\pi} \int_0^{2\pi} x \sin nx \, dx \]
\[ = \frac{2}{n^2\pi} x \cos nx \bigg|_0^{2\pi} - \frac{2}{n^2\pi} \int_0^{2\pi} \cos nx \, dx \]
\[ = \frac{4}{n^2}, \]
and
\[ b_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx \]
\[ = -\frac{1}{n\pi} x^2 \cos nx \bigg|_0^{2\pi} + \frac{2}{n\pi} \int_0^{2\pi} x \cos nx \, dx \]
\[ = -\frac{4\pi}{n} + \frac{2}{n^2\pi} x \sin nx \bigg|_0^{2\pi} - \frac{2}{n^2\pi} \int_0^{2\pi} \sin nx \, dx \]
\[ = -\frac{4\pi}{n}. \]

9. This is an optional problem.

(a) Assume that the Fourier coefficients of a continuous, $2\pi$-periodic function vanish identically. Show that this function must be the zero function. Hint: WLOG assume $f(0) > 0$. Use the relation
\[ \int_{-\pi}^{\pi} f(x)p(x)\, dx = 0, \]
where $p(x)$ is a trigonometric polynomial of the form $(\varepsilon + \cos x)^k$ for some small $\varepsilon$ and large $k > 0$.

(b) Use the result in (a) to show that if the Fourier series of a continuous, $2\pi$-periodic function converges uniformly, then it converges uniformly to the function itself.

(c) Apply (b) to Problem 3(a) to show
\[ \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \]

Solution

(a) Let $p_{\varepsilon,k}(x) = C_k(\varepsilon + \cos x)^{2k}$, where $C_k^{-1} = \int_{-\pi}^{\pi} (\varepsilon + \cos x)^{2k} \, dx$. Using $\int_{-\pi}^{\pi} p_{\varepsilon,k}(x)\, dx = 1$, one has
\[
\int_{-\pi}^{\pi} f(x)p_{\varepsilon,k}(x)\, dx - f(0) = \int_{-\pi}^{\pi} p_{\varepsilon,k}(x)(f(x) - f(0))\, dx
= \int_{-\delta}^{\delta} p_{\varepsilon,k}(x)(f(x) - f(0))\, dx + (\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi})[p_{\varepsilon,k}(x)(f(x) - f(0))]\, dx.
\]
Given $\eta > 0$, by continuity, there exists $\delta > 0$ such that
\[ |f(x) - f(0)| \leq \frac{1}{2}\eta, \quad \forall x \in [-\delta, \delta]. \]
Then
\[ \left| \int_{-\delta}^{\delta} p_{\varepsilon,k}(x)(f(x) - f(0))\,dx \right| \leq \frac{1}{2} \eta \int_{-\delta}^{\delta} p_{\varepsilon,k}(x)\,dx \leq \frac{1}{2} \eta \int_{-\pi}^{\pi} p_{\varepsilon,k}(x)\,dx \leq \frac{1}{2} \eta, \]

and
\[ |(\int_{-\delta}^{-\pi} + \int_{\delta}^{\pi})[p_{\varepsilon,k}(x)(f(x) - f(0))]\,dx| \leq \max_{\delta \leq |x| \leq \pi} p_{\varepsilon,k}(x) \int_{-\pi}^{\pi} |f(x) - f(0)|\,dx \leq \frac{1}{2} \eta, \]

First choose \( \varepsilon \) sufficiently small so that for \( \delta \leq |x| \leq \pi, |(\varepsilon + \cos x)| < 1 \), then choose \( k \) sufficiently large so that
\[ \max_{\delta \leq |x| \leq \pi} p_{\varepsilon,k}(x) \leq \frac{1}{1 + \int_{-\pi}^{\pi} |f(x) - f(0)|\,dx} \frac{\eta}{2}. \]

Hence one has
\[ \left| \int_{-\pi}^{\pi} p_{\varepsilon,k}(x)(f(x) - f(0))\,dx \right| \leq \eta. \]

Since \( \eta > 0 \) is arbitrary, this shows \( \int_{-\pi}^{\pi} f(x)p_{\varepsilon,k}(x)\,dx \to f(0) \), for suitably chosen \( \varepsilon \to 0, k \to \infty \). By Problem 4, the trigonometric polynomial \( p_{\varepsilon,k}(x) \) is a finite Fourier series, which can be written as \( p_{\varepsilon,k}(x) = \sum_{n=-k}^{k} c_n e^{inx} \). Suppose the Fourier coefficients of \( f(x) \) vanish identically, then \( \int_{-\pi}^{\pi} f(x)p_{\varepsilon,k}(x)\,dx = 0 \). This implies that \( f(0) = 0 \). By translations, it holds that for \( x \in [-\pi, \pi], f(x) = 0 \). Also refer to Stein-Shakarchi, Fourier Analysis, page 39-40 for a slightly variant proof.

(b) Suppose \( S_N(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \) is the partial sums of the Fourier series of a continuous, \( 2\pi \)-periodic function \( u \), and that \( S_N \) converges uniformly. Let \( v(x) = \lim_{N \to \infty} S_N(x) \). By the uniform convergence, \( v(x) \) is a continuous, \( 2\pi \)-periodic function. Furthermore,
\[ \int v(x) \cos nx \,dx = \lim_{N \to \infty} \int a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \cos nx \,dx = \pi b_n, \]

and the same holds for \( a_n \). Let \( w = u - v \). Then the Fourier coefficients of \( w \) vanish identically, and by (a) one has \( w \equiv 0 \). Hence \( u = v \) and that \( S_N \) converge uniformly to \( u \).

(c) By Problem 3(a), \( \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx \) are the Fourier series of the function \( u(x) = x^2 \), and the series converges uniformly. Hence by (b), it must converge to \( u(x) = x^2 \). One has
\[ 0 = u(0) = \frac{\pi^2}{3} - 4 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}, \]

and we obtain
\[ \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}. \]