Assignment 3

1. Optional. Let $f$ be a function defined on $(a, b]$ which is integrable on $[c, b]$ for all $c \in (a, b)$. It is called improperly integrable over $(a, b]$ if

$$\lim_{c \to a^+} \int_c^b |f|$$

exists. When this happens,

$$\lim_{c \to a^+} \int_c^b f$$

also exists and we define the improper integral of $f$ over $(a, b]$ to be

$$\int_a^b f = \lim_{c \to a^+} \int_c^b f .$$

(a) Show that if $f$ is integrable on $[a, b]$, its improper integral also exists and is equal to its usual integral.

(b) Show that Riemann-Lebesgue Lemma holds for improperly integrable functions.

2. Optional. Show that

$$- \log |2 \sin \frac{x}{2}| \sim \cos x + \frac{\cos 2x}{2} + \frac{\cos 3x}{3} + \cdots .$$

Suggestion. Verify this function is $2\pi$-periodic and improperly integrable first. The calculation of $a_0$ is tricky, involving the definite integral $I = \int_0^{\pi/2} \log \sin t \, dt$. To evaluate it use $\sin t = 2 \sin t/2 \cos t/2$ and eventually show $I = -\frac{\pi}{2} \log 2$.

3. Let $a_n, b_n$ be the Fourier coefficients of some $f \in R_{2\pi}$.

(a) Show that for each $r \in [0, 1)$, the trigonometric series given by

$$a_0 + \sum_{k=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx)$$

is uniformly convergent to some function in $C_{2\pi}$. Denote this function by $f_r(x)$.

(b) Show that

$$f_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(z) f(x + z) \, dz,$$

where the Poisson kernel $P_r$ is given by

$$P_r(z) = \frac{1 - r^2}{1 - 2r \cos z + r^2} .$$

(c) Let $f$ be continuous at $x$. Show that $\lim_{r \to 1} f_r(x) = f(x)$.

The treatment is parallel to that for the Dirichlet kernel (the parameter $n$ is now replaced by $r$), but differs at the final step; we do not need Lipschitz continuity. Think about it.

4. (a) Can you find a cosine series which converges uniformly to the sine function on $[0, \pi]$? If yes, find one.
(b) Is the series in (a) unique?

(c) Can you find a cosine series which converges pointwisely to the sine function on \([-a, \pi]\)
where \(a\) is a number in \((0, \pi)\)?

5. Let \(f\) be an integrable function on \([-\pi, \pi]\). Show that for each \(\varepsilon > 0\), there exists a
trigonometric polynomial \(p\) satisfying \(p < f\) on \([-\pi, \pi]\) and
\[
\int_{-\pi}^{\pi} |f - p| < \varepsilon.
\]

6. Show that there is a countable subset of \(C[a, b]\) such that for each \(f \in C[a, b]\), there is
some \(\varepsilon > 0\) such that \(\|f - g\|_\infty < \varepsilon\) for some \(g\) in this set. Suggestion: Take this set to be
the collection of all polynomials whose coefficients are rational numbers.

7. Let \(f\) be continuously on \([a, b] \times [c, d]\). Show that for each \(\varepsilon > 0\), there exists a polynomial
\(p = p(x, y)\) so that
\[
\|f - p\|_\infty < \varepsilon, \quad \text{in} \ [a, b] \times [c, d].
\]
In fact, this result holds in arbitrary dimension.

8. (a) Let \(S\) be the vector subspace in \(C[0, 1]\) spanned by the polynomials \(1, x\), and \(x^2\). Find
an orthonormal set in \(S\) which spans \(S\).

(b) Find the quadratic polynomial that minimizing the \(L^2\)-distance from \(1/(1 + x)\) to \(S\).

9. The Legendre polynomials are given by
\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n[(x^2 - 1)^n]}{dx^n}, \quad n \geq 0.
\]

(a) Write down \(P_0, \ldots, P_4\).

(b) Show that
\[
\left\{ \sqrt{\frac{2n + 1}{2}} P_n \right\}_{n=0}^{\infty}
\]
forms an orthonormal set in \(R[-1, 1]\).

(c) Verify that each \(P_n\) is a solution to the differential equation
\[
\left( [(1 - x^2)] y' \right)' + n(n + 1)y = 0.
\]