THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2230A (First term, 2015–2016)
Complex Variables and Applications
Notes 4 Cauchy-Riemann

4.1 Geometric Meanings

In this section, we will discuss the geometric implication of complex differentiability at a point $z_0 \in \Omega$. All the results come from the additional special property of the differential matrix. For convenience, we write

$$Df_{(x_0,y_0)} \overset{\text{def}}{=\!=} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}_{(x_0,y_0)} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}.$$

Note that this is a matrix of real numbers and the phenomenon only occurs at the point z_0.

First, we will discuss ***why*** the function f takes perpendicularly intersecting curves to the same situation. In fact, more generally, f always ***preserves the intersecting angle*** of curves.

Exercise 4.1. Try to justify the above paragraph mathematically.

Recall in Linear Algebra, a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends the vectors

$$(1,0) \mapsto (a,c), \qquad (0,1) \mapsto (b,d).$$

The matrix $D = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ has a determinant p^2+q^2, which is zero if and only if $p = q = 0$. Thus, either

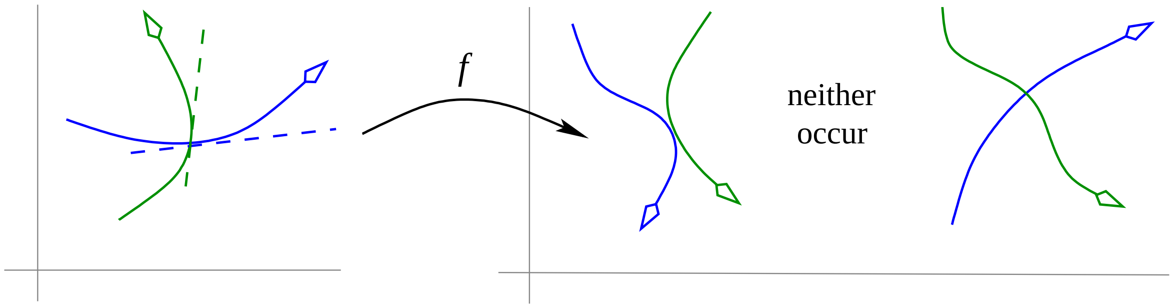
$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad D = \sqrt{p^2+q^2} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where $\cos\alpha = \dfrac{p}{\sqrt{p^2+q^2}}$ and $\sin\alpha = \dfrac{q}{\sqrt{p^2+q^2}}$. Therefore, the action of D is simply a rotation together with a change of length by a factor of $\sqrt{p^2+q^2}$. In other words, for any pair of vectors $\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2$, the length of their images $D(\vec{\boldsymbol{v}}_1), D(\vec{\boldsymbol{v}}_2)$ may change, but the angle between them remains the same.

Exercise 4.2. Apply the above facts in Linear Algebra to the differential matrix of $(x,y) \mapsto (u,v)$ to prove that the image curves will have the same intersecting angle of the original curves.

Finally, the matrix $D = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ also has determinant equals $p^2+q^2 \geq 0$. Thus, except the case that $D = 0$, the determinant is always positive. Suppose $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ are a pair of positively oriented linearly independent vectors (e.g., $\vec{\boldsymbol{e}}_1$ and $\vec{\boldsymbol{e}}_2$). Then their image vectors $D(\vec{\boldsymbol{v}}_1)$ and $D(\vec{\boldsymbol{v}}_2)$ are also positively oriented linearly independent. As a result, the following picture of curve mappings will not occur. The first case corresponds to a nonzero differential matrix with zero determinant; the second case is having negative determinant.

Exercise 4.3. Justify the above description mathematically.



4.2 Pointwise

We are considering $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ and looking at what happens at a point $z_0 \in \Omega$. Our convention always denote $z = x + \mathbf{i}y$ and $f(z) = u(x, y) + \mathbf{i}v(x, y)$. We are assuming the following statement.

STATEMENT (I). The function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$.

In such a setting, f is complex differentiable at z_0 if and only if the function $(x, y) \mapsto (u, v)$ is differentiable at (x_0, y_0) and the Cauchy-Riemann Equations are satisfied at (x_0, y_0), i.e.,

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}_{(x_0,y_0)} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}_{(x_0,y_0)}.$$

This is equivalent to the statement about $\mathbb{R}^2 \to \mathbb{R}^2$.

STATEMENT (IA). The function $(x, y) \mapsto (u, v)$ is differentiable at the corresponding point (x_0, y_0) and the Cauchy-Riemann Equations are satisfied at that point.

4.2.1 The Converse

At this moment, we also have a weaker statement which is only about f at a point.

STATEMENT (II). All partial derivatives of the function $(x, y) \mapsto (u, v)$ exist at the corresponding point (x_0, y_0) and the Cauchy-Riemann Equations are satisfied at that point.

In short, (I) $\iff$ (IA) $\implies$ (II). ***What*** about the converse? Recall that from multivariable calculus, the converse is not true. However, there is a similar situation, in which an additional condition of continuous partial derivatives implies differentiability.

STATEMENT (IIA). All partial derivatives of the function $(x, y) \mapsto (u, v)$ exist on a neighborhood $B(z_0, \delta)$ and are continuous at the point (x_0, y_0) and the Cauchy-Riemann Equations are satisfied at the point.

Let us suppose (IIA) holds and look at the rough ideas behind. Then we write

$$
\begin{aligned}
u(x,y)-u(x_0,y_0) &= u(x,y)-u(x_0,y)+u(x_0,y)-u(x_0,y_0)\\
&= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y)}(x-x_0)+\varepsilon_1+\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}(y-y_0)+\varepsilon_2\,,\\
&\qquad \text{where } \varepsilon_1,\varepsilon_2\to 0 \text{ "fast enough" by existence of partial derivatives;}\\
&= \left[\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+\varepsilon_3\right](x-x_0)+\varepsilon_1+\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}(y-y_0)+\varepsilon_2\,,\\
&\qquad \text{where } \varepsilon_3\to 0 \text{ (may not fast) by continuity of partial derivatives;}\\
&= \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}(x-x_0)+\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}(y-y_0)+\varepsilon_u\,,\\
&\qquad \text{where } \varepsilon_u=\varepsilon_1+\varepsilon_2+\varepsilon_3(x-x_0)\to 0 \text{ fast enough.}
\end{aligned}
$$

The last line is merely the differentiability of $(x,y)\mapsto u$ at the point (x_0,y_0). Similarly, we obtain the differentiability of $(x,y)\mapsto v$ at the same point. So far, we *have not used* the Cauchy-Riemann Equations. Let us further write

$$
\begin{aligned}
f(z)-f(z_0) &= u(x,y)-u(x_0,y_0)+\mathbf{i}\,[v(x,y)-v(x_0,y_0)]\\
&= \left[\left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)}+\mathbf{i}\left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}\right](x-x_0)+\left[\left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)}+\mathbf{i}\left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)}\right](y-y_0)+\varepsilon\\
&= [p+\mathbf{i}q]\cdot[(x-x_0)+\mathbf{i}(y-y_0)]+\varepsilon\,.
\end{aligned}
$$

It is only at the last step that we need the Cauchy-Riemann Equations at (x_0,y_0) to get p,q.

Therefore, we have established (IIA) $\underset{\not\Longleftarrow}{\Longrightarrow}$ (I) $\underset{\not\Longleftarrow}{\Longrightarrow}$ (II). Note that the backward implication cannot be concluded because (IIA) is not only about a point z_0.

4.3 On a neighborhood

In the above, the converse does not work because (I) is only a pointwise statement while (IIA) is not. ***What if*** we raise the level of (I) to a neighborhood?

STATEMENT (IC). The function $f:\ \Omega\subset\mathbb{C}\to\mathbb{C}$ is complex differentiable on $B(z_0,\delta)$ for some $\delta>0$.

Immediately, by the same reason as (I) $\Longrightarrow$ (II), a consequence of (IC) is: All partial derivatives of $(x,y)\mapsto(u,v)$ exist on $B(z_0,\delta)$ and the Cauchy-Riemann Equations are true on the neighborhood. This conclusion is different from (IIA) and it is hard to say which one is stronger. The strongest statement seems to be a total neighborhood version of (II), i.e.,

STATEMENT (IIC). All partial derivatives of $(x,y)\mapsto(u,v)$ are defined and continuous on $B(z_0,\delta)$ and the Cauchy-Riemann Equations are satisfied on the neighborhood.

Clearly, (IIC) $\Longrightarrow$ (IC). The ***first surprising fact*** is that (IIC) $\Longleftarrow$ (IC).

In other words, on a neighborhood, the *existence* of complex derivative guarantees the *continuity* of it. This certainly is not true for real functions. The ***second surprising fact*** is that the continuous derivative can be used to define ***second derivative***. Thus we have chain effect,

$$f' \text{ exists} \quad \Longrightarrow \quad f'' \text{ exists} \quad \Longrightarrow \quad \cdots \quad \Longrightarrow \quad f^{(n)} \text{ exists for all } n.$$

For the moment, we have only mentioned the two surprising facts. The proofs of them form the main content of this course. We will look at it from different points of view.

4.3.1 Analytic Functions

According to the above discussion, there is an important type of complex functions. These functions are mainly the object of study in this course. A function $f: \Omega \subset \mathbb{C} \to \mathbb{C}$ is *analytic* or *holomorphic* at $z_0 \in \Omega$ if there is $\delta > 0$ such that $f'(z)$ exists for every $z \in B(z_0, \delta)$. A function is analytic on Ω if it is analytic at every $z \in \Omega$.

Most of the functions that we come across are indeed analytic functions. But, at this stage, we only know the definition of these functions as real functions. Therefore, we are only giving a partial list of examples.

1. A polynomial in z is analytic on $\Omega = \mathbb{C}$.

   We call an analytic function on $\mathbb{C}$ an *entire function.*

2. The function $z \mapsto 1/z$ is analytic on $\Omega = \mathbb{C} \setminus \{0\}$.

3. A *rational function*, $z \mapsto \dfrac{P(z)}{Q(z)}$, where P, Q are polynomials, is analytic on $\Omega = \mathbb{C} \setminus \{z \in \mathbb{C}: \ Q(z) = 0\}$.

4. There are other well-known analytic functions, which we will define later. For example, e^z, $\sin z$, $\cos z$, are entire functions; other trigonometric functions or hyperbolic functions are analytic on suitable domains; the function $\log z$ is special that needs more discussion.

4.3.2 Harmonic Conjugate

Let $f = u + \mathbf{i}v: \ \Omega \subset \mathbb{C} \to \mathbb{C}$ be analytic on Ω and assume both u, v are of C^∞ (later, this is a consequence of analyticity). Then we have the Cauchy-Riemann Equations on Ω,

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x\,.$$

Futher differentiate the first wrt x and the second wrt y, we have

$$u_{xx} = v_{yx} \qquad \text{and} \qquad u_{yy} = -v_{xy}\,.$$

As a consequence of u, v are of C^2, we have

$$\triangle u \overset{\text{def}}{:=} u_{xx} + u_{yy} \equiv 0\,, \qquad \triangle v \overset{\text{def}}{:=} v_{xx} + v_{yy} \equiv 0 \quad \text{on } \Omega.$$

Both u, v are called *harmonic* functions on Ω. Moreover, v (and in fact, $v + c$) is a *harmonic conjugate* of u. Note that u is NOT a harmonic conjugate of v because $v + \mathbf{i}u$ does not satisfy the Cauchy-Riemann Equations.

Exercise 4.4. Show that $-u$ is a harmonic conjugate of v.

Question. Given a harmonic function u, is there a method to find its harmonic conjugate v?

In the following, we will show by an example of how to find a harmonic conjugate v from a given u. Then afterwards, we will discuss the rationale behind the method and whether it is always valid. Let $u(x,y) = x^3 - 3xy^2 - 2xy$ on $\Omega = \mathbb{C}$. It is easy to verify that $\triangle u \equiv 0$. According to the Cauchy-Riemann Equations,

$$v_y = u_x = 3x^2 - 3y^2 - 2y\,.$$

Therefore, for a fixed point (x_0, y_0), we may integrate wrt y to have

$$\begin{aligned} v(x,y) - v(x,y_0) &= \int_{y_0}^{y} v_y(x,t)\,dt = \int_{y_0}^{y} \left(3x^2 - 3t^2 - 2t\right)\,dt \\ &= 3x^2\,(y - y_0) - \left(y^3 - y_0^3\right) - \left(y^2 - y_0^2\right)\,. \end{aligned}$$

For simplicity, we group all the terms $v(x,y_0)$ and terms involving y_0 into $\varphi(x)$,

$$v(x,y) = 3x^2y - y^3 - y^2 + \varphi(x)\,.$$

Using the other one of Cauchy-Riemann Equations,

$$v_x(x,y) = 6xy - 0 - 0 + \varphi'(x) = -u_y = -\left(0 - 6xy - 2x\right)\,.$$

Thus, integrating writ x, we have

$$\varphi(x) - \varphi(x_0) = \int_{x_0}^{x} \varphi'(t)\,dt = \int_{x_0}^{x} 2t\,dt = x^2 - x_0^2\,.$$

By grouping $\varphi(x_0)$ and x_0^2 as a constant, we have

$$v(x,y) = 3x^2y - y^3 - y^2 + x^2 + C\,, \qquad \text{where } C \text{ is constant depending on } (x_0,y_0).$$

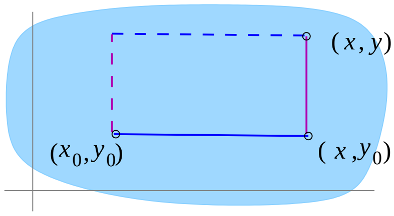
The method seems straight forward, but there is a rationale behind the method.

4.3.3 Integration rationale

In the above example, there are two major integration steps,

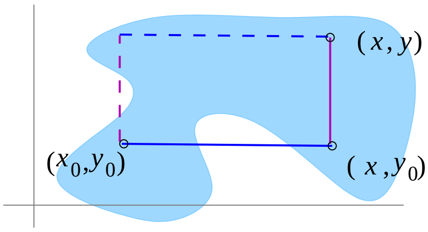
- Integrating v_y wrt y and get a formula of $v(x,y)$ involving $\varphi(x)$. Note that in this step, $\varphi(x)$ actually contains the term $v(x,y_0)$.
- Integrating $\varphi'(x)$ wrt x and get a formula of $\varphi(x)$ involving a constant C. This C actually contains information at the point (x_0,y_0).

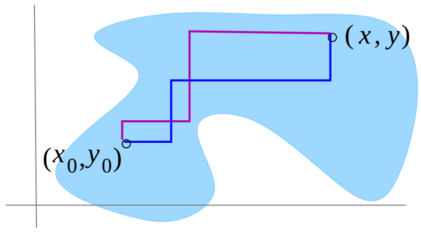
In principle, the second step corresponds to integration from (x_0,y_0) to (x,y_0) along the horizontal line in the picture below; and the first step corresponds to that from (x,y_0) to (x,y) along the vertical line.



EXERCISE 4.5. Try to use $v_x = -u_y$ first and then $v_y = u_x$ to work on the above example. This corresponds to integrating along the dotted lines. Show that the answers are the same.

However, this method works in the above because the function u is defined on the whole $\mathbb{C}$. Therefore, no matter where (x, y) is, the horizontal and vertical lines from (x_0, y_0) to (x, y) always stay inside Ω. If Ω is more complicated, then part of the lines may go outside of Ω where u is not defined, as in the left-hand picture below.





For such a situation, we need to use a sequence of horizontal and vertical lines, such as the right-hand picture above to get $v(x, y)$. But, this leads to another question: ***which*** path should we choose? Is the expression of $v(x, y)$ independent of the choice and only depends on information at (x_0, y_0)?

EXERCISE 4.6. Consider $u(x, y) = \dfrac{x}{x^2 + y^2}$ defined on $\Omega = \mathbb{C} \setminus \{0\}$ and take $z_0 = -1 - \mathbf{i}$. Show that the value of $v(x, y)$ by two different horizontal and vertical lines can still be the same.

However, the situation for $u(x, y) = \log \sqrt{x^2 + y^2}$ on $\Omega = \mathbb{C} \setminus \{0\}$ is different. Observe that

$$u_x = \frac{x}{x^2 + y^2}, \qquad u_y = \frac{y}{x^2 + y^2}.$$

Then, one may use the Cauchy-Riemann Equations to do the following integration,

$$v(1,1) - v(-1,1) = \int_{-1}^{1} v_x(s,1)\, ds = \int_{-1}^{1} -u_y(s,1)\, ds = \int_{-1}^{1} \frac{-(1)\, ds}{s^2 + 1} = -A$$

$$v(-1,1) - v(-1,-1) = \int_{-1}^{1} v_y(-1,t)\, dt = \int_{-1}^{1} u_x(-1,t)\, ds = \int_{-1}^{1} \frac{(-1)\, ds}{(-1)^2 + t^2} = -A$$

So, we have $v(1, 1) - v(-1, -1) = -2A$.

Note that in the above calculation, we first find $v(1, 1) - v(-1, 1)$, which is the difference when x varies from -1 to 1. Then we get $v(-1, 1) - v(-1, -1)$, which is varying y from -1 to 1.

EXERCISE 4.7. Show that the result is different along the other paths. That is, if one finds $v(1, 1) - v(1, -1)$ and $v(1, -1) - v(-1, -1)$, the result is $v(1, 1) - v(-1, -1) = 2A$.

From this example, one sees that the method must be used carefully. There is a theory behind, which will be discussed later.