1. For a set $S$ of numbers, a member $c$ of $S$ is called the maximum of $S$ provided that it is an upper bound for $S$. Prove that a set $S$ of numbers has a maximum if and only if it is bounded above and $\sup S$ belong to $S$. Give an example of a set $S$ of numbers that is nonempty and bounded above but has no maximum.

2. Let $S = \{1 - (-1)^n/n : n \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$.

3. Suppose that $A$ is a nonempty set of real numbers that is both bounded above and bounded below, and that $\inf A = \sup A$. Prove that the set $A$ consists of exactly one number.

4. Show that if $A$ and $B$ are bounded above in $\mathbb{R}$, then $A \cup B$ is a bounded above. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

5. (Optional) For a function $f(x)$, we can define another function

$$f^*(p) = \sup_{x \in \mathbb{R}} \{px - f(x)\}$$

If $g = e^x$, calculate $g^*$ and $g^{**}$.