MATH2020A Advanced Calculus II, 2015-16

MIDTERM EXAM

Honesty in Academic Work: *The Chinese University of Hong Kong places very high importance on honesty in academic work submitted by students, and adopts a policy of zero tolerance on cheating and plagiarism. Any related offence will lead to disciplinary action including termination of studies at the University.*

Answer all SIX questions.

Q1. *(25 points)* Point out TRUE or FALSE without any proof for each statement.

(a) The following identity is true:

$$\int_0^1 \int_{1+x}^{\sqrt{13-x^2}} dy\,dx = \int_1^3 \int_0^{y-1} dx\,dy + \int_3^{\sqrt{13}} \int_0^{\sqrt{13-y^2}} dx\,dy.$$

(b) Let R be the region enclosed by the curve

$$(2x-y)^2 + (x+y-1)^2 = 4$$

Then $\text{Area}(R) = \frac{4\pi}{3}$.

(c) The center of mass of the trapezoid region with vertices $(1,1)$, $(-1,1)$, $(3,-3)$ and $(-3,-3)$ of constant density is at $(0,-1)$.

(d) Let C be a smooth curve joining the point P_0 to the point P_1 in the plane and f be a smooth function with gradient $\vec{F} = \nabla f$, defined everywhere. Then

$$\int_C f\,ds = |\vec{F}(P_1)| - |\vec{F}(P_0)|.$$

(e) For the annulus region R: $1 \le x^2 + y^2 \le 4$, we have

$$\iint_R \frac{x^2 - y^2}{x^4 + y^4}\,dA = 0.$$

Q2. *(15 points)* Find the mass of a thin plate $0 \le x \le 1$, $0 \le y \le x$ of density $\delta(x,y) = \frac{\sin(1-y)}{1-y}$.

Q3. *(15 points)* Let $\vec{F} = (ax^2y + y^3 + 1)\mathbf{i} + (2x^3 + bxy^2 + 2)\mathbf{j}$ be a vector field, where a and b are constants.

(a) Find the values of a and b for which $\vec{F}$ is conservative.

(b) For these values of a and b, find $f(x,y)$ such that $\vec{F} = \nabla f$.

(c) Still using the values of a and b from part (a), compute $\int_C \vec{F} \cdot d\vec{r}$ along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

Q4. *(15 points)* Consider the rectangle R with vertices $(0,0), (1,0), (1,4)$ and $(0,4)$. The boundary of R is the curve C, consisting of C_1 the segment from $(0,0)$ to $(1,0)$, C_2 the segment from $(1,0)$ to $(1,4)$, C_3 the segment from $(1,4)$ to $(0,4)$, and C_4 the segment from $(0,4)$ to $(0,0)$. Consider the vector field

$$\vec{F} = (xy + \sin x \cos y)\mathbf{i} - (\cos x \sin y)\mathbf{j}$$

(a) Find the flux of $\vec{F}$ out of R through C.

(b) Find the total flux of $\vec{F}$ out of R through C_1, C_2 and C_3.

Q5. *(15 points)* Find the volume of the solid enclosed by the plane $z = 4$ and the surface

$$z = (2x - y)^2 + (x + y - 1)^2.$$

Q6. *(15 points)* Suppose $\vec{F}$ is a vector field that is defined in the annulus $1/100 \leq x^2 + y^2 \leq 100$ with $\operatorname{curl} \vec{F} = 0$. Suppose $\int_{C_1} \vec{F} \cdot d\vec{r} = 5$ along C_1 the upper half of the unit circle oriented from $(1, 0)$ to $(-1, 0)$, and $\int_{C_2} \vec{F} \cdot d\vec{r} = -4$ along C_2 the lower half of the unit circle oriented from $(1, 0)$ to $(-1, 0)$. Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ along

(a) the curve C that is the upper half of the ellipse $x^2 + \frac{y^2}{16} = 1$ from $(1, 0)$ to $(-1, 0)$.

(b) the circle C of radius 2 centered at the origin, in counter-clockwise direction.

(c) the circle C of radius 2 centered at $(3, 0)$, in counter-clockwise direction.

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