Review of ODE.

Only the following ODEs may appear in final.

A. $$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

Solu: $\Longleftrightarrow \begin{cases} x' = \lambda_1 x + a & ① \\ y' = \lambda_2 y + b. & ② \end{cases}$

$$① \Longleftrightarrow e^{-\lambda_1 t} x' = e^{-\lambda_1 t} \lambda_1 x + e^{-\lambda_1 t} a$$

$$\Longleftrightarrow e^{-\lambda_1 t} x' - e^{-\lambda_1 t} \lambda_1 x = e^{-\lambda_1 t} a$$

$$\Longleftrightarrow \frac{d}{dt}\left(x(t) e^{-\lambda_1 t} \right) = e^{-\lambda_1 t} a(t)$$

$$\Longleftrightarrow x(t) e^{-\lambda_1 t} - C_1 = \int_0^t e^{-\lambda_1 t} a(t) dt \qquad C_1 = x(0).$$

$$\Longleftrightarrow x(t) = C_1 e^{\lambda_1 t} + e^{\lambda_1 t} \int_0^t e^{-\lambda_1 t} a(t) dt. \ \square$$

B. $$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$

Solu: $\Longleftrightarrow \begin{cases} x'(t) = \lambda x(t) + y(t) + a(t) & ① \\ y'(t) = \lambda y(t) + b(t) & ② \end{cases}$

$$② \iff y(t) = C_1 e^{\lambda t} + B(t), \quad B(t) = e^{\lambda t}\int_0^t e^{-\lambda t} b(t)\,dt, \quad C_1 = y(0).$$

Then, $① \iff x'(t) = \lambda x(t) + C_1 e^{\lambda t} + a(t) + B(t)$

$$\iff x'(t)e^{-\lambda t} - e^{-\lambda t}\lambda x(t) = C_1 + (a(t)+B(t))e^{-\lambda t}$$

$$\iff \frac{d}{dt}\left(e^{-\lambda t}x(t)\right) = C_1 + (a(t)+B(t))e^{-\lambda t}$$

$$\iff e^{-\lambda t}x(t) - C_2 = \int_0^t C_1 + (a(t)+B(t))e^{-\lambda t}\,dt, \quad C_2 = x(0)$$

$$\iff e^{-\lambda t}x(t) - C_2 = C_1 t + \int_0^t (a(t)+B(t))e^{-\lambda t}\,dt$$

$$\iff x(t) = C_2 e^{\lambda t} + C_1 t e^{\lambda t} + e^{\lambda t}\int_0^t (a(t)+B(t))e^{-\lambda t}\,dt. \quad \square$$

c. $$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where A has two different real eigenvalues

Please refer to the tutorial notes 10.

(1st order ODE ①)

Review of ODE:

Only the following 2 situations may appear in the final:

$X' = AX$, A is a 2×2 matrix

① A has 2 different real eigenvalues

② A has a repeated eigenvalue.

$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ or $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

PDE.

A. Given $a, b \in \mathbb{R}$, $b \neq 0$.

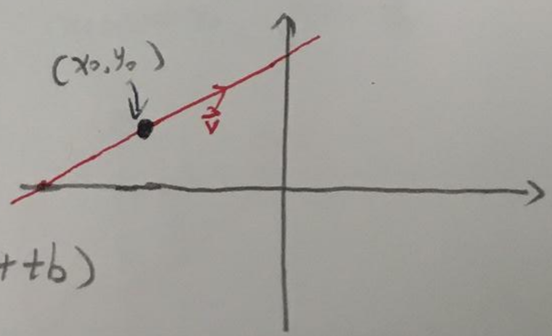
Consider
$$\begin{cases} a \frac{\partial}{\partial x} u(x,y) + b \frac{\partial}{\partial y} u(x,y) = 0, & \forall x, y \in \mathbb{R} \quad ① \\ u(x,0) = f(x), & \forall x \in \mathbb{R} \quad ② \end{cases}$$

① $\Rightarrow$ For any $(x_0, y_0) \in \mathbb{R}^2$, we have $D_{\vec{v}} u \big|_{(x_0,y_0)} = 0$

where $\vec{v} = \frac{1}{\sqrt{a^2+b^2}}(a,b)$.

$\Rightarrow$ u is constant along the line $L = \{(x_0,y_0) + t\vec{v} \mid t \in \mathbb{R}\}$

for any (x_0, y_0).



$\Rightarrow u(x,y) = u(x+ta, y+tb)$

$\forall t \in \mathbb{R}$.

$u(x,y) = u(x+ta, y+tb)$

$= u(x + (-\frac{y}{b})a, 0)$ (Let $t = -\frac{y}{b}$)

$= f(x - \frac{a}{b}y)$ (By ②).

B. Given $a(x,y)$, $b(x,y)$.

consider

$$\begin{cases} a(x,y)\frac{\partial}{\partial x}u(x,y) + b(x,y)\frac{\partial}{\partial y}u(x,y) = 0, & \forall x,y \in \mathbb{R} \quad ① \\ u(x,0) = f(x), & \forall x \in \mathbb{R} \quad ② \end{cases}$$

We want to find $r(t) = (x(t), y(t))$ such that

$$\boxed{r'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a(x(t), y(t)) \\ b(x(t), y(t)) \end{pmatrix}.} \rightsquigarrow \text{ODE}$$

If we can find such $r(t)$, then

$$D_{r'(t)} u \Big|_{(x(t), y(t))} = \nabla f \cdot r'(t)$$

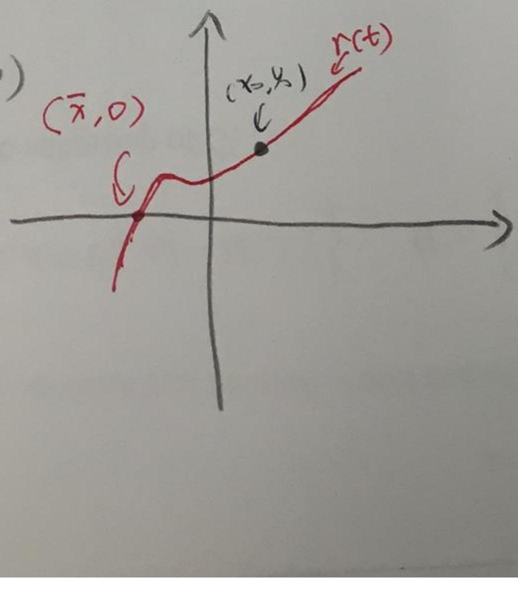
$$= a(x,y)\frac{\partial u}{\partial x}(x,y) + b(x,y)\frac{\partial u}{\partial y}(x,y) = 0$$

$\Rightarrow$ For any (x_0, y_0), u is constant along the curve $r(t) = (x(t), y(t))$ where $x(0) = x_0$, $y(0) = y_0$.

$\Rightarrow u(x_0, y_0) = u(x(t), y(t))$

$= u(\bar{x}, 0)$

$= f(\bar{x})$.



Example: $\begin{cases} (x+1)\dfrac{\partial u}{\partial x} + 3y\dfrac{\partial u}{\partial y} = 0 , & x \geq 0. \\ u(0, y) = (y+1)^6 \end{cases}$

Solu: I. We want to find $r(t) = (x(t), y(t))$ s.t. $\begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$

$$r'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} x(t)+1 \\ 3y(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \wedge$$

$$\Leftrightarrow \begin{cases} x'(t) = x(t) + 1 & ① \\ y'(t) = 3y(t) & ② \end{cases}$$

① : $(x(t)+1)' = x(t) + 1$

$$e^{-t}(x(t)+1) = x_0 + 1)$$

$$x(t) = (x_0+1)e^t - 1$$

② $y'(t) = 3y(t)$

$$e^{-3t} y(t) = y_0$$

$$y(t) = y_0 e^{3t}$$

$$\Leftrightarrow r(t) = ((x_0+1)e^t - 1, y_0 e^{3t})$$

II. $u(x, y) = u((x+1)e^t - 1, ye^{3t})$

$$= u(0, y \cdot (\tfrac{1}{x+1})^3) \qquad (\text{Let } (x+1)e^t - 1 = 0)$$

$$= (\frac{y}{(x+1)^3} + 1)^6 \quad . \square$$

Let G be a group in which every element has order at most 2. Show that G is abelian.