1st Order ODE

$X' = AX$ (1), (A 2×2 matrix)

① If A has two real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then, the general solution of (1) is given by

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2$$

② If A has repeated eigenvalues λ and V_{10}, V_{20} are two linearly independent solutions of $(A-\lambda I_2)^2 V = 0$, set $V_{11} = (A-\lambda I_2) V_{10}$ and $V_{21} = (A-\lambda I_2) V_{20}$. Then, the general solution of (1) is given by $X(t) = \alpha(e^{\lambda t} V_{10} + t e^{\lambda t} V_{11}) + \beta(e^{\lambda t} V_{20} + t e^{\lambda t} V_{21})$.

③ If A has two complex eigenvalues $\lambda, \bar{\lambda}$ ($\lambda = u + iv$) and associated eigenvectors V_1, V_2. Then, the general solution of (1) is given by $X(t) = \alpha e^{\lambda t} V_1 + \beta e^{\bar{\lambda} t} V_2$.

Example:

①
$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \qquad |A - \lambda I_2| = (\lambda-1)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$$

Then, $\lambda_1 = 3$, $\lambda_2 = -1$.

$$(A - \lambda_1 I_2)V = 0 \iff \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0 \quad \text{Then, } V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)V = 0 \iff \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0 \quad \text{Then, } V_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

The general solution is $X(t) = \alpha e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \beta e^{-t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

~~4. Let G be a group in which every element has order at most 2. Show that G is abelian.~~

② $A=\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}$ $\quad |A-\lambda I_2| = (1-\lambda)(3-\lambda)+1 = (\lambda-2)^2$

Then, $\lambda_1=\lambda_2=2$.

$(A-2I_2)^2 V = 0 \iff \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=0$

Then, $V_{10}=\begin{pmatrix}1\\0\end{pmatrix}$, $V_{20}=\begin{pmatrix}0\\1\end{pmatrix}$.

$V_{11}=(A-2I_2)V_{10}=\begin{pmatrix}-1\\-1\end{pmatrix}$ $\quad V_{21}=(A-2I_2)V_{20}=\begin{pmatrix}1\\1\end{pmatrix}$

Then, $X(t)=\alpha e^{2t}\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-1\\-1\end{pmatrix}t\right)+\beta e^{2t}\left(\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\1\end{pmatrix}t\right)$.

③ $A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ $\quad |A-\lambda I_2| = \lambda^2+1=0$

Then, $\lambda_1=i$ $\quad \lambda_2=-i$

$\lambda_1=i$, $\quad (A-\lambda_1 I_2)V=0 \iff \begin{pmatrix}-i & 1\\ -1 & -i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=0$. Then, $V_1=\begin{pmatrix}1\\ i\end{pmatrix}$

$\lambda_2=-i$, $\quad (A-\lambda_2 I_2)V=0 \iff \begin{pmatrix}i & 1\\ -1 & i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=0$. Then, $V_2=\begin{pmatrix}1\\ -i\end{pmatrix}$

Then, $X(t)=\alpha e^{it}\begin{pmatrix}1\\ i\end{pmatrix}+\beta e^{-it}\begin{pmatrix}1\\ -i\end{pmatrix}$

$$=\alpha(\cos t+i\sin t)\begin{pmatrix}1\\ i\end{pmatrix}+\beta(\cos t-i\sin t)\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$=\alpha\left(\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}+i\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\right)+\beta\left(\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}-i\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}\right)$$

$$=\alpha'\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}+\beta'\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

$(\alpha'=\alpha+\beta,\ \beta'=i\alpha-i\beta)$.

$$X' = AX + F(t) \quad (2)$$

Let $X_0(t)$ be the general solution of $X' = AX$ and $X^*(t)$ is a solution of (2), then any solution of (2) can be written as $X(t) = X_0(t)C + X^*(t)$.

Proof: For any solution of (2) $X(t)$, we have

$X(t) - X^*(t)$ is a solution of $X' = AX$.

Then, $X(t) - X^*(t) = X_0(t)C$. □.

The method to find such a $X^*(t)$:

Assume $X^*(t) = X_0(t)C(t)$, (3)

then take (3) in (2),

$$X_0'(t)C(t) + X_0(t)C'(t) = A(t)X_0(t)C(t) + F(t)$$

Since $X_0'(t) = A(t)X_0(t)$, then

$$X_0(t)C'(t) = F(t).$$

$$C'(t) = X_0^{-1}(t)F(t).$$

$$C(t) = \int_{t_0}^{t} X_0^{-1}(t)F(t)\,dt$$

Now we get such a $X^*(t) = X_0(t)C(t)$. □