1. (a) Let $\vec e_1, \vec e_2, \ldots , \vec e_n$ be the standard basis vectors of $\mathbb{R}^n$:

$$\vec e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots , \vec e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$ 

Let $\vec f_1, \vec f_2, \ldots , \vec f_m$ be the standard basis vectors of $\mathbb{R}^m$.

Let $A$ be an $m \times n$ matrix.

- For $1 \leq j \leq n$, show that $A\vec e_j$ is equal to the $j$-th column of $A$.
- For $1 \leq i \leq m$, show that $\vec f_i^\top A$ is equal to the $i$-th row of $A$.

(b) Show that, given two $m \times n$ matrices $A$ and $B$, the condition $A\vec v = B\vec v$ for all $\vec v \in \mathbb{R}^n$ implies that $A = B$, i.e.:

$$A_{ij} = B_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$ 

2. Let $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear map which first reflects a vector over the $xz$-plane, and then rotates it counter-clockwise about the $z$-axis by $\pi/6$. Find the matrix corresponding to $A$.

(Here, we identify the $x$, $y$ and $z$-axes with the axes pointing in the directions of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively. The “clock” in counter-clockwise faces in the direction of the $z$-axis.)

3. (a) Let:

$$A = \begin{pmatrix} 4 & 10 \\ -7 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -7 \\ 10 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 9 & -2 \\ -3 & 1 \end{pmatrix}.$$
Verify that:

\[ A(B + C) = AB + AC. \]

(b) From the definition of matrix addition and multiplication:

\[ (A + B)_{ij} = A_{ij} + B_{ij}, \quad (AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}, \]

show that, for any \( m \times n \) matrix \( A \), and \( n \times l \) matrices \( B, C \), we have:

\[ A(B + C) = AB + AC. \]

4. (a) Let:

\[
A = \begin{pmatrix}
1 & -1 & 3 & -5 \\
2 & 0 & -1 & 3 \\
7 & 9 & -4 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 4 \\
1 & 5 \\
-3 & 0 \\
0 & -6
\end{pmatrix}, \\
C = \begin{pmatrix}
-2 & 0 & 3 & 1 \\
5 & -7 & 0 & 4
\end{pmatrix}.
\]

Verify that \((AB)C = A(BC)\).

(b) \((Optional)\) Show that for any \( m \times n \) matrix \( A \), \( n \times l \) matrix \( B \), and \( l \times r \) matrix \( C \), we have:

\[ A(BC) = (AB)C. \]

5. Find all \( 2 \times 2 \) matrices \( A \) such that \( A^2 = 0 \), the zero matrix.

6. Solve the following system of linear equations by performing row reduction on the associated augmented matrix:

\[
x_1 - x_2 + 5x_3 + 7x_4 = -23 \\
2x_1 + 4x_3 - 4x_4 = -16 \\
3x_2 - 2x_4 = 0 \\
5x_1 - x_4 = 10
\]

7. For what values of \( a, b, c \in \mathbb{R} \) would the following matrix equation have a unique solution \( \vec{x} \in \mathbb{R}^3 \)?

\[
\begin{pmatrix}
2 & 0 & 4 \\
1 & 1 & 3 \\
a & b & c \\
-1 & -5 & -7
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]
8. (Optional) An (undirected) graph consists of two sets of data: A set of points, called vertices, and a set of unordered pairs of vertices, called edges. For example, the graph with vertices \( \{V_1, V_2, V_3, V_4, V_5\} \) and edges
\[
\{\{V_1, V_2\}, \{V_1, V_3\}, \{V_1, V_4\}, \{V_4, V_5\}\}
\]
may be visualized as follows:

![Graph Diagram]

The adjacency matrix of a graph with \( n \) vertices is an \( n \times n \) matrix \( A = (A_{ij}) \) defined by:

\[
A_{ij} = \begin{cases} 
1 & \text{if } \{V_i, V_j\} \text{ is an edge of the graph,} \\
0 & \text{if there is no edge connecting } V_i \text{ and } V_j. 
\end{cases}
\]

In the example above, the corresponding adjacency matrix is:

\[
A = \begin{pmatrix}
0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

Notice that it is symmetric, i.e. \( A^\top = A \).

A walk in a graph is a sequence of edges linking one vertex to another. The number of edges in the sequence is called the length of the walk. In the example above, the sequence \( \{V_1, V_4\}, \{V_4, V_5\} \) is a walk of length two from \( V_1 \) to \( V_5 \).

Prove the following theorem:
Theorem. Let $A = (A_{ij})$ be the adjacency matrix of a graph. Show that, for any integer $k \geq 1$, the number $(A^k)_{ij}$ (the $ij$-th entry of $A^k$) is equal to the number of walks of length $k$ from $V_i$ to $V_j$. 