Materials & Tutorial for 1010a,b Weeks 4 (Tues)

Topics covered in lectures (either group A or group B)

A Word on Notations:
In the following, we will use '$\lim_{x\to c}$' and '$\lim\limits_{x\to c}$' interchangeably.

(Must-know)

(1) Definition of Function, domain, co-domain, range, injective functions, conditions guaranteeing existence of inverse function
(2) Examples of functions, (i) polynomials (each degree n polynomial equation has at most n roots), (ii) sine, cosine functions; (iii) exp, log functions; <u>log</u> as 'inverse' function of <u>exp</u> function
(3) The relationship between
$$e \stackrel{\text{def}}{=} 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \qquad \& \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
(4) The relationship between $\exp(x)$ & e^x
(5) Definition of derivative as a 'limit'
(6) 'Derivative exists at c' implies '<u>continuous</u> at c'
(7) Need of the concept of 'limit' in order to define 'derivative'
(8) 'Function f has derivative at c'
$\Longrightarrow$ f satisfies '$\lim_{x\to c}(f(x) - f(c)) = 0$',
$\Longrightarrow$ '$\lim_{x\to c} f(x) = f(c)$'.
(9) The last sentence in the point (8) above means:
 (i) $\lim\limits_{x\to c,\ x<0} f(x)$ exists (& <u>finite</u>);
 (Notations: $\boxed{\lim\limits_{x\to c,\ x<0}}$ is written as $\boxed{\lim\limits_{x\to 0^-}}$)
 (ii) $\lim\limits_{x\to c,\ x>0} f(x)$ exists (& <u>finite</u>);
 (Notations: $\boxed{\lim\limits_{x\to c,\ x>0}}$ is written as $\boxed{\lim\limits_{x\to 0^+}}$) AND
 (iii) 'both' of these limits are <u>equal to</u> the <u>VALUE</u> of f at c.
 (iv)* A function f satisfying (i),(ii) & (iii) is called '<u>continuous</u> at the point $x = c$'

i.e. $\boxed{\lim_{x\to c} f(x) = f(c)}$ means $\boxed{f \text{ is } \underline{\text{continous}} \text{ at } x = c}$

(10) If 'limit' exists, it is unique
(Comments:
- Note how mathematicians usually 'formulate' uniqueness!)
- To show 'uniqueness' of limits, we may need the $\epsilon - N$ definition in point (3) below.)

(11) $+, -, \times, \div$ properties of 'limits' (Proofs omitted!)

(Good-to-Know, but NOT "Must-know")

(1) Generalized Binomial Theorem
(2) Euler's Formula $\exp(\sqrt{-1} \cdot x) = \cos x + \sqrt{-1} \cdot \sin x$, ($x$ is any real no., measured in 'radians')

(You don't need to know yet)

* $\epsilon - N$ definition of the phrase '$\lim_{x\to\infty} f(x) = L$'

(Topics mentioned, not yet fully discussed.
BUT used in Assmt 1)

- The fact: '$f' > 0$' implies 'f is strictly increasing'
  Questions:
  (1) What is the 'definition' of 'strictly increasing'?
  (2) For what 'values' of x is $f' > 0$? For what values of x is f strictly increasing?
- Method of graph (i.e. 'curve' defined by $y = f(x)$) sketching
- Concept of asymptotes
- '2nd derivative > 0' (& '1st derivative $= 0$') implies 'local minimum' (also known as 'relative minimum')
- '2nd derivative < 0' (& '1st derivative $= 0$') implies 'local maximum' (also known as 'relative maximum')
- Definition of local minimum (maximum)

Assignments for Week 3 Tutorial

Topics

(1) Derivative from First Principle (also known as 'from Definition')
(2) Simple Curve sketching
(3) Method of using strictly increasing/decreasing function to get strict inequality

Assignments

1. Consider the function $f(x) = (x-1)|x|$ defined on the domain $D = (-\infty, \infty)$.
   (a) Compute the limit $\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}$, when $c = 0$
   (b) Compute the limit $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$, when $c = 0$
   (c) Determine from (i) & (ii) above whether $f(x) = (x-1)|x|$ has derivative at $x = c = 0$.

2. (Skip if there isn't much time) Repeat the above question for the function $f(x) = (x-1)^b|x|$ where $b = 2, 3, 4, \cdots$

3. (Application of derivative to show '(strictly) increasing/decreasing(ness)' of a function')
   Let $f : (0, \infty) \to \mathbb{R}$ be a function given by
$$f(x) = 1 + \frac{x}{2} - \sqrt{1+x}$$
   (a) Show that $f'(x) > 0$
   (b) Deduce from (a) that the following inequality holds for any $x > 0$:
$$1 + \frac{x}{2} > \sqrt{1+x}$$
   (c) Similarly, show (by considering a certain function) that
$$\sqrt{1+x} > 1 + \frac{x}{2} - \frac{x^3}{8}, \quad \text{for any } x > 0.$$

4. (Comment: In this question, we will use the notation:
$$\lim_{x \to \infty} f(x) = \infty$$

to mean 'as x increases indefinitely, $f(x)$ increases indefinitely'. The meaning of

$$\lim_{x\to\infty} f(x) = -\infty$$

is similar.)

Consider the function $f(x) = \dfrac{x^n(x-1)^n}{(x-3)^n}$.

(a) $\boxed{n = 1 \text{ case}}$

(i) State what the maximal domain of f is.

(ii) State where the function has zeros (i.e for what x is it true that $f(x) = 0$)

(iii) State where the function is > 0, is < 0

(iv) State (and give simple reasons) what this limit is:
$$\lim_{x\to\infty} f(x)$$

(v) State (and give simple reasons) what this limit is:
$$\lim_{x\to-\infty} f(x)$$

(vi) What are the local maximum (or <u>local minimum</u>) value of the function f (if there is <u>any</u>) ? Give reasons.

(vii) Sketch the "graph" of the function (i.e. "draw rough shape of the <u>curve</u> $y = f(x)$)

5. $\boxed{n = 2 \text{ case}}$ Repeat the questions above.

(<u>Comment</u>:) This simple method works for all rational functions of the form

$$r(x) = \frac{(x-a_1)^{b_1}(x-a_2)^{b_2}\cdots(x-a_k)^{b_k}}{(x-c_1)^{d_1}(x-c_2)^{d_2}\cdot\cdots\cdot(x-c_m)^{b_m}},$$

where the exponents are all natural numbers.