Notes for 1010a,b Weeks 2 and 3

Topics:

($\boldsymbol{f \circ g}$, exp, ln)

1. Review of Logarithm

In school, we learned that $\log_{10} x$ is given by

$$^{*}10^y = x \iff y = \log_{10} x$$

Supposing that the above equation has always a solution. We can argue (by 'analogy') that 10 is not the only choice for "base" as any positive real number works. So if we choose e (see "Comments" below for a definition of e) to be the base and obtain

$$e^y = x,$$

then we have

$$e^{\log_e x} = x.$$

Notation

We will often abbreviate $\log_e x$ by the symbol $\ln x$.

Comments

(1) Assuming the equation $e^{\ln x} = x,\ x > 0$, together with the fact that (we'll prove it in point (3) below!)

$$e^w = \exp(w),$$

where w is any real no., we will show that the functions "ln" and "exp" are inverse functions of each other.

(In the Appendix section, i.e. section 3 below, we will discuss two things: (i) the concept of "inverse function" and (ii) under what conditions there exists an inverse function to a given function).

(2) What is the number e? We define (there are other definitions!) e to be the number given by the infinite sum

$$e = \exp(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots .$$

*How do we know this equation has solution y if the given x is not a natural no.?

(3) Using the above definition of e (which is approximately 2.718...), we can make sense of what we mean by e^w (at least) in the case when w is a natural number. In such a case, e^w is just given by the following expression:

$$
\begin{aligned}
e^w &= \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^w, \text{ where } w = 0, 1, 2, \cdots \\
&= \lim_{z\to\infty}\left(1+\frac{w}{z}\right)^z, \qquad (1.1)
\end{aligned}
$$

where $z \stackrel{\text{def}}{=} nw$ (which is also a natural no.)

However, if we want to obtain e^w for <u>any</u> real no. w, (not only for natural nos.!) then we have to use the "Generalized Binomial Theorem (GBT)", † where z is now <u>any real</u> no., to compute the term $\left(1+\frac{w}{z}\right)^z$ in (1.1).

Doing this, we get a completely analogous formula to (1.1) above, i.e.

$\boxed{\text{For simplicity, we assume } w > 0!}$

$$\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^w = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n\,w}$$

Now, let $z = nw$, then $\frac{1}{n} = \frac{w}{z}$ (or $n = \frac{z}{w}$), and we get

$$\left(1+\frac{1}{n}\right)^{nw} = \left(1+\frac{w}{z}\right)^z$$

As $n \to \infty$ we obtain $z \to \infty$, hence

$$e^w = \lim_{z\to\infty}\left(1+\frac{w}{z}\right)^z \qquad (1.2)$$

(• But now $z \stackrel{\text{def}}{=} nw$ is a <u>real</u> no., because w is a real no.!)

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†GBT says: $(1+x)^\alpha = 1 + \alpha + \frac{\alpha\cdot(\alpha-1)}{2!}x^2 + \frac{\alpha\cdot(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$, where in the 'numerator', one can put <u>any real no.</u>, but in the denominator, <u>only</u> natural numbers. (Question) For what values of x is this formula valid?

Conclusion Since

$$
\begin{aligned}
\exp(w) &= 1 + w + \frac{w}{2!} + \frac{w^3}{3!} + \cdots \\
&= \lim_{\widetilde{\boldsymbol{z}}\to\infty}\left(1+\frac{w}{\widetilde{\boldsymbol{z}}}\right)^{\widetilde{\boldsymbol{z}}}, \text{where } \widetilde{\boldsymbol{z}} \text{ is any natural no.} \qquad (1.3)\\
&= \lim_{\hat{z}\to\infty}\left(1+\frac{w}{z}\right)^{z} \text{ where } z \text{ is any real no.} \qquad (1.4)\\
&= e^w
\end{aligned}
$$

therefore, we conclude that

$$\exp(w) = e^w \text{ for any real no. } w. \qquad (1.5)$$

Comment: (1.3) = (1.4) because 'limit', if it 'exists' is unique.

Consequence 1
By (1.5), the equation

$$e^{\ln x} = x, \forall x > 0$$

is the same as

$$\exp(\ln x) = x, \forall x > 0$$

That is, "exp" is the inverse function of "ln" function.

Consequence 2
Using this, we get the formula $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$.

Now, we proceed to "prove" this (the proof is correct, but there are a lot of "holes" to be filled in!)

• Summary So Far
We have learned (i) polynomial (functions), (2) cosine, sine functions, (3) exp, ln functions, (4) $+, -, \times, \div$ of functions (• Note: when using $\div$, make sure that the denominator isn't zero!).

2. "Proof" of the Formula for logarithm

You can skip this section if you want. But do remember the formula for ln(1+x). It's important!

We start by considering the equation

$$e^{\ln(1+x)} = 1 + x$$

for each $1 + x > 0$.

By using the "product" definition of e^w (i.e. (1.2)), the left-hand side becomes

$$\lim_{n\to\infty}\left(1 + \frac{\ln(1+x)}{n}\right)^n,$$

hence for large n, we have

$$\left(1 + \frac{\ln(1+x)}{n}\right)^n \approx 1 + x.$$

Taking nth root on both sides of this, we obtain

$$1 + \frac{\ln(1+x)}{n} \approx (1+x)^{1/n}$$

Now when the Generalized Binomial Theorem is applied to the right-hand side of it, we obtain

$$1 + \frac{\ln(1+x)}{n} \approx 1 + \frac{\frac{1}{n}}{1!}\cdot x + \frac{\frac{1}{n}(\frac{1}{n}-1)}{2!}x^2 + \frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)}{3!}x^3 + \cdots$$

implying (after canceling the term "1" on both sides)

$$\frac{\ln(1+x)}{n} \approx \frac{\frac{1}{n}}{1!}\cdot x + \frac{\frac{1}{n}(\frac{1}{n}-1)}{2!}x^2 + \frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)}{3!}x^3 + \cdots$$

and implying (after multiplying by n on both sides)

$$\begin{aligned}
\ln(1+x) &\approx n\left\{\frac{\frac{1}{n}}{1!}\cdot x + \frac{\frac{1}{n}(\frac{1}{n}-1)}{2!}x^2 + \frac{\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2)}{3!}x^3 + \cdots\right\} \\
&\approx x + \frac{(\frac{1}{n}-1)}{2!}x^2 + \frac{(\frac{1}{n}-1)(\frac{1}{n}-2)}{3!}x^3 + \cdots \\
&\approx x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots
\end{aligned}$$

when n is very large (because all the terms involving $1/n$ go to zero as <u>n becomes indefinitely large</u> (In symbol, the underlined phrase is written as $\boxed{n \to \infty}$)). $\square$

<u>Comment</u>

To make the above proof watertight, one has to put a lot more effort. We will not go into this, however!

3. Appendix

In this Appendix, we give some remarks on the inverse function g of any given function $f : D \to E$. (As always, D, E are subsets of the set of real nos., i.e. the set denoted by $\mathbb{R}$ (or written as $(-\infty, \infty)$, or $\{x | -\infty < x < \infty\}$)

Inverse Function
Let $f : D \to E$ be a function, then another function $g : E \to D$ is an inverse function of f, provided that

(1) The range $R(f)$ (defined below) of f is equal to the co-domain E;

$\boxed{\text{This ensures that the domain of } g \text{ is } E.}$

(2) The function f is an injective function.

$\boxed{\text{This ensures that each } y \in E \text{ has 1 and only 1 output, i.e. } g(y)}$

In point (1) above, we mentioned the word "range". We now define it.

Range of a Function.
Let $f : D \to E$, then the range of f, denoted by the symbol $R(f)$ is the set

$$R(f) \stackrel{def}{=} \{y \in E \mid y = f(x), \text{ for some } x \in D\}$$

<u>Comment</u>
In daily language, the range of f is "the set of those y's reachable by all x in D". It is instructive to compare this way of talking about $R(f)$ with the one which we wrote down above.

3.1. **Conditions for Inverse Function.**

3.1.1. ***First Condition for Inverse Function: $R(f) = E$.*** This condition says:

Given <u>any</u> $y \in E$, we can <u>always</u> find $x \in D$ satisfying

$$y = f(x)$$

(In other words, the equation $y = f(x)$ has always a solution $x \in D$).

*** Digression (How to Sketch Inverse Function)**
In school math, we learned that to sketch inverse function of a given function f – one just "reflect" the curve (or "graph") of the function f (i.e. $y = f(x)$) about the straight line $y = x$. This method can be justified by (1) rotating the two axes (i.e. the $x-$ and the $y-$ axis counterclockwise by 90 degrees, then (2) change the negative $y-$ axis back to the position of the positive $y-$ axis by "reflecting" about the vertical axis.)[‡]

3.1.2. ***Second Condition for Inverse Function – Injectivity***. In order for a function $f : D \to E$ to have inverse function, f needs also to be <u>injective</u>. By this we mean any $f(x)$ in the range of f originates from <u>one and only one</u> element x.

The above description is, however, not too useful, so we reformulate it as

*** Another Way of Defining Injective Function**
$f : D \to E$ is <u>injective</u> if whenever $f(x_1) = f(x_2)$, then $x_1 = x_2$. Or more formally,

$$\forall x_1, x_2 \in D \mid \text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

A convenient condition to guarantee injectivity: df/dx > 0
Let $f : (a, b) \to \mathbb{R}$. Suppose that $\frac{df}{dx}\Big|_{x=c} > 0$ for each c in (a, b). Then we will show that f is <u>strictly</u> increasing for each c in (a, b).

<u>Comment</u>:
The same holds if we replace > 0 by < 0 throughout the domain.

[‡]The rotation of a point (x, y) in the $xy-$plane can be expressed using something called 2×2 matrix. If we denote a point (x, y) by a 2×1 (or "column") matrix, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, then the 90 degrees counterclockwise rotation can be shown to be given by the following object (known as 2×2 matrix):

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Since 'f is strictly increasing in (a,b)' $\implies$ 'f is injective' (Question: Why? Can you prove it?), to show injectivity, we only need to show $\frac{df}{dx}\big|_{x=c} > 0$ for each point c in the domain. This motivates us to study derivative of a function.

E.g.
Show that the function $f : (0,\infty) \to \mathbb{R}$ given by $f(x) = \frac{x^2-1}{x}$ has inverse function $g : \mathbb{R} \to (0,\infty)$.

Solution

(1) the range $R(f)$ is equal to the set $\mathbb{R}$, because for each $y \in \mathbb{R}$, we have

$$\begin{aligned} y &= \frac{x^2-1}{x} \\ &\implies x^2 - xy - 1 = 0 \\ &\implies x = \frac{y \pm \sqrt{y^2+4}}{2} \\ &\implies x = \frac{y + \sqrt{y^2+4}}{2} \end{aligned}$$

(the last line comes from our requirement that $x > 0$). Hence x can be solved for each $y > 0$

(2) Next, we can compute (alternatively, we can see from the formula in point (1) that x is uniquely determined by y)

$$\frac{d(x-(1/x))}{dx}$$

at each $x \in (0,\infty)$ to see that $\frac{df}{dx} > 0$, hence f is injective.

Conclusion
The function $f(x) = (x^2-1)/x$ from the domain $(0,\infty)$ to the range $\mathbb{R}$ has an inverse function.
(Question: What is the formula for this inverse function?)

4. A Word on Piecewise Defined Functions

Some students asked me the question: "What is the absolute value function?" This function, just as many other functions in this course, are known as "piecewise defined" functions. They are defined by different rules on different regions of the domain.

E.g.

$$f(x) = |x| \overset{\text{def}}{=} \begin{cases} -x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ x & \text{if} \quad x > 0 \end{cases}$$

<u>Comment:</u> This function is from the domain $\mathbb{R}$ to the co-domain $\mathbb{R}$. In the region $(-\infty, 0)$ the function is given by $-x$, in the region $\{0\}$, the function is given by 0 and in the region $(0, \infty)$, it is given by x.

This function gives rise to a curve $y = |x|$ which is $V-$shaped.

Question. Sketch the curve given by $y = |1 - |x||$. Compare it with the curve $y = (1 - x^2)^2$.