Eg 5.8 (iii)

initial tableau

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	1	3	4	1	0	30
x_5	1	4	-1	0	1	10
	-2	-7	3	0	0	0

optimal tableau

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	0	-1	5	1	-1	20
x_1	1	4	-1	0	1	10
x_0	0	1	1	0	2	20

$$\begin{array}{cc} & \begin{array}{cc}\vec{a}_4 & \vec{a}_1\end{array} \\ B = & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\end{array}$$

$$B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Suppose

$$A = \begin{pmatrix} 1 & 3 & 4 & 1 & 0 \\ 1 & 4 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \hat{A} = \begin{array}{c} \quad\ \hat{a}_2 \\ \begin{pmatrix} 1 & 1 & 4 & 1 & 0 \\ 1 & 3 & -1 & 0 & 1 \end{pmatrix}\end{array}$$

$$Y = \begin{pmatrix} 0 & -1 & 5 & 1 & -1 \\ 1 & 4 & -1 & 0 & 1 \end{pmatrix} \longrightarrow \hat{Y} = \begin{array}{c} \quad\ \hat{y}_2 \\ \begin{pmatrix} 1 & -2 & 4 & 1 & 0 \\ 1 & 3 & -1 & 0 & 1 \end{pmatrix}\end{array}$$

$$\left(\because \hat{A} = B\hat{Y} \Rightarrow \hat{a}_2 = B\hat{y}_2 \Rightarrow \hat{y}_2 = B^{-1}\hat{a}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right)$$

$$\begin{aligned} \vec{z} - \vec{c} &= \vec{c}_B^{\,T} Y - \vec{c} \\ &= (0, 2)\, Y - \vec{c} \\ &= (2\ 8\ {-2}\ 0\ 0) - (2\ 7\ {-3}\ 0\ 0) \\ &= (0\ 1\ 1\ 0\ 0) \end{aligned} \longrightarrow \begin{aligned} \hat{z} - \vec{c} &= \vec{c}_B^{\,T} \hat{Y} - \vec{c} \\ &= (0, 2)\, \hat{Y} - \vec{c} \\ &= (2\ 6\ {-2}\ 0\ 0) - (2\ 7\ {-3}\ 0\ 0) \\ &= (0, -1, 1, 0, 0) \end{aligned}$$

Thus the new tableau is

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	0	-2	5	1	-1	20
x_1	1	3*	-1	0	1	10
x_0	0	-1	1	0	2	20

After 1 iteration, we get the optimal tableau

	x_1	x_2	x_3	x_4	x_5	$\vec{b}$
x_4	$2/3$	0	$13/3$	1	$-1/3$	$80/3$
x_2	$1/3$	1	$-1/3$	0	$1/3$	$10/3$
x_0	$1/3$	0	$2/3$	0	$7/3$	$70/3$

#

Adding a new variable:

original

$\max \vec{c}^T\vec{x}$

$\begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0} \end{cases}$

new

$\max \vec{c}^T\vec{x} + c_n \cdot x_n$

$\begin{cases} A\vec{x} + \vec{a}_n x_n = \vec{b} \\ \vec{x}, x_n \geq 0 \end{cases}$

A	$\vec{b}$
$-\vec{c}^T$	0

$\Downarrow$

Y	$\vec{x}_B = B^{-1}\vec{b}$
$-(\vec{c}^T - \vec{z}^T)$	$x_0 = \vec{c}_B^T \vec{x}_B$

A	$\vec{a}_n$	$\vec{b}$
$-\vec{c}^T$	$-c_n$	0

$\Downarrow$

Y	$\vec{y}_n$	$\vec{x}_B = B^{-1}b$
$-(\vec{c}^T - \vec{z}^T)$	$-(c_n - z_n)$	x_0

$x_n = 0$

$\Downarrow$

$A\vec{x} = \vec{b}$

$\Downarrow$

$\vec{x}_B$ is still feasible

$\begin{cases} \vec{y}_n = B^{-1}\vec{a}_n, \\ z_n = \vec{c}_B^T B^{-1}\vec{a}_n \end{cases}$

feasible but not necessarily optimal.

Eg 5.8 Suppose

initial

						x_6	
x_4	1	3	4	1	0	1	30
x_5	1	4	-1	0	1	1	10
	-2	-7	3	0	0	-4	0

$\longrightarrow$

optimal

						x_6	
x_4	0	-1	5	1	-1	0	20
x_1	1	4	-1	0	1	1	10
	0	1	1	0	2	-2	20

feasible but not optimal

$$\vec{y}_6 = B^{-1}\vec{a}_6 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$z_6 = \vec{c}_B^T \vec{y}_6 = (0, 2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2$$

$$-(c_6 - z_6) = -(4-2) = -2$$

Constrained Optimization

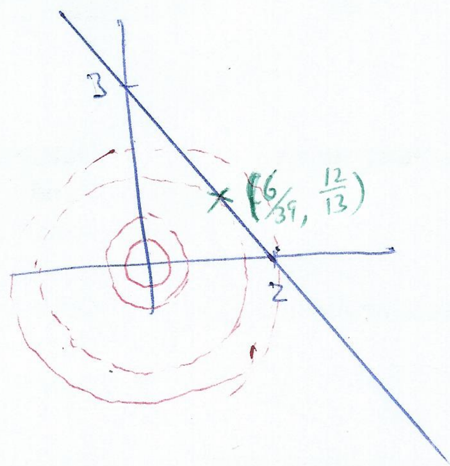
$\min f(x)$

s.t. $g(x) = 0$

Lagrange Multiplier method

$\min \mathcal{J}(x) = f(x) + \lambda g(x)$ unconstrained optimization

$$\begin{cases} \frac{\partial \mathcal{J}(x)}{\partial x} = f'(x) + \lambda g'(x) = 0 \\ \frac{\partial \mathcal{J}(x)}{\partial \lambda} = g(x) = 0 \end{cases}$$



Eg 1. $\min x^2 + y^2$

s.t. $3x + 2y = 6$

$\min \mathcal{J}(x, y) = x^2 + y^2 + \lambda(3x + 2y - 6)$

$\frac{\partial \mathcal{J}}{\partial x} = 2x + 3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}x \Rightarrow x = -\frac{3}{2}\lambda$

$\frac{\partial \mathcal{J}}{\partial y} = 2y + 2\lambda = 0 \Rightarrow \lambda = -y \Rightarrow y = -\lambda$

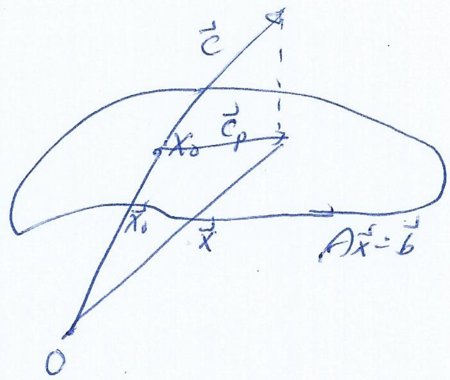
$\frac{\partial \mathcal{J}}{\partial \lambda} = 3x + 2y - 6 = 0 \Rightarrow 3(-\frac{3}{2}\lambda) + 2(-\lambda) = 6 \Rightarrow -\frac{13}{2}\lambda = 6 \Rightarrow \lambda = -\frac{12}{13}$

$\therefore x = +\frac{6}{39}, \ y = +\frac{12}{13}$

Given $A\vec{x}_0 = \vec{b}$

$$\min \ \| \vec{x} - (\vec{x}_0 + \vec{c}) \|_2^2$$

$$A\vec{x} = \vec{b}$$



$$\Rightarrow \min_{A\vec{x}=b} \left\{ \| \vec{x} - (\vec{x}_0 + \vec{c}) \|_2^2 + \vec{\lambda}^T (A\vec{x} - \vec{b}) \right\}$$

$$\Rightarrow \begin{cases} \dfrac{\partial}{\partial x} = \vec{x} - (\vec{x}_0 + \vec{c}) + A^T\vec{\lambda} = 0 & (1) \\ \dfrac{\partial}{\partial \lambda} = A\vec{x} - \vec{b} = 0 & (2) \end{cases}$$

$$\Rightarrow \quad \vec{x} \overset{(1)}{=} \vec{x}_0 + \vec{c} - A^T\vec{\lambda} \qquad (3)$$

$$\overset{(2)}{\Rightarrow} \quad A(\vec{x}_0 + \vec{c}) - AA^T\vec{\lambda} = \vec{b}$$

$$\Rightarrow \quad A\vec{c} - AA^T\vec{\lambda} = \vec{0}$$

$$\Rightarrow \quad \vec{\lambda} = (AA^T)^{-1}A\vec{c} \qquad (4)$$

$$\overset{(3)}{\Rightarrow} \quad \vec{x} = \vec{x}_0 + \vec{c} - A^T(AA^T)^{-1}A\vec{c}$$

$$= \vec{x}_0 + \left(I - A^T(AA^T)^{-1}A\right)\vec{c}$$

$$\Rightarrow \quad \vec{c}_p = \left(I - A^T(AA^T)^{-1}A\right)\vec{c} \qquad \#$$

$\text{Min} \quad z = x_1 - 2x_2 \qquad \vec{c}^T = (1, -2, 0)$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \\ x_1, x_2, x_3 \geq 0 \end{cases} \qquad A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Iteration 1 initial solution $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

To transform $\vec{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ to $\tilde{x} = (1, 1, 1)$

$$\underset{\vec{x}}{\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}} = \underbrace{\begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & 1/3 \end{pmatrix}}_{D} \underset{\tilde{x}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$$\tilde{x} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} x$$

$\text{Min} \quad z = 1/3\,\tilde{x}_1 - 2/3\,\tilde{x}_2$

$$\begin{cases} 1/3\,\tilde{x}_1 - 2/3\,\tilde{x}_2 + 1/3\,\tilde{x}_3 = 0 \\ 1/3\,\tilde{x}_1 + 1/3\,\tilde{x}_2 + 1/3\,\tilde{x}_3 = 1 \\ \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \geq 0 \end{cases}$$

$$\tilde{A} = AD = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & 1/3 \end{pmatrix} = \begin{pmatrix} 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\tilde{c} = Dc = \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & 1/3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/3 \\ 0 \end{pmatrix}$$

$$P = I - \tilde{A}^T (\tilde{A}\tilde{A}^T)^{-1} \tilde{A}$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \left[\begin{pmatrix} 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \\ 1/3 & 1/3 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$\vec{c}_p = P\tilde{c} = \begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1/3 \\ -2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/6 \\ 0 \\ -1/6 \end{pmatrix}$$

$$\tilde{x}_{new} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\alpha}{\left(\frac{1}{6}\right)} \begin{pmatrix} 1/6 \\ 0 \\ -1/6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(Note that $\tilde{A}\tilde{x}_{new} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ \forall \alpha$ & $\tilde{x}_{new} \geq 0 \quad \forall \alpha \leq 1$)

Let's choose $\alpha = \frac{1}{2}$, then new $\hat{x}$ is

$$\hat{x}_{new} = \begin{pmatrix} 3/2 \\ 1 \\ 1/2 \end{pmatrix} \Rightarrow \vec{x}_{new} = D\hat{x}_{new} = \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & 1/3 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}$$

Iteration 2: Initial solution is $x = (1/2, 1/3, 1/6)$

To transform $\vec{x} = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ to $\hat{x} = (1, 1, 1)$

$$\underset{\vec{x}}{\begin{pmatrix} 1/2 \\ 1/3 \\ 1/6 \end{pmatrix}} = \underset{D}{\begin{pmatrix} 1/2 & & \\ & 1/3 & \\ & & 1/6 \end{pmatrix}} \underset{\hat{x}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \Rightarrow \hat{x} = \begin{pmatrix} 2 & & \\ & 3 & \\ & & 6 \end{pmatrix} x$$

$$\Downarrow$$

$$\hat{A} = AD = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1/2 & & \\ & 1/3 & \\ & & 1/6 \end{pmatrix} = \begin{pmatrix} 1/2 & -2/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \end{pmatrix}$$

$$\min \; Z = \tfrac{1}{2}\hat{x}_1 - \tfrac{2}{3}\hat{x}_2$$

$$\begin{cases} \frac{1}{2}\hat{x}_1 - \frac{2}{3}\hat{x}_2 + \frac{1}{6}\hat{x}_3 = 0 \\ \frac{1}{2}\hat{x}_1 + \frac{1}{3}\hat{x}_2 + \frac{1}{6}\hat{x}_3 = 1 \end{cases}$$

$$\hat{x}_1, \hat{x}_2, \hat{x}_3 \geq 0$$

$$\hat{c} = DC = \begin{pmatrix} 1/2 & & \\ & 1/3 & \\ & & 1/6 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -2/3 \\ 0 \end{pmatrix}$$

$$P = I - \hat{A}^T(\hat{A}\hat{A}^T)^{-1}\hat{A} = \begin{pmatrix} 1/10 & 0 & -3/10 \\ 0 & 0 & 0 \\ -3/10 & 0 & 9/10 \end{pmatrix}$$

$$\vec{c}_p = P\hat{c} = \begin{pmatrix} 1/10 & 0 & -3/10 \\ 0 & 0 & 0 \\ -3/10 & 0 & 9/10 \end{pmatrix}\begin{pmatrix} 1/2 \\ -2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/20 \\ 0 \\ -3/20 \end{pmatrix}$$

$$\hat{x}_{new} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\alpha}{\left(\frac{3}{20}\right)}\begin{pmatrix} 1/20 \\ 0 \\ -3/20 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \alpha\begin{pmatrix} 1/3 \\ 0 \\ -1 \end{pmatrix}$$

(Note that $\hat{A}\hat{x}_{new} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; \forall \alpha$
$\hat{x}_{new} \geq 0 \quad \forall \alpha \leq 1$)

Choosing $\alpha = \frac{1}{2}$, $\hat{x}_{new} = \begin{pmatrix} 7/6 \\ 1 \\ 1/2 \end{pmatrix} \Rightarrow \vec{x}_{new} = \begin{pmatrix} 1/2 & & \\ & 1/3 & \\ & & 1/6 \end{pmatrix}\begin{pmatrix} 7/6 \\ 1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 7/12 \\ 1/3 \\ 1/12 \end{pmatrix}$ #

primal

$$\max \ \vec{c}^T \vec{x}$$

$$\text{s.t.} \begin{cases} A\vec{x} = \vec{b} \\ \vec{x} \geq \vec{0} \end{cases}$$

Dual

$$\min \ \vec{b}^T \vec{y} \qquad \qquad \min \ \vec{b}^T \vec{y}$$

$$\text{s.t.} \begin{cases} A^T\vec{y} \geq \vec{c} \\ \vec{y} \text{ free} \end{cases} \Rightarrow \text{s.t.} \begin{cases} A^T\vec{y} - \vec{s} = \vec{c} \\ \vec{s} \geq \vec{0} \end{cases}$$

Duality gap : at optimal $\quad x_i s_i = 0 \quad \forall i$

not at optimal $\quad x_i \cdot s_i \geq 0 \quad \forall i$

Define:

$$\left.\begin{cases} A^T\vec{y} - \vec{s} - \vec{c} = \vec{0} \\ A\vec{x} - \vec{b} = \vec{0} \end{cases}\right\} ①$$

$$\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\vec{x} \geq 0, \ \vec{s} \geq 0$$

Solve

$$g(\vec{x}, \vec{y}, \vec{s}) = \begin{pmatrix} A^T\vec{y} - \vec{s} - \vec{c} \\ A\vec{x} - \vec{b} \\ (X)(S)\vec{e} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{s.t.} \quad \vec{x} \geq \vec{0}, \ \vec{s} \geq \vec{0} \qquad ②$$

$$F = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : ① \ ② \text{ are satisfied} \right\}$$

$$F^i = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : ① \text{ is satisfied } + \vec{s} > 0, \ \vec{x} > 0 \right\}$$

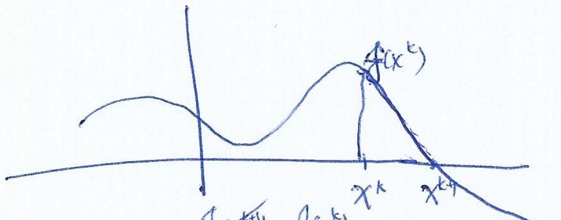
Naive Interior Point Method.

① start with $(x^0, y^0, z^0) \in \mathcal{F}^i$

② determine direction $(\delta_x^k, \delta_y^k, \delta_z^k)$

$$\begin{pmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{pmatrix} = \begin{pmatrix} x^k \\ y^k \\ z^k \end{pmatrix} + \begin{pmatrix} \delta_x^k \\ \delta_y^k \\ \delta_z^k \end{pmatrix}$$ by Newton's Method.

$$g(\vec{x}) = \vec{0}$$



$$\frac{\cancel{g(x^{k+1})} - g(x^k)}{x^{k+1} - x^k} = g'(x_k)$$

$$-\frac{g(x^k)}{g'(x^k)} = x^{k+1} - x^k$$

$$x^{k+1} = x^k - (g'(x^k))^{-1} g(x_k)$$

$$\begin{pmatrix} \delta_x^k \\ \delta_y^k \\ \delta_z^k \end{pmatrix} = -(g'(\vec{x}_k))^{-1} g(\vec{x}_k)$$

$$(g'(\vec{x}_k)) \begin{pmatrix} \delta_x^k \\ \delta_y^k \\ \delta_z^k \end{pmatrix} = -g(\vec{x}_k)$$

$$\begin{pmatrix} 0 & A^T & -I \\ A & 0 & 0 \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \delta_x^k \\ \delta_y^k \\ \delta_z^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -X^k S^k e \end{pmatrix}$$ if $\vec{x}_k \in \mathcal{F}^i$.

But $\begin{pmatrix} x^{k+1} \\ y^{k+1} \\ z^{k+1} \end{pmatrix}$ may $\notin \mathcal{F}$