Eg 5.6

primal optimal

$$\begin{array}{l} \text{structural} \\ \\ \text{slack} \end{array} \begin{pmatrix} 0 \\ \frac{1}{2} \\ \hline 0 \\ 0 \end{pmatrix}$$

dual optimal

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ \hline 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \hline 2 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \text{structural} \\ \\ \text{surplus} \end{array}$$

Eg 5.5

primal optimal

$$\begin{array}{l} \text{structural } x_1 \\ \phantom{\text{structural }} x_2 \\ \\ \text{slack} \end{array} \begin{pmatrix} 4 \\ 3 \\ \hline 2 \\ 5 \\ 0 \\ 0 \\ 4 \end{pmatrix}$$

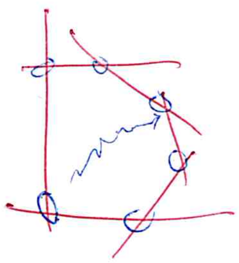
dual optimal

$$\begin{pmatrix} 0 \\ 0 \\ \hline 0 \\ 0 \\ 5/2 \\ 1/2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 5/2 \\ 1/2 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix} \begin{array}{l} \text{structural} \\ \\ \text{surplus} \end{array}$$

---

at optimal $\underset{\text{Thm 5.8}}{\overset{\text{Thm 5.7}}{\Longleftrightarrow}}$ $\underset{\text{dual surplus}}{\underbrace{(u_s^*)_i}} \cdot \underset{\text{primal structure}}{\underbrace{x_i^*}} = 0 \quad \forall i$

≥ 0 (before optimal)



$\underset{\text{dual structure}}{\underbrace{(u^*)_i}} \underset{\text{primal slack}}{\underbrace{(x_s^*)_i}} = 0 \quad \forall i$

≥ 0 (before optimal)

Duality Gap

Interior-pt method

Suppose $\exists \vec{u}_0 \in FR_d$ s.t. $\vec{b}^T \vec{u}_0$ is finite

$\Rightarrow$ optimal value for the primal is finite

$\vec{c}^T \vec{x} \le \vec{b}^T \vec{u}_0$ (weak Duality Thm)

$\forall \vec{x} \in FR_p$

$\Rightarrow \underbrace{\vec{c}^T \vec{x}^*}_{\text{optimal}} \le \vec{b}^T \vec{u}_0 = \text{finite}$

---

primal is feasible , ~~dual is~~ $FR_d = \phi$

$\Rightarrow \underbrace{\vec{c}^T \vec{x}^*}_{\text{optimal}} \to \infty.$

pf: By contradiction. if $\exists \vec{x}^*$ s.t. $\underbrace{\vec{c}^T \vec{x}^*}_{\text{finite}} \ge \vec{c}^T \vec{x} \ \forall \vec{x} \in FR_p$

By the proof the strong duality Thm

$\vec{u}_0 \equiv B^{-1} \vec{c}_B$ feasible & optimal

Contradiction

Thm 5.7 If $\vec{X}, \vec{u}$ are optimal ($\vec{X}_s, \vec{u}_s$ are optimal)

then $x_i \cdot u_{s_i} = 0 \quad \forall i$

$u_i \cdot x_{s_i} = 0 \quad \forall i$

pf: $\vec{u}^T A \vec{x} + \vec{u}^T \vec{X}_s = \vec{u}^T \vec{b} = \vec{b}^T \vec{u}$

$\vec{x}^T A \vec{u} - \vec{x}^T \vec{u}_s = \vec{x}^T \vec{c} = \vec{c}^T \vec{x}$

$\vec{b}^T \vec{u} = \vec{c}^T x$

$\Updownarrow$ (optimal weak Duality Th)

$\vec{u}^T \vec{x}_s + \vec{x}^T \vec{u}_s = 0$

$\Downarrow$ $\Updownarrow$

$\vec{u}^T \vec{x}_s = 0$ & $\vec{x}^T \vec{u}_s = 0$

$\Downarrow$ $\Downarrow$ $\Updownarrow$

$\forall i \quad u_i \cdot x_{s_i} = 0, \quad x_i \cdot u_{s_i} = 0$ ※

Thm 5.8 $x_i \cdot u_{s_i} = 0, \; u_i \cdot x_{s_i} = 0 \Rightarrow \vec{u}, \vec{x}$ are optimum.

Primal

initial table

$$\left[\begin{array}{ccccc|c} 2 & +2 & 1 & 1 & 0 & 4 \\ 1 & +2 & 2 & 0 & 1 & 6 \\ \hline -1 & -4 & -3 & 0 & 0 & 0 \end{array}\right] \leftarrow \text{feasible} \qquad \frac{4}{2},\ \frac{6}{2}$$

not optimal

enter

$$\left[\begin{array}{c|c} X & \begin{matrix}1\\2\end{matrix} \\ \hline 2\ 0\ 0\ 1\ 1 & 10 \end{array}\right] \quad \text{feasible}$$

optimal

Dual

$$\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 2 & 2 & -4 \\ 1 & 2 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 4 & 6 & 0 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{leaving} \\ \text{not} \\ \text{feasible} \end{array}$$

optimal

$$\left[\begin{array}{c|c} X^T & \begin{matrix}2\\0\\0\\1\\1\end{matrix} \\ \hline 1\ 2 & 10 \end{array}\right] \quad \text{feasible}$$

optimal

Dual Simplex Method

primal: not feasible but optimal

$\rightsquigarrow$ feasible & optimal

Part 2. Determine the entering basic variable: Select the nonbasic variable whose coefficient in Eq. (0) reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to Eq. (0). This selection is made by checking the nonbasic variables with *negative coefficients* in that equation (the one containing the leaving basic variable) and selecting the one with the smallest ratio of the Eq. (0) coefficient to the absolute value of the coefficient in that equation.

Part 3. Determine the new basic solution: Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination (see Appendix 4). When we set the nonbasic variables equal to zero, each basic variable (and Z) equals the new right-hand side of the one equation in which it appears (with a coefficient of $+1$).

3. *Feasibility test:* Determine whether this solution is feasible (and therefore optimal): Check to see whether all the basic variables are *nonnegative*. If they are, then this solution is feasible, and therefore optimal, so stop. Otherwise, go to the iterative step.

To fully understand the dual simplex method, you must realize that the method proceeds just as if the *simplex method* were being applied to the complementary basic solutions in the *dual problem*. (In fact, this interpretation was the motivation for constructing the method as it is.) Part 1, determining the leaving basic variable, is equivalent to determining the entering basic variable in the dual problem. The variable with the largest negative value corresponds to the largest negative coefficient in Eq. (0) of the dual problem (see Table 6.3). Part 2, determining the entering basic variable, is equivalent to determining the leaving basic variable in the dual problem. The coefficient in Eq. (0) that reaches zero first corresponds to the variable in the dual problem that reaches zero first. The two criteria for stopping the algorithm are also complementary.

We shall now illustrate the dual simplex method by applying it to the *dual problem* for the Wyndor Glass Co. (see Table 6.1). Normally this method is applied directly to the problem of concern (a primal problem). However, we have chosen this problem because you have already seen the simplex method applied to *its* dual problem (namely, the primal problem[1]) in Table 4.8 so you can compare the two. To facilitate the comparison, we shall continue to denote the decision variables in the problem being solved by y_i rather than x_j.

In *maximization* form, the problem to be solved is

$$\text{Maximize} \quad Z = -4y_1 - 12y_2 - 18y_3,$$

subject to

$$y_1 + 3y_3 \geq 3$$
$$2y_2 + 2y_3 \geq 5$$

and

$$y_1 \geq 0, \quad y_2 \geq 0, \quad y_3 \geq 0.$$

After the functional constraints are converted to $\leq$ form and the slack variables are introduced, the initial set of equations is that shown for iteration 0 in Table 9.2. Notice that all the coefficients in Eq. (0) are nonnegative, so the solution is optimal if it is feasible.

Table 9.2 **Dual Simplex Method Applied to Wyndor Glass Co. Dual Problem**

Iteration	Basic Variable	Eq. No.	Coefficient of: Z	y_1	y_2	y_3	y_4	y_5	Right Side
0	Z	0	1	4	12	18	0	0	0
	y_4	1	0	-1	0	-3	1	0	-3
	y_5	2	0	0	-2	-2	0	1	-5
1	Z	0	1	4	0	6	0	6	-30
	y_4	1	0	-1	0	-3	1	0	-3
	y_2	2	0	0	1	1	0	$-\frac{1}{2}$	$\frac{5}{2}$
2	Z	0	1	2	0	0	2	6	-36
	y_3	1	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	0	1
	y_2	2	0	$-\frac{1}{3}$	1	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{3}{2}$

The initial basic solution is $y_1 = 0$, $y_2 = 0$, $y_3 = 0$, $y_4 = -3$, $y_5 = -5$, with $Z = 0$, which is not feasible because of the negative values. The leaving basic variable is y_5 ($5 > 3$), and the entering basic variable is y_2 ($\frac{12}{2} < \frac{18}{2}$), which leads to the second set of equations, labeled as iteration 1 in Table 9.2. The corresponding basic solution is $y_1 = 0$, $y_2 = \frac{5}{2}$, $y_3 = 0$, $y_4 = -3$, $y_5 = 0$, with $Z = -30$, which is not feasible.

The next leaving basic variable is y_4, and the entering basic variable is y_3 ($\frac{6}{3} < \frac{4}{1}$), which leads to the final set of equations in Table 9.2. The corresponding basic solution is $y_1 = 0$, $y_2 = \frac{3}{2}$, $y_3 = 1$, $y_4 = 0$, $y_5 = 0$, with $Z = -36$, which is feasible and therefore optimal.

Notice that the optimal solution for the *dual* of this problem[1] is $x_1^* = 2$, $x_2^* = 6$, $x_3^* = 2$, $x_4^* = 0$, $x_5^* = 0$, as was obtained in Table 4.8 by the simplex method. We suggest that you now trace through Tables 9.2 and 4.8 simultaneously and compare the complementary steps for the two mirror-image methods.

9.3 Parametric Linear Programming

At the end of Sec. 6.7 we described *parametric linear programming* and its use for conducting sensitivity analysis systematically by gradually changing various model parameters simultaneously. We shall now present the algorithmic procedure, first for the case where the c_j parameters are being changed and then where the b_i parameters are varied.

Systematic Changes in the c_j Parameters

For the case where the c_j parameters are being changed, the *objective function* of the ordinary linear programming model,

$$Z = \sum_{j=1}^{n} c_j x_j,$$

[1] The *complementary optimal basic solutions property* presented in Sec. 6.3 indicates how to read the optimal solution for the dual problem from row 0 of the final simplex tableau for the primal problem. This ... of whether the simplex method or the dual simplex method is used to

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Given any feasible dual solution $[\mathbf{u}, \mathbf{u}_s]$ and any column n-vector $\mathbf{x}$, (5.4) implies that

$$\mathbf{x}^T A^T \mathbf{u} - \mathbf{x}^T \mathbf{u}_s = \mathbf{x}^T \mathbf{c} = \mathbf{c}^T \mathbf{x}, \tag{5.6}$$

Since the cost coefficients of all slack and surplus variables are zero, we see that if $[\mathbf{x}_0, \mathbf{x}_{0s})]$ and $[\mathbf{u}_0, \mathbf{u}_{0s})]$ are optimal solutions to the primal and the dual problems, then

$$\mathbf{c}^T \mathbf{x}_0 = \mathbf{b}^T \mathbf{u}_0.$$

Hence by (5.5) and (5.6), we have

$$\mathbf{u}_0^T \mathbf{x}_{0s} + \mathbf{x}_0^T \mathbf{u}_{0s} = 0.$$

Using the fact that $\mathbf{u}_0$, $\mathbf{x}_{0s}$, $\mathbf{x}_0$, $\mathbf{u}_{0s} \geq \mathbf{0}$, we finally have $\mathbf{u}_0^T \mathbf{x}_{0s} = 0 = \mathbf{x}_0^T \mathbf{u}_{0s}$. □

We remark that the converse of Theorem 5.7 is also true.

Theorem 5.8. *Let $[\mathbf{x}_0, \mathbf{x}_{0s}]$ be feasible solution to primal and $[\mathbf{u}_0, \mathbf{u}_{0s}]$ be feasible solution to dual. Suppose that*

(i) for each i, $i = 1, 2, \cdots, m$, the product of the ith primal slack variable and ith dual variable is zero, and

(ii) for each j, $j = 1, 2, \cdots, n$, the product of the jth primal variable and jth surplus dual variable is zero.

Then $[\mathbf{x}_0, \mathbf{x}_{0s}]$ and $[bou_0, \mathbf{u}_{0s}]$ are optimal solutions to the primal and the dual respectively.

Proof. By assumption, $\mathbf{u}_0^T \mathbf{x}_{0s} + \mathbf{x}_0^T \mathbf{u}_{0s} = \mathbf{0}$. Hence

$$\mathbf{u}_0^T \mathbf{x}_{0s} = -\mathbf{x}_0^T \mathbf{u}_{0s} = -\mathbf{u}_{0s}^T \mathbf{x}_0.$$

Adding the term $\mathbf{u}_0^T A \mathbf{x}_0$ to both sides, we have

$$\mathbf{u}_0^T (A\mathbf{x}_0 + \mathbf{x}_{0s}) = (\mathbf{u}_0^T A - \mathbf{u}_{0s}^T)\mathbf{x}_0.$$

Since $[\mathbf{x}_0, \mathbf{x}_{0s}]$ and $[\mathbf{u}_0, \mathbf{u}_{0s}]$ are feasible solutions to the primal and the dual,

$$\mathbf{u}_0^T \mathbf{b} = \mathbf{c}^T \mathbf{x}_0.$$

Thus by Theorem 5.3, both solutions are optimal solutions. □

To see why we have the complementary slackness, suppose that the jth surplus variable of the dual problem is positive. Then by Theorem 5.5, the reduced cost coefficient of the jth structural variable of the primal problem is negative (because it is equal to the negation of the jth surplus variable of the dual problem). Hence the jth primal structural variable should be equal to zero if it is at the optimum. For if not, then we can set it to zero and thus increase the objective value.

Example 5.6. (Dual Prices) Let the primal be given by

$$\begin{array}{rl} \max & x_1 + 4x_2 + 3x_3 \\ \text{subject to} & 2x_1 + 2x_2 + \ \ x_3 \leq 4 \\ & x_1 + 2x_2 + 2x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

FCF $\longrightarrow$

3 structural variables 2 slack variables

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Its dual is

(0,0) optimal but not feasible

$$\begin{array}{rl} \min & 4u_1 + 6u_2 \\ \text{subject to} & 2u_1 + \ \ u_2 \geq 1 \\ & 2u_1 + 2u_2 \geq 4 \\ & u_1 + 2u_2 \geq 3 \\ & u_1, u_2 \geq 0 \end{array}$$

3 surplus variable 3 artificial variables

Initial Tableau:

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	2	2	1	1	0	4
x_5	1	2	2	0	1	6
x_0	-1	-4	-3	0	0	0

(2,0,0) optimal surplus variables

Optimal Tableau:

primal structural slack

	x_1	x_2	x_3	x_4	x_5	**b**
x_2	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1
x_3	-1	0	1	-1	1	2
x_0	2	0	0	1	1	10

optimal 2 dual structural variables

$(u_1, u_2) = (1, 1)$

Thus the optimal primal solution is $\mathbf{x}^* = [0, 1, 2, 0, 0]$, and by the duality theorem, the optimal dual solution is $\mathbf{u}^* = [1, 1, 2, 0, 0]$. Let us check for the complementary slackness for these two dual solutions.

$$\begin{array}{lllllll} u_1^* > 0 & \Rightarrow & x_4^* = 0 & \Rightarrow & 2x_1^* + 2x_2^* + x_3^* = 4 & \text{i.e.} & 2(0) + 2(1) + 2 = 4 \\ y_2^* > 0 & \Rightarrow & x_5^* = 0 & \Rightarrow & x_1^* + 2x_2^* + 2x_3^* = 6 & \text{i.e.} & 0 + 2(1) + 2(2) = 6 \\ x_1^* = 0 & \Rightarrow & u_3^* \geq 0 & \Rightarrow & 2u_1^* + u_2^* \geq 1 & \text{i.e.} & 2(1) + 1 = 3 \geq 1 \\ x_2^* > 0 & \Rightarrow & u_4^* = 0 & \Rightarrow & 2u_1^* + 2u_2^* = 4 & \text{i.e.} & 2(1) + 2(1) = 4 \\ x_3^* > 0 & \Rightarrow & u_5^* = 0 & \Rightarrow & u_1^* + 2u_2^* = 3 & \text{i.e.} & 1 + 2(1) = 3 \end{array}$$

5.4 Dual Simplex Method

In the usual simplex method, which will be called *primal method* for distinction, we start with a primal BFS $\mathbf{x}$, maintain primal feasibility $\{x_{i0} \geq 0\}_{i=1}^m$ and strive for non-positivity of the reduced cost coefficients (which is equivalent to $\{x_{0j} \geq 0\}_{j=1}^n$). However, by Theorem 5.5, the entries in the x_0 row give the values of the dual variables at optimal. Thus the nonnegativity of $\{x_{0j} \geq 0\}_{j=1}^n$ is equivalent to the feasibility of the dual variables.

In the *dual method*, we start with a dual BFS $\mathbf{u}$, maintain dual feasibility $\{u_{j0} \geq 0\}_{j=1}^n$ (which is equivalent to $\{x_{0j} \geq 0\}_{j=1}^n$) and strive for nonnegativity of $\{u_{0i} \geq 0\}_{i=1}^m$ (which is equivalent to primal feasibility $\{x_{i0} \geq 0\}_{i=1}^m$).

Since at any iteration, both the primal and the dual solutions have the same objective value, by the duality theorem, we see that if both solutions are feasible, then we have reached optimality.

Algorithm for the dual simplex method

1. Given a *dual* BFS $\mathbf{x}_B$, if $\mathbf{x}_B \geq \mathbf{0}$, then the current solution is optimal; otherwise select an index r such that the component x_r of $\mathbf{x}_B$ is negative.

2. If $y_{rj} \geq 0$ for all $j = 1, 2, \cdots, n$, then the dual is unbounded; otherwise determine an index s such that
$$-\frac{y_{0s}}{y_{rs}} = \min_j \left\{ -\frac{y_{0j}}{y_{rj}} | y_{rj} < 0 \right\} .$$

3. Pivot at element y_{rs} and return to step 1.

Example 5.7. Consider the problem:

Primal

$$\begin{array}{rr} \min & 3x_1 + 4x_2 + 5x_3 \\ \text{subject to} & x_1 + 2x_2 + 3x_3 \geq 5 \\ & 2x_1 + 2x_2 + x_3 \geq 6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Dual → is in FCF

In canonical form, it is

$$\begin{array}{rr} \max & -3x_1 - 4x_2 - 5x_3 \\ \text{subject to} & -x_1 - 2x_2 - 3x_3 \leq -5 \\ & -2x_1 - 2x_2 - x_3 \leq -6 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Slack variables

Canonical form

not feasible canonical form

The initial tableau is

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -5 \\ -6 \end{pmatrix}$$ optimal not feasible

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	-1	-2	-3	1	0	-5
x_5	$+2^*$	$+2$	$+1$	0	-1	$+6$
x_0	3	4	5	0	0	0
ratios	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{1}$	–	–	

not feasible → leaving

optimal

entering

min ⇒ next one is still optimal

After one iteration

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	0	$+1^*$	$+\frac{5}{2}$	-1	$+\frac{1}{2}$	$+2$
x_1	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3
x_0	0	1	$\frac{7}{2}$	0	$\frac{3}{2}$	-9
ratios	–	$\frac{1}{1}$	$\frac{7}{5}$	–	$\frac{3}{1}$	

← non feasible leaving

optimal

entering

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Optimal Tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_2	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2
x_1	1	0	-2	1	-1	1
x_0	0	0	1	1	1	-11

} feasible

optimal

Since both the primal and the dual solutions are feasible, we have reached the optimal solution. The primal optimal solution is given by $\mathbf{x}^* = [1, 2, 0]$, the dual optimal solution is $\mathbf{u}^* = [1, 1]$ and the optimal objective value is 11 for the original problem is a minimization problem.

5.5 Post-Optimality or Sensitivity Analysis

Given an LP problem, suppose that we have found the optimal feasible solution by the simplex (or dual simplex) method. *Post-optimality* or *sensitivity analysis* is the study of how the changes in the original LP problem would affect the feasibility and optimality of the current optimal solution. Before we analyze the method, we first recall the following criteria for determining the optimal primal solutions.
Primal feasibility:

$$A\mathbf{x} = \mathbf{b} \Leftrightarrow B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{b}. \tag{5.7}$$

Primal optimality:

$$\mathbf{z}^T - \mathbf{c}^T = \mathbf{c}_B^T B^{-1} A - \mathbf{c}^T \geq \mathbf{0}. \tag{5.8}$$

In the following we will consider changes in the original problem that can affect only one of these criteria. For in these cases, we can obtain the new optimal solution without redoing the whole simplex method for the new LP.

(1) *Changes in resource vector* $\mathbf{b}$.
From (5.7) and (5.8), we see that changes in $\mathbf{b}$ will affect the feasibility but not the optimality of the current optimal solution. Thus if the current optimal solution satisfies the old constraints with the new right hand sides, then it will be the new optimal solution. By the duality theory, the changes in $\mathbf{b}$ will affect the optimality but not the feasibility of the dual optimal solution. In fact, the cost vector for the dual problem is given by $\mathbf{b}$.

(2) *Changes in cost/profit vector* $\mathbf{c}$.
By the duality theory, (or from (5.7) and (5.8) again), we see that changes in the cost vector $\mathbf{c}$ will affect only the optimality of the primal optimal solution and the feasibility of the dual optimal solution. Thus if the current optimal solution satisfies the criteria that the new x_0 row is nonnegative, then it will be the optimal solution for the new LP.

(3) *Changes in technology matrix* A.
If the changes in A occur at the basic variables, then B will be changed. From (5.7) and (5.8), we see that both the feasibility and the optimality of the current optimal solution may be violated. In that case, we have to redo the whole problem. However, if the changes of A are restricted to columns of nonbasic variables (i.e. N in (5.7)), then we see that only dual feasibility (or equivalently primal optimality) will be affected because $\mathbf{x}_N = \mathbf{0}$.

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(4) *Addition of a new primal variable/dual constraint* $a_{ij} + c_j$.
This case is essentially the same as considering *simultaneously* changes in the objective function coefficient as well as the corresponding technological coefficients of nonbasic variable. (One can assume that the a_{ij} and the c_j are originally there with values equal to zero.) Consequently, the addition of a new variable can only affect the optimality of the problem. This means that the new variable will enter the solution if, and only if, it improves the objective function value. Otherwise the new variable becomes just another nonbasic variable ($= 0$).

(5) *Addition of a new primal constraint/dual variable* $a_{ij} + b_i$.
A new constraint can affect the feasibility of the current optimal solution only if it is *active*, i.e. it is not redundant with respect to the current optimal solution. Consequently, the first step would be to check whether the new constraint is satisfied by the current optimal solution. If it is satisfied, the new constraint is redundant and the optimal solution remains unchanged. Otherwise, the new constraint must be added to the system and the dual simplex method is used to clear the primal infeasibility (dual optimality).

Example 5.8. Consider the LPP given by the following tableau:

Original tableau

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	1	3	4	1	0	30
x_5	0	4	-1	0	1	10
x_0	-2	-7	3	0	0	0

A I

$\vec{c} = (2\ 7\ 3\ 0\ 0)$

$\vec{c}_B = (0, 2)$ $x_0 = \vec{c}_B \cdot \vec{x}_B$

The optimal tableau is:

$B = (\vec{a}_4, \vec{a}_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $B^{-1} = (\quad)$ $= (0,2) \begin{pmatrix} 20 \\ 10 \end{pmatrix}$

optimal tableau

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	0	-1	5	1	-1	20
x_1	1	4	-1	0	1	10
x_0	0	1	1	0	2	20

B^{-1}

$\begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{matrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 20 \\ 0 \end{pmatrix}$ $= 20$

(1) *Changes in resource vector* **b**.
Let the new $\hat{\mathbf{b}} = [10, 20]^T$. Then the new basic solution is given by

$$\text{new}\quad \hat{\mathbf{x}}_B = B^{-1}\hat{\mathbf{b}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} -10 \\ 20 \end{bmatrix} < 0 \text{ non-feasible}$$

Thus it is no longer feasible. The new objective value is

$$\hat{x}_0 = \mathbf{c}_B^T \hat{\mathbf{x}}_B = [0, 2] \begin{bmatrix} -10 \\ 20 \end{bmatrix} = 40.$$

We then need to apply the dual simplex method to restore primal feasibility with a new **b** column of $[-10, 20, 40]^T$. The new starting tableau is given by:

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enter

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	0	-1^*	5	1	-1	-10
x_1	1	4	-1	0	1	20
x_0	0	1	1	0	2	40

leave

→ dual simplex

(2) *Changes in cost/profit vector* **c**.
Let the new $\hat{\mathbf{c}} = [3, 6, -3, 0, 0]^T$. The new x_0 row is given by

$$\hat{\mathbf{z}}^T - \hat{\mathbf{c}}^T = \hat{\mathbf{c}}_B^T B^{-1} A - \hat{\mathbf{c}}^T = [0, 6, 0, 0, 3] \geq \mathbf{0}.$$

(Recall that $B^{-1}A$ is just the last tableau.) This indicates primal optimality. Thus the primal optimal solution is unchanged. Looking at the dual, the new dual variables are

$$\hat{\mathbf{u}} = B^{-T}\hat{\mathbf{c}}_B = B^{-T}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

(3) *Changes in technology coefficients* a_{ij}.
In the optimal tableau, x_1 and x_4 are basic. Thus we can only change the entries of $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_5$. Let the new $\hat{\mathbf{a}}_2 = [1, 5]^T$. While the primal feasibility remains, we need to calculate the new reduced cost coefficient for x_2.

$$\hat{z}_2 - c_2 = \mathbf{c}_B^T B^{-1}\hat{\mathbf{a}}_2 - c_2 = [0, 2]\begin{bmatrix} 1 \\ 5 \end{bmatrix} - 7 = 3 \geq 0\,.$$

Thus the current basis remains optimal with $\mathbf{x}_B = (10, 20)^T$ and $x_0 = 20$ unchanged. However, if $\hat{\mathbf{a}}_2 = [1, 3]^T$, then $\hat{z}_2 - c_2 = -1 \leq 0$, indicating non-optimality. In this case, we need to replace the column under x_2 in the (previously optimal) tableau by

$$\hat{\mathbf{y}}_2 = B^{-1}\hat{\mathbf{a}}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

and pivot in the x_2 column once (to have x_2 become basic) to restore optimality. The new optimal $\mathbf{x}_B = [x_4, x_2]^T = [26\frac{2}{3}, 3\frac{1}{3}]^T$ with $x_0 = 23\frac{1}{3}$.

Example 5.9. (Adding extra constraints) Consider the following LLP problem:

	x_1	x_2	x_3	x_4	**b**
x_3	-1	-1	1	0	-1
x_4	-2	-3	0	1	-2
x_0	3	1	0	0	0

We note that we have primal infeasibility and dual feasibility. Using the dual simplex method, we get the following optimal tableau.