Math 3210

Tutorial 8

M method and two stage method:

Introduction to M methods: $\text{Max}: c_1x_1 + \cdots + c_nx_n$

for constrain

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \Rightarrow a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 = b_1$ ← don't tou… nice ca…

$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ge b_m \Rightarrow a_{11}x_1 + a_{12}x_2 + \cdots + a_{mn}x_n - s_m + \boxed{x_{m_1}} = b_m$

correspond to the M method

$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k \Rightarrow a_{k1}x_1 + \cdots + a_{kn}x_n + \boxed{x_{m_k}} = b_k$

Example 1:

Use M-method to solve the following LPP:

minimize $\quad z = 4x_1 + x_2$

subject to
$$3x_1 + x_2 = 3 \quad -M$$
$$4x_1 + 3x_2 \ge 6 \quad -M$$
$$x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$

$\text{Max}: Z_0 = c_1x_1 + \cdots + c_nx_n + 0\cdot s_1 + 0\cdot s_m \cdots + -Mx_{m_1} - \cdots - M\cdots$

then try our best to remove the M.

set $x_0 = -c_1x_1 - \cdots - c_nx_n + Mx_{m_1} + \cdots + Mx_{m_k}$

Idea:

$$\begin{aligned} 3x_1 + x_2 \quad + x_4 \quad &= 3 \\ 4x_1 + 3x_2 - x_3 + x_5 \quad &= 6 \\ x_1 + 2x_2 \quad + x_6 &= 4 \end{aligned}$$

$$Z_0 = 4x_1 + x_2 - 0(x_3) \mp Mx_4 - Mx_5 + 0(x_6)$$
$$= 4x_1 + x_2 - Mx_4 - Mx_5 \qquad c = \begin{pmatrix} 4 \\ 1 \\ 0 \\ -M \\ -M \\ 0 \end{pmatrix}$$

(1)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	3	1	0	1	0	0	3
x_5	4	3	-1	0	1	0	6
x_6	1	2	0	0	0	1	4
x_0	4	1	0	m	m	0	0

(2)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	3	1	0	1	0	0	3
x_5	4	3•	-1	0	1	0	6
x_6	1	2	0	0	0	1	4
x_0	4-7m	1-4m	m	0	0	0	-9m

$(C_j - Z_j)$

(3)	x_1	x_2	x_3	x_4	x_6	b
x_4	5/3•	0	1/3	1	0	1
x_2	4/3	1	-1/3	0	0	2
x_6	-5/3	0	2/3	0	1	0
x_0	(8-5m)/3	0	(1-m)/3	0	0	-m-2

$C_j - Z_j$

(4)	x_1	x_2	x_3	x_6	b
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
x_6	0	0	1•	1	1
x_0	0	0	-1/5	0	-18/5

$C_j - Z_j$

(5)	x_1	x_2	x_3	x_6	b
x_1	1	0	0	-1/5	2/5
x_2	0	1	0	3/5	9/5
x_3	0	0	1	1	1
x_0	0	0	0	1/5	-17/5

$C_j - Z_j$

Thus optimal solution is (2/5,9/5) with optimal value 17/5.

Example 2:

Use M-method to solve the following LPP:

(a) maximize $\quad z = 2x_1 + 3x_2 - 5x_3$

(b) minimize $\quad z = 4x_1 - 8x_2 + 3x_3$

subject to

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

a)

Left as exercise

b)

(1)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	1	1	1	0	1	0	7
x_6	2	-5	1	-1	0	1	10
x_0	4	-8	3	0	m	m	0

(2)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	1	1	1•	0	1	0	7
x_6	2	-5	1	-1	0	1	10
x_0	4-3m	4m-8	3-2m	m	0	0	-17m

(3)	x_1	x_2	x_3	x_4	x_6	b
x_3	1	1	1	0	0	7
x_6	1•	-6	0	-1	1	3
x_0	1-m	6m-11	0	m	0	-3m-21

(4)	x_1	x_2	x_3	x_4	b
x_3	0	7•	1	1	4
x_1	1	-6	0	-1	3
x_0	0	-5	0	1	-24

(5)	x_1	x_2	x_3	x_4	b
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
x_0	0	0	5/7	12/7	-148/7

Thus optimal solution is (4/7,45/7,0) with optimal value 148/7.

Introduction to two phase methods:

for constrains:

$a_{11}x_1 + \cdots + a_{1n}x_n \le b_1 \to a_{11}x_1 + \cdots + a_{1n}x_n + s_1 = b_1$

$a_{m1}x_1 + \cdots + a_{mn}x_m \ge b_m \to a_{m1}x_1 + \cdots + a_{1n}x_n - s_m + \boxed{a_1} = b_m$

$a_{k1}x_1 + \cdots + a_{kn}x_m = b_k \to a_{k1}x_1 + \cdots + a_{kn}x_n + \boxed{a_k} = b_k$

1st max $Z^* = -a_1 + \cdots + (-a_k)$ / minimised: $Z^* = a_1 + \cdots + a_k$

2nd remove them and carry on the usual to simplex methods.

note if $Z^* < 0$, no feasible solution exist

Example 3:

Use Two Phase method to solve the following LPP:

(a) ~~minimize~~ max $z = 2x_1 + 3x_2 - 5x_3$

(b) ~~maximize~~ min $z = 4x_1 - 8x_2 + 3x_3$

subject to:

$x_1 + x_2 + x_3 = 7$

$2x_1 - 5x_2 + x_3 \geq 10$

$x_1, x_2, x_3 \geq 0$

Part a)

(1)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	1	1	1	0	1	0	7
x_6	2	-5	1	-1	0	1	10
x_0	0	0	0	0	-1	-1	0

(2)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	1	1	1*	0	1	0	7
x_6	2	-5	1	-1	0	1	10
x_0	3	-4	2	-1	0	0	17

(3)	x_1	x_2	x_3	x_4	x_6	b
x_3	1	1	1	0	0	7
x_6	1*	-6	0	-1	1	3
x_0	1	-6	0	-1	0	3

(4)	x_1	x_2	x_3	x_4	b
x_3	0	7*	1	1	4
x_1	1	-6	0	-1	3
x_0	0	0	0	0	0

(5)	x_1	x_2	x_3	x_4	b
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
x_0	0	0	50/7	1/7	102/7

Thus optimal solution is (4/7,45/7) with optimal value 102/7.

idea:

$x_1 + x_2 + x_3 \quad + x_5 \quad = 7$

$2x_1 - 5x_2 + x_3 - x_4 \quad + x_6 = 10.$

Part b)

(1)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	1	1	1	0	1	0	7
x_6	2	-5	1	-1	0	1	10
x_0	0	0	0	0	-1	-1	0

(2)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_5	1	1	1*	0	1	0	7
x_6	2	-5	1	-1	0	1	10
x_0	3	-4	2	-1	0	0	17

(3)	x_1	x_2	x_3	x_4	x_6	b
x_3	1	1	1	0	0	7
x_6	1*	-6	0	-1	1	3
x_0	1	-6	0	-1	13	

(4)	x_1	x_2	x_3	x_4	b
x_3	0	7*	1	1	4
x_1	1	-6	0	-1	3
x_0	4	-8	3	0	0

(5)	x_1	x_2	x_3	x_4	b
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7
x_0	0	0	5/7	12/7	-148/7

Thus optimal solution is (4/7,45/7,0) with optimal value 148/7.

Special case when applying such methods:

Empty feasible ~~solution~~ region: No way to kick out the M.

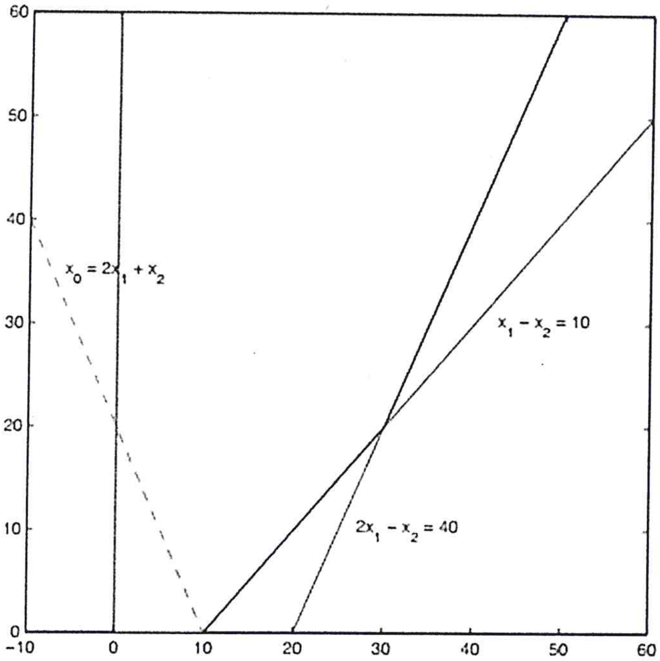
Unbound feasible region: An entire column with entries ≤ 0.

Optimal solution: zero non basic $C_j - Z_j$

Meaning of unbounded feasible region:

$$\max \quad x_0 = 6x_1 - 2x_2$$

$$\text{subject to} \quad \begin{cases} 2x_1 - x_2 \leq 2 \\ x_1 \leq 4 \\ x_1, \; x_2 \geq 0 \end{cases}$$



Example 4:

Show how the M-method and two-phase method will conclude that the following problem has no feasible solution.

$$\begin{aligned} &\text{Maximize} && x_0 = 2x_1 + 5x_2 \\ &\text{Subject to} && 3x_1 + 2x_2 \geq 6 \\ & && 2x_1 + x_2 \leq 2 \\ & && x_1, x_2 \geq 0 \end{aligned}$$

M-method:

(1)	x_1	x_2	x_3	x_4	x_5	b
x_4	3	2	-1	1	0	6
x_5	2	1	0	0	1	2
x_0	-2	-5	0	m	m	0

(2)	x_1	x_2	x_3	x_4	x_5	b
x_4	3	2	-1	1	0	6
x_5	2	1•	0	0	1	2
x_0	-2-5m	-5-3m	m	0	0	-8m

(3)	x_1	x_2	x_3	x_4	x_5	b
x_4	-1	0	-1	1	-2	2
x_2	2	1	0	0	1	2
x_0	8+m	0	m	0	5+3m	10-2m

Two-phase method:

(1)	x_1	x_2	x_3	x_4	x_5	b
x_4	3	2	-1	1	0	6
x_5	2	1	0	0	1	2
x_0	0	0	0	-1	-1	0

(2)	x_1	x_2	x_3	x_4	x_5	b
x_4	3	2	-1	1	0	6
x_5	2	1•	0	0	1	2
x_0	5	3	1	0	0	8

(3)	x_1	x_2	x_3	x_4	x_5	b
x_4	-1	0	-1	1	-2	2
x_2	2	1	0	0	1	2
x_0	-1	0	1	0	-3	2

Example 5:

Consider the following LPP:

Maximize $\quad x_0 = 40x_1 + 20x_2 + 2x_3$

Subject to

$$3x_1 - 3x_2 + 5x_3 \le 50$$
$$x_1 + x_3 \le 10$$
$$x_1 - x_2 + 4x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

1) At which direction, the solution space is unbounded ?

2) Solve the above LPP.

(1)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	3	-3	5	1	0	0	50
x_5	1	0	1	0	1	0	10
x_6	1*	-1	4	0	0	1	2
x_0	-40	-20	-2	0	0	0	0

(2)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	0	0	-7	1	0	-3	44
x_5	0	1*	-3	0	1	-1	8
x_1	1	-1	4	0	0	1	2
x_0	0	-60	158	0	0	40	80

(3)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	0	0	-7	1	0	-3	44
x_2	0	1	-3	0	1	-1	8
x_1	1	0	1*	0	1	0	10
x_0	0	0	-22	0	60	-20	560

(4)	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	7	0	0	1	7	-3	114
x_2	3	1	0	0	4	-1	38
x_3	1	0	1	0	1	0	10
x_0	22	0	0	0	82	-20	780

Thus the LPP is unbounded.

Example 6:

Max

$$x_0 = -x_2 - 2x_3 + 1$$

subject to

$$2x_2 - 1x_3 + x_5 = 2$$
$$x_2 + 2x_3 - x_4 + x_6 = 2$$
$$x_i \geq 0; i = 2, 3, 4, 5, 6$$

(1)	x_2	x_3	x_4	x_5	x_6	b
x_5	2	-1	0	1	0	2
x_6	1	2	-1	0	1	2
x_0	0	0	0	0	-1	0

(2)	x_2	x_3	x_4	x_5	x_6	b
x_5	2	-1	0	1	0	2
x_6	1	2•	-1	0	1	2
x_0	1	2	-1	0	0	2

(3)	x_2	x_3	x_4	x_5	x_6	b
x_5	5/2	0	-1/2	1	1/2	3
x_3	1/2	1	-1/2	0	1/2	1
x_0	0	0	0	0	-1	0

(4)	x_2	x_3	x_4	x_5	b
x_5	5/2	0	-1/2	1	3
x_3	1/2	1	-1/2	0	1
x_0	1	2	0	0	0

(5)	x_2	x_3	x_4	x_5	b
x_5	5/2	0	-1/2	1	3
x_3	1/2	1	-1/2	0	1
x_0	0	0	1	0	-2

(5)•	x_2	x_3	x_4	x_5	b
x_2	1	0	-1/5	2/5	6/5
x_3	0	1	-2/5	-1/5	2/5
x_0	0	0	1	0	-2

(a)The feasible region is unbounded at the direction of x_4.