Math 3210 Homework 2

Due Date: Oct. 13th, before 18:00. Please put your homework in the assignment box before the due time. The box is on the 2nd floor of Lady Shaw Building opposite to Rm 221. The solution to the homework problems will be uploaded to the course homepage right after the due time. Because of this, *any late homework submission will not be accepted.*

1. A set S in the Euclidean space $\mathbb{R}^n$ is called a **star shaped domain** if there exists a distinguished point $x_0 \in S$ such that for all $x \in S$, the line segment from x_0 to x lies in S. Construct a star-shaped domain S in $\mathbb{R}^2$ so that S is not a convex set.

2. Given a convex set S in $\mathbb{R}^n$. A non-zero vector $\mathbf{v}$ is called a **direction** of the set if for each point $x_0 \in S$, the ray $\{x_0 + \lambda \mathbf{v}\}$ belongs to S for all $\lambda \geq 0$.
(a) Show that a bounded convex set can't have a direction.
(b) Let S be the strip defined by $\{(x, y) | -1 \leq y \leq 1\}$ in $\mathbb{R}^2$, write down the directions of S.
(c) Find a convex set S in $\mathbb{R}^2$ such that S has a unique extreme point and infinitely many directions.

3. Consider the curve C defined by $y = x^2$ in $\mathbb{R}^2$, which separate $\mathbb{R}^2$ into two regions. Consider the region containing te point $(0, 1) \in \mathbb{R}^2$, denote its closure by S. Show that every point on ∂S is an extreme point of S.

4. Convert the following LPP to its standard form:
maximize $z = 3x_1 + 2x_2 - x_3 + x_4$ subject to

$$\begin{cases} x_1 + 2x_2 + x_3 - x_4 & \leq 5, \\ -2x_1 - 4x_2 + x_3 + x_4 & \leq -1, \\ x_1 & \geq 0, \\ x_2 & \leq 0. \end{cases}$$

5. Convert the LPP below to canonical form.
minimize $z = 3x_1 - 2x_3$ subject to

$$\begin{cases} x_1 - 2x_2 + x_3 & = 1, \\ x_1 + x_2 & \geq 4, \\ x_1, x_2 & \geq 0, \\ x_3 & \leq 3. \end{cases}$$

6. Consider the following LPP:
maximize $z = 2x + 5y$ subject to

$$\begin{cases} -3x + 2y & \leq 6, \\ x + 2y & \geq 2, \\ x, y & \geq 0. \end{cases}$$

(a) Sketch the feasible set of the problem.
(b) Explain why this LPP does not have optimal solution.

7. Consider the following LPP:
minimize $z = 3x + 5y$ subject to

$$\begin{cases} -3x + 2y & \leq 6, \\ x + 2y & \geq 2, \\ x, y & \geq 0. \end{cases}$$

(a) Find all the extreme points of the feasible set.
(b) Solve the LPP by finding an optimal solution.

8. Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$, consider the linear system $A\mathbf{x} = \mathbf{b}$. Given a feasible solution $\mathbf{x} = (2, 3, 2, 3, 3)^T$,
(a) move it to a basic feasible solution $\mathbf{x}_1 = (0, 0, 4, 6, 8)^T$,
(b) move from $\mathbf{x}_1$ to another basic feasible that is different from $\mathbf{x}_1$ by only one basic variable.