1. Show that \( f \) is continuous from \((X, d)\) to \((Y, \rho)\) if and only if \( f^{-1}(F) \) is closed in \( X \) whenever \( F \) is closed in \( Y \).

**Solution.** Use \( f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \) to reduce to the statement: \( f \) is continuous iff \( f^{-1}(G) \) is open for open \( G \).

2. Identify the boundary points, interior points, interior and closure of the following sets in \( \mathbb{R} \):
   (a) \([1, 2) \cup (2, 5) \cup \{10\} \).
   (b) \([0, 1] \cap \mathbb{Q} \).
   (c) \( \bigcup_{k=1}^{\infty} \left(1/(k+1), 1/k\right) \).
   (d) \( \{1, 2, 3, \cdots \} \).

**Solution.**
   (a) Boundary points \( 1, 2, 5, 10 \). Interior points \((1, 2), (2, 5)\). Interior \((1, 2) \cup (2, 5)\). Closure \([1, 5] \cup \{10\} \).
   (b) Boundary points: all points in \([0, 1] \cap \mathbb{Q}\). No interior point. Interior \( \emptyset \). Closure \([0, 1]\).
   (c) Boundary points \( \{1/k : k \geq 1\} \cup \{0\} \). Interior points: all points in this set. Interior: This set (because it is an open set). Closure \([0, 1]\).
   (d) Boundary points \( 1, 2, 3, \cdots \). No interior points. Interior \( \emptyset \). Closure: the set itself (it is a closed set).

3. Identify the boundary points, interior points, interior and closure of the following sets in \( \mathbb{R}^2 \):
   (a) \( R = [0, 1] \times [2, 3] \cup \{0\} \times (3, 5) \).
   (b) \( \{(x, y) : 1 < x^2 + y^2 \leq 9\} \).
   (c) \( \mathbb{R}^2 \setminus \{(1, 0), (1/2, 0), (1/3, 0), (1/4, 0), \cdots \} \).

**Solution.**
   (a) Boundary points: the geometric boundary of the rectangle and the segment \( \{0\} \times [3, 5] \). Interior points: all points inside the rectangle. Interior \((0, 1) \times (3, 5)\). Closure \([0, 1] \times [3, 5] \cup \{0\} \times [3, 5]\).
   (b) Boundary points: all \((x, y)\) satisfying \( x^2 + y^2 = 1 \) or \( x^2 + y^2 = 9 \). Interior points: all points satisfying \( 1 < x^2 + y^2 < 9 \). Interior \( \{(x, y) : 1 < x^2 + y^2 < 9\} \). Closure \( \{(x, y) : 1 \leq x^2 + y^2 \leq 9\} \).
   (c) Boundary points: The set together with \( \{(0, 0)\} \). Interior points: None. Interior \( \emptyset \). Closure \( \{(0, 0), (1, 0), (1/2, 0), (1/3, 0), \cdots \} \).

4. Describe the closure and interior of the following sets in \( C[0, 1] \):
   (a) \( \{f : f(x) > -1, \forall x \in [0, 1]\} \).
   (b) \( \{f : f(0) = f(1)\} \).

**Solution.**
(a) Closure \( \{ f \in C[0,1] : f(x) \geq -1, \forall x \in [0,1] \} \). Interior: The set itself. It is an open set.

(b) Closure: The set itself. It is a closed set. Interior: \( \phi \). For any \( f \) satisfying \( f(0) = f(1) \), there are many \( g \in C[0,1] \) satisfying \( \| g - f \|_{\infty} < \varepsilon \) but \( g(0) \neq g(1) \).

5. Let \( A \) and \( B \) be subsets of \((X,d)\). Show that \( A \cup B = A \cup B \).

Solution. We have \( A \subset B \) whenever \( A \subset B \) right from definition. So \( A \cup B \subset A \cup B \). Conversely, if \( x \in A \cup B \), \( B_\varepsilon(x) \) either has non-empty intersection with \( A \) or \( B \). So there exists \( \varepsilon_j \to 0 \) such that \( B_\varepsilon(x) \) has nonempty intersection with \( A \) or \( B \), so \( x \in A \cup B \).

6. Show that \( E = \{ x \in X : d(x,E) = 0 \} \) for every non-empty \( E \subset X \).

Solution. Let \( x \in \overline{E} \). By definition, for each \( n \) there exists some \( y_n \in E \) such that \( y_n \in B_{1/n}(x) \). It follows that \( d(x,E) \leq d(x,y_n) \to 0 \) which implies \( d(x,E) = 0 \). On the other hand, if \( d(x,E) = 0 \), there exists \( \{ x_n \} \subset E \) such that \( d(x,x_n) \to 0 \), so \( x \in \overline{E} \).

7. Show that \( f \) is continuous from \((X,d)\) to \((Y,\rho)\) if and only if for every \( E \subset X \), \( f(E) \subset \overline{f(E)} \).

Solution. Let \( y_0 = f(x_0), x_0 \in \overline{E} \). We can find \( x_n \in E, \ x_n \to x_0 \). By continuity, \( f(x_n) \to f(x_0) = y_0 \). As \( f(x_n) \in f(E) \), \( y_0 = f(x_0) \in \overline{f(E)} \). Conversely, if for some \( x_n \to x_0 \) but \( f(x_n) \) does not tend to \( f(x_0) \), there exists some \( B_\rho(f(x_0)) \) such that there are infinitely many \( f(x_n) \) not belonging to \( B_\rho(f(x_0)) \). WLOG assume the whole \( \{ f(x_n) \} \) does, that is, \( \{ f(x_n) \} \cap B_\rho(f(x_0)) = \phi \) for all \( n \). Now consider the set \( F = \{ x_1, x_2, \cdots \} \). By assumption, \( f(F) \subset \overline{f(F)} \). In particular, \( f(x_0) \in \overline{f(F)} \), that is, \( B_\rho(f(x_0)) \cap \{ f(x_n) \} \neq \phi \) for some \( n \), contradiction holds.