In Problems 1 through 10, find the gradient vector ∇f at the indicated point P.

1. $f(x, y) = 3x - 7y$; $\quad P(17, 39)$
2. $f(x, y) = 3x^2 - 5y^2$; $\quad P(2, -3)$
3. $f(x, y) = \exp(-x^2 - y^2)$; $\quad P(0, 0)$
4. $f(x, y) = \sin \frac{1}{4}\pi xy$; $\quad P(3, -1)$
5. $f(x, y, z) = y^2 - z^2$; $\quad P(17, 3, 2)$
6. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$; $\quad P(12, 3, 4)$
7. $f(x, y, z) = e^x \sin y + e^y \sin z + e^z \sin x$; $\quad P(0, 0, 0)$
8. $f(x, y, z) = x^2 - 3yz + z^3$; $\quad P(2, 1, 0)$
9. $f(x, y, z) = 2\sqrt{xyz}$; $\quad P(3, -4, -3)$
10. $f(x, y, z) = (2x - 3y + 5z)^5$; $\quad P(-5, 1, 3)$

In Problems 11 through 20, find the directional derivative of f at P in the direction of $\mathbf{v}$; that is, find

$$D_{\mathbf{u}} f(P), \quad \text{where} \quad \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}.$$

11. $f(x, y) = x^2 + 2xy + 3y^2$; $\quad P(2, 1)$, $\mathbf{v} = \langle 1, 1 \rangle$
12. $f(x, y) = e^x \sin y$; $\quad P(0, \pi/4)$, $\mathbf{v} = \langle 1, -1 \rangle$
13. $f(x, y) = x^3 - x^2y + xy^2 + y^3$; $\quad P(1, -1)$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$
14. $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$; $\quad P(-3, 3)$, $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$
15. $f(x, y) = \sin x \cos y$; $\quad P(\pi/3, -2\pi/3)$, $\mathbf{v} = \langle 4, -3 \rangle$
16. $f(x, y, z) = xy + yz + zx$; $\quad P(1, -1, 2)$, $\mathbf{v} = \langle 1, 1, 1 \rangle$
17. $f(x, y, z) = \sqrt{xyz}$; $\quad P(2, -1, -2)$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
18. $f(x, y, z) = \ln(1 + x^2 + y^2 - z^2)$; $\quad P(1, -1, 1)$, $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$
19. $f(x, y, z) = e^{xyz}$; $\quad P(4, 0, -3)$, $\mathbf{v} = \mathbf{j} - \mathbf{k}$
20. $f(x, y, z) = \sqrt{10 - x^2 - y^2 - z^2}$; $\quad P(1, 1, -2)$, $\mathbf{v} = \langle 3, 4, -12 \rangle$

In Problems 21 through 28, find the maximum directional derivative of f at P and the direction in which it occurs.

21. $f(x, y) = 2x^2 + 3xy + 4y^2$; $\quad P(1, 1)$
22. $f(x, y) = \arctan\left(\dfrac{y}{x}\right)$; $\quad P(2, -3)$
23. $f(x, y) = \ln(x^2 + y^2)$; $\quad P(3, 4)$
24. $f(x, y) = \sin(3x - 4y)$; $\quad P(\pi/3, \pi/4)$
25. $f(x, y, z) = 3x^2 + y^2 + 4z^2$; $\quad P(1, 5, -2)$

26. $f(x, y, z) = \exp(x - y - z)$; $P(5, 2, 3)$

27. $f(x, y, z) = \sqrt{xy^2z^3}$; $P(2, 2, 2)$

28. $f(x, y, z) = \sqrt{2x + 4y + 6z}$; $P(7, 5, 5)$

In Problems 29 through 34, use the normal gradient vector to write an equation of the line (or plane) tangent to the given curve (or surface) at the given point P.

29. $\exp(25 - x^2 - y^2) = 1$; $P(3, 4)$

30. $2x^2 + 3y^2 = 35$; $P(2, 3)$

31. $x^4 + xy + y^2 = 19$; $P(2, -3)$

32. $3x^2 + 4y^2 + 5z^2 = 73$; $P(2, 2, 3)$

33. $x^{1/3} + y^{1/3} + z^{1/3} = 1$; $P(1, -1, 1)$

34. $xyz + x^2 - 2y^2 + z^3 = 14$; $P(5, -2, 3)$

The properties of gradient vectors listed in Problems 35 through 38 exhibit the close analogy between the gradient operator ∇ and the single-variable derivative operator D. Verify each, assuming that a and b are constants and that u and v are differentiable functions of x and y.

35. $\nabla(au + bv) = a\nabla u + b\nabla v$.   36. $\nabla(uv) = u\nabla v + v\nabla u$.

37. $\nabla\left(\dfrac{u}{v}\right) = \dfrac{v\nabla u - u\nabla v}{v^2}$ if $v \neq 0$.

38. If n is a positive integer, then $\nabla u^n = nu^{n-1}\nabla u$.

39. Show that the value of a differentiable function f decreases the most rapidly at P in the direction of the vector $-\nabla f(P)$, directly opposite to the gradient vector.

40. Suppose that f is a function of three independent variables x, y, and z. Show that $D_{\mathbf{i}}f = f_x$, $D_{\mathbf{j}}f = f_y$, and $D_{\mathbf{k}}f = f_z$.

41. Show that the equation of the line tangent to the conic section $Ax^2 + Bxy + Cy^2 = D$ at the point (x_0, y_0) is

$$(Ax_0)x + \tfrac{1}{2}B(y_0x + x_0y) + (Cy_0)y = D.$$

42. Show that the equation of the plane tangent to the quadric surface $Ax^2 + By^2 + Cz^2 = D$ at the point (x_0, y_0, z_0) is

$$(Ax_0)x + (By_0)y + (Cz_0)z = D.$$

43. Show that an equation of the plane tangent to the paraboloid $z = Ax^2 + By^2$ at the point (x_0, y_0, z_0) is $z - z_0 = 2Ax_0x + 2By_0y$.

44. Suppose that the temperature at the point (x, y, z) in space, with distance measured in kilometers, is given by

$$w = f(x, y, z) = 10 + xy + xz + yz$$

(in degrees Celsius). Find the rate of change (in degrees Celsius per kilometer) of temperature at the point $P(1, 2, 3)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$.

45. Suppose that the function

$$w = f(x, y, z) = 10 + xy + xz + yz$$

of Problem 44 gives the temperature at the point (x, y, z) of space. (Units in this problem are in kilometers, degrees Celsius, and minutes.) What time rate of change (in degrees Celsius per minute) will a hawk observe as it flies through $P(1, 2, 3)$ at a speed of 2 km/min, heading directly toward the point $Q(3, 4, 4)$?

46. Suppose that the temperature w (in degrees Celsius) at the point (x, y) is given by

$$w = f(x, y) = 10 + (0.003)x^2 - (0.004)y^2.$$

In what direction $\mathbf{u}$ should a bumblebee at the point $(40, 30)$ initially fly in order to get warmer the most quickly? Find the directional derivative $D_{\mathbf{u}}f(40, 30)$ in this optimal direction $\mathbf{u}$.

47. Suppose that the temperature W (in degrees Celsius) at the point (x, y, z) in space is given by

$$W = 50 + xyz.$$

(a) Find the rate of change of temperature with respect to distance at the point $P(3, 4, 1)$ in the direction of the vector $\mathbf{v} = \langle 1, 2, 2\rangle$. (The units of distance in space are feet.) (b) Find the maximal directional derivative $D_{\mathbf{u}}W$ at the point $P(3, 4, 1)$ and the direction $\mathbf{u}$ in which that maximum occurs.

48. Suppose that the temperature (in degrees Celsius) at the point (x, y, z) in space is given by the formula

$$W = 100 - x^2 - y^2 - z^2.$$

The units of distance in space are meters. (a) Find the rate of change of temperature at the point $P(3, -4, 5)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$. (b) In what direction does W increase most rapidly at P? What is the value of the maximal directional derivative at P?

49. Suppose that the altitude z (in miles above sea level) of a certain hill is described by the equation $z = f(x, y)$, where

$$f(x, y) = \frac{1}{10}(x^2 - xy + 2y^2).$$

(a) Write an equation (in the form $z = ax + by + c$) of the plane tangent to the hillside at the point $P(2, 1, 0.4)$. (b) Use $\nabla f(2, 1)$ to approximate the altitude of the hill above the point $(2.2, 0.9)$ in the xy-plane. Compare your result with the actual altitude at this point.

50. Find an equation for the plane tangent to the paraboloid $z = 2x^2 + 3y^2$ and, simultaneously, parallel to the plane $4x - 3y - z = 10$.

51. The cone with equation $z^2 = x^2 + y^2$ and the plane with equation $2x + 3y + 4z + 2 = 0$ intersect in an ellipse. Write an equation of the plane normal to this ellipse at the point $P(3, 4, -5)$ (Fig. 12.8.11).

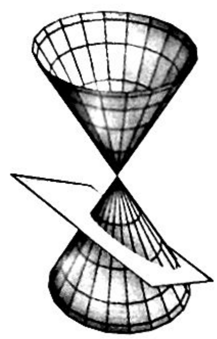

FIGURE 12.8.11 The cone and plane of Problems 51 and 52.

52. It is apparent from geometry that the highest and lowest points of the ellipse of Problem 51 are those points where its tangent line is horizontal. Find those points.

53. Show that the sphere $x^2 + y^2 + z^2 = r^2$ and the cone $z^2 = a^2x^2 + b^2y^2$ are orthogonal (that is, have perpendicular tangent planes) at every point of their intersection (Fig. 12.8.12).

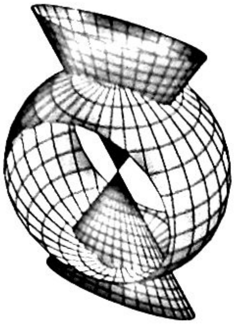

FIGURE 12.8.12 A cut-away view of the cone and sphere of Problem 53.

54. Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are planes tangent to the circular ellipsoid $x^2 + y^2 + 2z^2 = 2$ at the two points P_1 and P_2 having the same z-coordinate. Show that $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect the z-axis at the same point.

55. A plane tangent to the surface $xyz = 1$ at a point in the first octant cuts off a pyramid from the first octant. Show that any two such pyramids have the same volume.

In Problems 56 through 61, the function $z = f(x, y)$ describes the shape of a hill; $f(P)$ is the altitude of the hill above the point $P(x, y)$ in the xy-plane. If you start at the point $(P, f(P))$ of this hill, then $D_{\mathbf{u}}f(P)$ is your rate of climb (rise per unit of horizontal distance) as you proceed in the horizontal *direction* $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$. *And the angle at which you climb while you walk in this direction is $\gamma = \tan^{-1}(D_{\mathbf{u}}f(P))$, as shown in Fig. 12.8.13.*

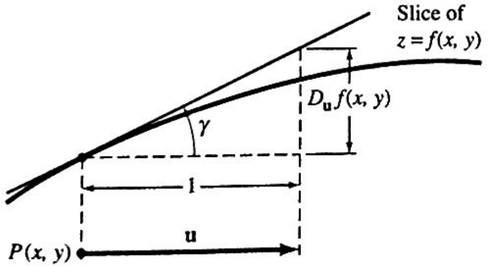

FIGURE 12.8.13 The cross section of the part of the graph above **u** (Problems 56 through 61).

56. You are standing at the point $(-100, -100, 430)$ on a hill that has the shape of the graph of

$$z = 500 - (0.003)x^2 - (0.004)y^2,$$

with x, y, and z given in feet. (a) What will be your rate of climb (*rise* over *run*) if you head northwest? At what angle from the horizontal will you be climbing? (b) Repeat part (a), except now you head northeast.

57. You are standing at the point $(-100, -100, 430)$ on the hill of Problem 56. In what direction (that is, with what compass heading) should you proceed in order to climb the most steeply? At what angle from the horizontal will you initially be climbing?

58. Repeat Problem 56, but now you are standing at the point $P(100, 100, 500)$ on the hill described by

$$z = \frac{1000}{1 + (0.00003)x^2 + (0.00007)y^2}.$$

59. Repeat Problem 57, except begin at the point $P(100, 100, 500)$ of the hill of Problem 58.

60. You are standing at the point $(30, 20, 5)$ on a hill with the shape of the surface

$$z = 100 \exp\left(-\frac{x^2 + 3y^2}{701}\right).$$

(a) In what direction (with what compass heading) should you proceed in order to climb the most steeply? At what angle from the horizontal will you initially be climbing? (b) If, instead of climbing as in part (a), you head directly west (the negative x-direction), then at what angle will you be climbing initially?

61. (a) You are standing at the point where $x = y = 100$ (ft) on the side of a mountain whose height (in feet above sea level) is given by

$$z = \frac{1}{1000}(3x^2 - 5xy + y^2),$$

with the x-axis pointing east and the y-axis pointing north. If you head northeast, will you be ascending or descending? At what angle (in degrees) from the horizontal? (b) If you head 30° north of east, will you be ascending or descending? At what angle (in degrees) from the horizontal?

62. Suppose that the two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ both pass through the point P where both gradient vectors $\nabla f(P)$ and $\nabla g(P)$ exist. (a) Show that the two surfaces are tangent at P if and only if $\nabla f(P) \times \nabla g(P) = \mathbf{0}$. (b) Show that the two surfaces are orthogonal at P if and only if $\nabla f(P) \cdot \nabla g(P) = 0$.

63. Suppose that the plane vectors $\mathbf{u}$ and $\mathbf{v}$ are not collinear and that the function $f(x, y)$ is differentiable at P. Show that the values of the directional derivatives $D_{\mathbf{u}}f(P)$ and $D_{\mathbf{v}}f(P)$ determine the value of the directional derivative of f at P in every other direction.

64. Show that the function $f(x, y) = (\sqrt[3]{x} + \sqrt[3]{y})^3$ is continuous at the origin and has directional derivatives in all directions there, but is not differentiable at the origin.