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THE CONCEPT OF HETEROGENEOUS SCATTERING COEFFICIENTS AND ITS APPLICATION IN INVERSE MEDIUM SCATTERING*

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Abstract. This work investigates the scattering coefficients for inverse medium scattering problems. It shows some fundamental properties of the coefficients such as symmetry and tensorial properties. The relationship between the scattering coefficients and the far-field pattern is also derived. Furthermore, the sensitivity of the scattering coefficients with respect to changes in the permittivity and permeability distributions is investigated. In the linearized case, explicit formulas for reconstructing permittivity and permeability distributions from the scattering coefficients is proposed. They relate the exponentially ill-posed character of the inverse medium scattering problem at a fixed frequency to the exponential decay of the scattering coefficients. Moreover, they show the stability of the reconstruction from multifrequency measurements. This provides a new direction for solving inverse medium scattering problems.

Key words. inverse medium scattering, scattering coefficients, heteregeneous inclusions, far-field measurements, sensitivity, reconstruction algorithm

AMS subject classifications. 35R30, 35B30

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1. Introduction.

In this work we will be concerned with the following transverse magnetic polarized wave scattering problem

$$\nabla \cdot \frac{1}{\mu}\nabla u + \omega^2 \varepsilon u = 0 \text{ in } \mathbb{R}^2, \tag{1.1}$$

where μ, $\varepsilon > 0$ are the respective permittivity and permeability coefficients of the medium. We consider an inhomogeneous medium Ω contained inside a homogeneous background medium, and assume that Ω is an open bounded connected domain with a $\mathcal{C}^{1,\alpha}$ boundary for some $0 < \alpha < 1$. Let ν denote the outward normal vector at $\partial\Omega$, and μ_0, $\varepsilon_0 > 0$ be the medium coefficients of the homogeneous background medium. Suppose that $\mu, \varepsilon \in L^\infty$ and $\mu - \mu_0$ and $\varepsilon - \varepsilon_0$ are supported in Ω. Moreover, there exist positive constants $\underline{\mu}$ and $\underline{\varepsilon}$ such that $\mu(x) \geq \underline{\mu}$ and $\varepsilon(x) \geq \underline{\varepsilon}$ in Ω. Under these settings, we can write (1.1) as follows:

$$\begin{cases} \nabla \cdot \frac{1}{\mu}\nabla u + \omega^2 \varepsilon u = 0 & \text{in } \Omega, \\ \Delta u + k_0^2 u = 0 & \text{in } \mathbb{R}^2 \backslash \overline{\Omega}, \\ u^+ = u^- & \text{on } \partial\Omega, \\ \frac{1}{\mu_0}\frac{\partial u^+}{\partial \nu} = \frac{1}{\mu}\frac{\partial u^-}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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Here and throughout this paper, the superscripts $\pm$ indicate the limits from outside and inside of Ω, respectively, and $\partial/\partial\nu$ denotes the normal derivative. We shall complement the system (1.2) by the physical outgoing Sommerfeld radiation condition

$$\frac{\partial}{\partial r}(u-u_0) - ik_0(u-u_0) = o(|x|^{-\frac{1}{2}}) \text{ as } |x| \to \infty, \tag{1.3}$$

where $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is the wavenumber and u_0 is an incident field, solving the homogeneous Helmholtz equation $(\Delta + k_0^2)u_0 = 0$ in $\mathbb{R}^d$. The solution u to the system (1.2) and (1.3) represents the total field due to the scattering from the inclusion Ω corresponding to the incident field u_0.

The notion of scattering coefficients was previously studied for homogeneous electromagnetic inclusions [13] (see also [12]) in order to enhance near cloaking. The purpose of this paper is twofold. We first introduce the concept of inhomogeneous scattering coefficients and investigate some of their important properties and their sensitivity with respect to changes in the physical parameters. Then we make use of this new concept for solving the inverse medium scattering problem and understanding the associated fundamental issues of stability and resolution. The inhomogeneous scattering coefficients can be obtained from the far-field data by a least-squares method [14]. Explicit reconstruction formulas of the inhomogeneous electromagnetic parameters from the scattering coefficients at a fixed frequency or at multiple frequencies are derived in the linearized case. These formulas show that the exponentially ill-posed characteristics of the inverse medium scattering problem at a fixed frequency [4, 21, 24] are due to the exponential decay of the scattering coefficients. Moreover, they highlight the stability of the reconstruction from multifrequency measurements shown in [17, 18, 22, 23] since low-order scattering coefficients can be used to form a highly resolved image from multifrequency measurements. Based on the decay property of the inhomogeneous scattering coefficients, a resolution analysis analogous to the one in [7] can be easily derived. The resolving power, i.e., the number of scattering coefficients which can be stably reconstructed from the far-field measurements, can be expressed in terms of the signal-to-noise ratio in the far-field measurements. The scattering-coefficient-based approach introduced in this paper is a new promising direction for solving the long-standing inverse scattering problem with heterogeneous inclusions. It could be combined with the continuation method developed in [16, 19] for achieving a good resolution and stability for the image reconstruction.

For the sake of simplicity, we shall restrict ourselves to the scattering problem in two dimensions, but all the results and analyses hold true also for three dimensions.

The paper is organized as follows. In section 2 we introduce the notion of inhomogeneous scattering coefficients. Section 3 provides integral representations of the scattering coefficients and shows their exponential decay. This property is the root cause of the exponentially ill-posed character of the inverse medium scattering problem. In section 4 we prove that the scattering coefficients are nothing else but the Fourier coefficients of the far-field pattern, then derive transformation formulas for the scattering coefficients under rigid transformations and scaling in section 5. In section 6 we provide a sensitivity analysis with respect to the electromagnetic parameters for the scattering coefficients. In section 7 we derive new reconstruction formulas from the scattering coefficients at one frequency and at multiple frequencies as well. A few concluding remarks are given in section 8. Appendix A is used to construct a Neumann function for the inhomogeneous Helmholtz equation on a bounded domain. Appendices B and C are used to show the existence of some functions used in the derivation of the explicit reconstruction formulas in the linearized case.

2. Integral representation and scattering coefficients. In this section we define the scattering coefficients of inhomogeneous inclusions. The idea of scattering coefficients for inclusions with homogeneous permittivity and permeability was initially introduced in [13]. We extend this idea and define such a notion for inhomogeneous inclusions following the idea in [6, 13]. We first derive the fundamental representation of the solution u to the system (1.2)–(1.3). For $k_0 > 0$, let Φ_{k_0} be the fundamental solution to the Helmholtz operator $\Delta + k_0^2$ in two dimensions satisfying

$$(\Delta + k_0^2)\Phi_{k_0}(x) = \delta_0(x)$$

subject to the outgoing Sommerfeld radiation condition

$$\frac{\partial}{\partial r}\Phi_{k_0} - ik_0\Phi_{k_0} = o(|x|^{-\frac{1}{2}}) \quad \text{as } |x| \to \infty.$$

Then Φ_{k_0} is given by

$$\Phi_{k_0}(x) = -\frac{i}{4}H_0^{(1)}(k_0|x|), \tag{2.1}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. We can easily deduce from Green's formula that if u is the solution to (1.2)–(1.3), then we have for $x \in \mathbb{R}^2\backslash\overline{\Omega}$ that

$$\begin{aligned}(u-u_0)(x) &= \int_{\partial\Omega} \Phi_{k_0}(x-y)\frac{\partial(u-u_0)^+}{\partial\nu}(y)d\sigma(y) \\ &\quad - \int_{\partial\Omega} \frac{\partial\Phi_{k_0}(x-y)}{\partial\nu_y}(u-u_0)^+(y)d\sigma(y) \\ &= \int_{\partial\Omega} \Phi_{k_0}(x-y)\frac{\partial u^+}{\partial\nu}(y)d\sigma(y) - \int_{\partial\Omega} \frac{\partial\Phi_{k_0}(x-y)}{\partial\nu_y}u^+(y)d\sigma(y) \\ &= \int_{\partial\Omega} \left(\frac{\mu_0}{\mu}\right)\Phi_{k_0}(x-y)\frac{\partial u^-}{\partial\nu}(y)d\sigma(y) - \int_{\partial\Omega} \frac{\partial\Phi_{k_0}(x-y)}{\partial\nu_y}u^-(y)d\sigma(y),\end{aligned} \tag{2.2}$$

where the second equality holds since u_0 satisfies the homogeneous Helmholtz equation. Let $g = \frac{1}{\mu}\frac{\partial u^-}{\partial\nu}$. Then we define the Neumann-to-Dirichlet (NtD) map $\Lambda_{\mu,\varepsilon}$: $H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ such that for any $g \in H^{-\frac{1}{2}}(\partial\Omega)$, $u = \Lambda_{\mu,\varepsilon}g \in H^{\frac{1}{2}}(\partial\Omega)$ is the trace of the solution to the following system:

$$\nabla\cdot\frac{1}{\mu}\nabla u + \omega^2\varepsilon u = 0 \text{ in } \Omega; \quad \frac{1}{\mu}\frac{\partial u}{\partial\nu} = g \text{ on } \partial\Omega. \tag{2.3}$$

We remark that $\Lambda_{\mu_0,\varepsilon_0}$ is well-defined if $\omega\sqrt{\mu_0\varepsilon_0}$ is not a Neumann eigenvalue of $-\Delta$ on Ω. For general distributions ε and μ, in order to ensure the well-posedeness of $\Lambda_{\mu,\varepsilon}$, one should assume, throughout this paper, that 0 is not a Neumann eiganvalue of $\nabla\cdot(1/\mu)\nabla + \omega^2\varepsilon$ in Ω.

With this definition of $\Lambda_{\mu,\varepsilon}$, we have $\Lambda_{\mu,\varepsilon}[g] = u^-$ and $\frac{1}{\mu_0}\Lambda_{\mu_0,\varepsilon_0}[\frac{\partial\Phi_{k_0}}{\partial\nu}] = \Phi_{k_0}$ on $\partial\Omega$. We can therefore rewrite (2.2) as

$$(u-u_0)(x) = \int_{\partial\Omega}\mu_0\Phi_{k_0}(x-y)g(y)d\sigma(y) - \int_{\partial\Omega}\mu_0\Lambda^{-1}_{\mu_0,\varepsilon_0}[\Phi_{k_0}](x-y)\Lambda_{\mu,\varepsilon}[g](y)d\sigma(y).$$

One can check that $\Lambda_{\mu,\varepsilon}$ is self-adjoint under the duality pair $\langle\cdot,\cdot\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}$ on $\partial\Omega$. So, we can further write

$$(2.4)\quad (u-u_0)(x) = \mu_0 \int_{\partial\Omega} \Phi_{k_0}(x-y)\Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu_0,\varepsilon_0}-\Lambda_{\mu,\varepsilon})[g](y)d\sigma(y), \quad x\in\mathbb{R}^d\backslash\overline{\Omega}.$$

We now use Graf's addition formula [27] to derive a representation of $u-u_0$ for $|x|>R$, assuming that $\Omega\subset B_R(0)$ for some $R>0$. For the fundamental solution (2.1), we recall the Graf's addition formula for $|x|>|y|$:

$$(2.5)\qquad H_0^{(1)}(k_0|x-y|) = \sum_{n\in\mathbb{Z}} H_n^{(1)}(k_0|x|)e^{in\theta_x}J_n(k_0|y|)e^{-in\theta_y},$$

where x is in polar coordinate $(|x|,\theta_x)$, and the same for y. Now we define

$$(2.6)\qquad (u_0)_m(y) := J_m(k_0|y|)e^{im\theta_y},$$

and let u_m be the total field corresponding to the incident field $(u_0)_m$, namely, the solution to (1.2)–(1.3) with the incident field u_0 replaced by $(u_0)_m$. If we write

$$(2.7)\qquad g_m := \frac{1}{\mu}\frac{\partial u_m^-}{\partial\nu},$$

then for any incident field u_0 admitting the expansion

$$(2.8)\qquad u_0(y) = \sum_{m\in\mathbb{Z}} a_m J_m(k_0|y|)e^{im\theta_y},$$

we have

$$(2.9)\qquad g = \frac{1}{\mu}\frac{\partial u^-}{\partial\nu} = \sum_{m\in\mathbb{Z}} a_m g_m.$$

Noting that $\Omega\subset B_R(0)$, and putting (2.5) and (2.9) into (2.4), we derive the following representation:

$$(2.10)\qquad (u-u_0)(x) = -\frac{i\mu_0}{4}\sum_{m,n\in\mathbb{Z}}\int_{\partial\Omega} a_m H_n^{(1)}(k_0|x|)J_n(k_0|y|)e^{in(\theta_x-\theta_y)} \times \Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu_0,\varepsilon_0}-\Lambda_{\mu,\varepsilon})[g_m](y)d\sigma(y)$$

for $|x|>R$. This motivates us to introduce the following definition.

Definition 2.1. *The scattering coefficients $\{W_{nm}\}_{m,n\in\mathbb{Z}}$ at frequency ω of the inhomogeneous scatterer Ω with the permittivity and permeability distributions ε, μ are defined by*

(2.11)

$$W_{nm} = W_{nm}[\varepsilon,\mu,\omega,\Omega] := \mu_0\int_{\partial\Omega} J_n(k_0|y|)e^{-in\theta_y}\Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu_0,\varepsilon_0}-\Lambda_{\mu,\varepsilon})[g_m](y)d\sigma(y).$$

With this definition and the derivations above, we immediately come to the following representation theorem from (2.10).

Theorem 2.2. *Assume that $\Omega\subset B_R(0)$ for some $R>0$. Then for an incident field of the form $u_0(y)=\sum_{m\in\mathbb{Z}} a_m J_m(k_0|y|)e^{im\theta_y}$, the total field u (i.e., the solution of* (1.2)–(1.3)*) has the following representation:*

$$(2.12)\qquad (u-u_0)(x) = -\frac{i}{4}\sum_{m,n} a_m H_n^{(1)}(k_0|x|)e^{in\theta_x}W_{nm} \quad \textit{for} \quad |x|>R.$$

3. Representation and decay property of scattering coefficients. In this section we would like to represent the scattering coefficients using layer potentials and study their decay properties. In order to do this, we first introduce the Neumann function of the Helmholtz equation and the single and double layer potentials.

Let $N_{\mu,\varepsilon}(x, y)$ be the fundamental solution to the problem (2.3), i.e., for each fixed $z \in \Omega$, $N_{\mu,\varepsilon}(\cdot, z)$ is the solution to

$$\nabla \cdot \frac{1}{\mu}\nabla N_{\mu,\varepsilon}(\cdot, z) + \omega^2 \varepsilon N_{\mu,\varepsilon}(\cdot, z) = -\delta_z(\cdot) \text{ in } \Omega; \quad \frac{1}{\mu}\frac{\partial}{\partial\nu} N_{\mu,\varepsilon}(\cdot, z) = 0 \text{ on } \partial\Omega. \tag{3.1}$$

Let $\mathcal{N}_{\mu,\varepsilon}[g](x) := \int_{\partial\Omega} N_{\mu,\varepsilon}(x, y)g(y)d\sigma(y)$ for $x \in \Omega$. Then we can see that $\mathcal{N}_{\mu,\varepsilon}[g](x)$ is the solution to (2.3), and that

$$\Lambda_{\mu,\varepsilon}[g](x) = \mathcal{N}_{\mu,\varepsilon}[g](x), \quad x \in \partial\Omega, \tag{3.2}$$

by noting the relation (cf. [10])

$$\frac{1}{\mu}\frac{\partial}{\partial\nu}\mathcal{N}_{\mu,\varepsilon}[g] = g \quad \text{on } \partial\Omega.$$

Let $\mathcal{S}_{k_0}[\phi]$ and $\mathcal{D}_{k_0}[\phi]$ be the following single and double layer potentials on $\partial\Omega$:

$$\mathcal{S}_{k_0}[\phi](x) = \int_{\partial\Omega} \Phi_{k_0}(x - y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2, \tag{3.3}$$

and

$$\mathcal{D}_{k_0}[\phi](x) = \int_{\partial\Omega} \frac{\partial\Phi_{k_0}}{\partial\nu_y}(x - y)\phi(y)d\sigma(y), \quad x \in \mathbb{R}^2\backslash\partial\Omega. \tag{3.4}$$

Then the layer potentials $\mathcal{S}_{k_0}$ and $\mathcal{D}_{k_0}$ satisfy the following jump conditions:

$$\frac{\partial}{\partial\nu}\left(\mathcal{S}_{k_0}[\phi]\right)^{\pm} = \left(\pm\frac{1}{2}I + \mathcal{K}^*_{k_0,\Omega}\right)[\phi], \quad \left(\mathcal{D}_{k_0}[\phi]\right)^{\pm} = \left(\mp\frac{1}{2}I + \mathcal{K}_{k_0,\Omega}\right)[\phi], \tag{3.5}$$

where $\mathcal{K}_{k_0,\Omega}$ is the boundary integral operator defined by

$$\mathcal{K}_{k_0,\Omega}[\phi](x) = \int_{\partial\Omega} \frac{\partial\Phi_{k_0}}{\partial\nu_y}(x - y)\phi(y)d\sigma(y)$$

and $\mathcal{K}^*_{k_0,\Omega}$ is the L^2 adjoint of $\mathcal{K}_{k_0,\Omega}$ with L^2 being equipped with the real inner product. Note that $\frac{1}{2}I + \mathcal{K}^*_{k_0,\Omega}$ is invertible if k_0^2 is not a Dirichlet eigenvalue of $-\Delta$ on Ω; see [9, 10]. From (2.4) and the transmission conditions (1.2), we can see that the solution u to (1.2)–(1.3) can be represented as

$$u(x) = u_0(x) + \mu_0\mathcal{S}_{k_0}[\phi] \text{ for } x \in \mathbb{R}^d\backslash\overline{\Omega}; \quad u(x) = \mathcal{N}_{\mu,\varepsilon}[\psi] \text{ for } x \in \Omega \tag{3.6}$$

for some density pair $(\phi, \psi) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ which satisfies

$$u_0 = \Lambda_{\mu,\varepsilon}[\psi] - \mu_0\mathcal{S}_{k_0}[\phi] \quad \text{and} \quad \frac{1}{\mu_0}\frac{\partial u_0}{\partial\nu} = -\left(\frac{1}{2}I + \mathcal{K}^*_{k_0,\Omega}\right)[\phi] + \psi \quad \text{on} \quad \partial\Omega.$$

If we define

$$A := \begin{pmatrix} -\mu_0\mathcal{S}_{k_0} & \Lambda_{\mu,\varepsilon} \\ -(\frac{1}{2}I + \mathcal{K}^*_{k_0,\Omega}) & I \end{pmatrix}, \tag{3.7}$$

then we can write (ϕ, ψ) as the solution to the following equation,

$$A \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} u_0 \\ \frac{1}{\mu_0} \frac{\partial u_0}{\partial \nu} \end{pmatrix}, \tag{3.8}$$

and show the following result.

LEMMA 3.1. *The operator* $A : L^2(\partial\Omega)\times L^2(\partial\Omega) \to L^2(\partial\Omega)\times L^2(\partial\Omega)$ *is invertible.*

Proof. Let $(\phi, \psi) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ be such that $A\binom{\phi}{\psi} = 0$. Let u be defined by

$$u = \begin{cases} \mathcal{N}_{\mu,\varepsilon}[\psi] & \text{in } \Omega, \\ \mu_0 \mathcal{S}_{k_0}[\phi] & \text{in } \mathbb{R}^d \backslash \overline{\Omega}. \end{cases}$$

From the jump conditions

$$\begin{cases} \mu_0 \mathcal{S}_{k_0}[\phi] = \mathcal{N}_{\mu,\varepsilon}[\psi] & \text{on } \partial\Omega, \\ \mu_0(\frac{1}{2}I + \mathcal{K}^*_{k_0,\Omega}) = \frac{\partial}{\partial\nu}\mathcal{N}_{\mu,\varepsilon}[\psi] = \mu_0\psi & \text{on } \partial\Omega, \end{cases}$$

one can see that u satisfies the Helmholtz equation (1.1) together with the outgoing Sommerfeld radiation condition

$$\frac{\partial}{\partial r}u - ik_0 u = o(|x|^{-\frac{1}{2}}) \text{ as } |x| \to \infty. \tag{3.9}$$

Uniqueness of a solution to (1.1) subject to the Sommerfeld radiation condition (3.9) shows that $u = 0$ in $\mathbb{R}^d$. Then, since k_0^2 is not a Dirichlet eigenvalue of $-\Delta$ on Ω, we have $\phi = 0$, hence $\psi = 0$ as well. This shows the injectivity of A.

Next, since $\frac{1}{\mu_0}\Phi_{k_0}(|x-y|)$ and $N_{\mu,\varepsilon}(x, y)$ have the same singularity type (i.e., of logarithmic type) as $|x - y| \to 0$ [11] (see Appendix A) and $\mathcal{K}^*_{k_0,\Omega}$ is a compact operator on $L^2(\partial\Omega)$, it follows that A is a compact perturbation of the invertible operator on $L^2(\partial\Omega) \times L^2(\partial\Omega)$ which is given by

$$\begin{pmatrix} -\mu_0\mathcal{S}_{k_0} & \mu_0\mathcal{S}_{k_0} \\ -\frac{1}{2}I & I \end{pmatrix}.$$

Therefore, the Fredholm alternative holds and injectivity of A shows its invertibility. ∎

We define (ϕ_m, ψ_m) as the pair of solutions to the above equation (3.8) corresponding to the incident field $u_0(y) = (u_0)_m(y) := J_m(k_0|y|)e^{im\theta_y}$ defined as in (2.6); then W_{nm} can be simply expressed as

$$W_{nm} = \mu_0 \int_{\partial\Omega} J_n(k_0|y|)e^{-in\theta_y}\phi_m(y)d\sigma(y) = \mu_0\langle (u_0)_n, \phi_m\rangle_{L^2(\partial\Omega)}. \tag{3.10}$$

Using this expression, we can derive the decay property of scattering coefficients. Again from the fact that the functions $\frac{1}{\mu_0}\Phi_{k_0}(|x-y|)$ and $N_{\mu,\varepsilon}(x, y)$ have the same logarithmic type singularity as $|x - y| \to 0$ [11], we obtain from (3.8) that

$$\|\phi_m\|_{L^2(\partial\Omega)} + \|\psi_m\|_{L^2(\partial\Omega)} \le C\left(\|(u_0)_m\|_{L^2(\partial\Omega)} + \left\|\frac{\partial}{\partial\nu}(u_0)_m\right\|_{L^2(\partial\Omega)}\right). \tag{3.11}$$

Using the asymptotic behavior of the Bessel function J_m [2],

$$J_m(t)\Big/\frac{1}{\sqrt{2\pi|m|}}\left(\frac{et}{2|m|}\right)^{|m|} \to 1 \tag{3.12}$$

as $m \to \infty$, we have

$$\|(u_0)_n\|_{L^2(\partial\Omega)} \leq \frac{C_1^{|n|}}{|n|^{|n|}} \quad \text{and} \quad \|\phi_m\|_{L^2(\partial\Omega)} \leq \frac{C_2^{|m|}}{|m|^{|m|}}$$

for some constants C_1 and C_2. Therefore, we deduce from (3.10) that

$$|W_{nm}| = |\mu_0\langle (u_0)_m, \phi_m\rangle_{L^2(\partial\Omega)}| \leq \|(u_0)_n\|_{L^2(\partial\Omega)}\|\phi_m\|_{L^2(\partial\Omega)} \leq \frac{C^{|m|+|n|}}{|m|^{|m|}|n|^{|n|}}$$

for some constant C, leading to the following theorem.

THEOREM 3.2. *There exists a constant C depending on $(\mu, \varepsilon, \omega)$ such that*

$$|W_{nm}| \leq \frac{C^{|m|+|n|}}{|m|^{|m|}|n|^{|n|}} \quad \textit{for all } n, m \in \mathbb{Z}. \tag{3.13}$$

4. Far-field pattern.

In this section we shall derive the far-field pattern of the scattered field in terms of the scattering coefficients.

We consider the incident field u_0 as a plane wave of the form $u_0 = e^{ik_0\xi\cdot x}$ with ξ being on the unit circle. We recall the Fourier mode $(u_0)_m(y) := J_m(k_0|y|)e^{im\theta_y}$ in (2.6), and the solution pair (ϕ_m, ψ_m) to (3.8) corresponding to the incident field $(u_0)_m$. Then by the well-known Jacobi–Anger decomposition, we have the following decomposition of the plane wave in terms of $(u_0)_m$:

$$u_0 = e^{ik_0\xi\cdot x} = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\xi)} J_m(k_0|x|)e^{im\theta_x} = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\xi)}(u_0)_m, \tag{4.1}$$

where $\xi = (\cos\theta_\xi, \sin\theta_\xi)$ and $x = |x|(\cos\theta_x, \sin\theta_x)$.

Let (ϕ, ψ) be the solution pair to (3.8) corresponding to the incident field $u_0 = e^{ik_0\xi\cdot x}$, then using (4.1) and the principle of superposition we have

$$\phi = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\xi)}\phi_m \quad \text{and} \quad \psi = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_\xi)}\psi_m. \tag{4.2}$$

Then we can derive the far-field pattern in terms of the scattering coefficients. In fact, using (3.6), together with the following well-known asymptotic expansion of $\Phi_{k_0}(x-y)$,

$$\Phi_{k_0}(x-y) = e^{-i\frac{\pi}{4}}\sqrt{\frac{2}{\pi k_0|x|}}e^{ik_0(|x|-|y|\cos(\theta_x-\theta_y))} + O(|x|^{-\frac{3}{2}}) \quad \text{as} \quad |x| \to \infty, \tag{4.3}$$

and the Jacobi–Anger identity,

$$e^{-ik_0|y|\cos(\theta_x-\theta_y)} = \sum_n J_n(k_0|y|)e^{-in(\theta_y+\frac{\pi}{2})}e^{in\theta_x}, \tag{4.4}$$

we deduce the following representation:

$$u(x) - e^{ik_0\xi\cdot x} = -ie^{-i\frac{\pi}{4}}\frac{\mu_0 e^{ik_0|x|}}{\sqrt{8\pi k_0|x|}}\sum_{m,n\in\mathbb{Z}} i^{(m-n)}e^{-im\theta_\xi}e^{in\theta_x}$$
$$\times \int_{\partial\Omega} J_n(k_0|y|)e^{-in\theta_y}\phi_m(y)d\sigma(y) + O(|x|^{-\frac{3}{2}}).$$

Comparing this with the representation of W_{nm} in (3.10), we infer that

$$u(x) - e^{ik_0\xi\cdot x} = -ie^{-i\frac{\pi}{4}}\frac{\mu_0 e^{ik_0|x|}}{\sqrt{8\pi k_0|x|}}\sum_{m,n\in\mathbb{Z}} i^{(m-n)}e^{-im\theta_\xi}e^{in\theta_x}W_{nm} + O(|x|^{-\frac{3}{2}}). \tag{4.5}$$

This motivates us with the following definition of the far-field pattern.

DEFINITION 4.1. *For the total field u satisfying* (1.2)–(1.3) *with the incident field $u_0(x) = e^{ik_0\xi\cdot x}$, the far-field pattern $A_\infty[\varepsilon,\mu,\omega](\theta_\xi,\theta_x)$ can be defined by*

$$u(x) - e^{ik_0\xi\cdot x} = -ie^{-i\frac{\pi}{4}}\frac{\mu_0 e^{ik_0|x|}}{\sqrt{8\pi k_0|x|}}A_\infty[\varepsilon,\mu,\omega](\theta_\xi,\theta_x) + O(|x|^{-\frac{3}{2}}) \quad as \quad |x|\to\infty. \tag{4.6}$$

By comparing (4.6) with (4.5) we come to the following theorem.

THEOREM 4.2. *Let θ_ξ and θ_x be, respectively, the incident and the scattered direction. Then the far-field pattern $A_\infty[\varepsilon,\mu,\omega](\theta_\xi,\theta_x)$ defined by* (4.6) *can be expressed in the explicit form*

$$A_\infty[\varepsilon,\mu,\omega](\theta_\xi,\theta_x) = \sum_{m,n\in\mathbb{Z}} i^{(m-n)}e^{-im\theta_\xi}e^{in\theta_x}W_{nm}[\varepsilon,\mu,\omega]. \tag{4.7}$$

It is easy to see that the bounds in (3.13) ensure the convergence of the above series uniformly with respect to θ_ξ and θ_x, so $A_\infty[\varepsilon,\mu,\omega]$ is well-defined. Moreover, one can see that reconstructing the scattering coefficients from the far-field pattern is an exponentially ill-posed problem if the measurements of A_∞ are corrupted with noise.

5. Transformation rules and properties of scattering coefficients. In this section, we derive more properties, including some transformation rules for the scattering coefficients. To do so, we first represent the scattering coefficients in terms of an exterior NtD map. For any $g\in H^{-\frac{1}{2}}(\partial\Omega)$, the action of the exterior NtD map $\Lambda^e_{\mu_0,\varepsilon_0} : H^{-\frac{1}{2}}(\partial\Omega)\to H^{\frac{1}{2}}(\partial\Omega)$ is defined by the trace $u = \Lambda^e_{\mu_0,\varepsilon_0}g \in H^{\frac{1}{2}}(\partial\Omega)$ of the solution u to the system

$$\begin{cases} \frac{1}{\mu_0}\Delta u + \varepsilon_0\omega^2 u = 0 & \text{in } \mathbb{R}^d\backslash\overline{\Omega}, \\ \frac{1}{\mu_0}\frac{\partial u}{\partial\nu} = g & \text{on } \partial\Omega, \\ \frac{\partial}{\partial r}u - ik_0 u = o(|x|^{-\frac{1}{2}}) & \text{as } |x|\to\infty. \end{cases} \tag{5.1}$$

With the help of the exterior NtD map $\Lambda^e_{\mu_0,\varepsilon_0}$, we can derive some new representation of the scattering coefficients.

LEMMA 5.1. *Let $(u_0)_n$ and the scattering coefficients W_{nm} be defined as in* (2.6) *and* (2.11), *respectively, and let $\Lambda_{\mu,\varepsilon}$ and $\Lambda^e_{\mu_0,\varepsilon_0}$ be the interior and exterior NtD maps. Then the scattering coefficients W_{nm} can be expressed as*

$$W_{nm} = \langle (u_0)_n, \mathcal{A}_{\mu,\varepsilon}(u_0)_m\rangle_{L^2(\partial\Omega)} \quad \textit{for all } n,m\in\mathbb{Z}, \tag{5.2}$$

where the operator $\mathcal{A}_{\mu,\varepsilon}$ is given by

$$\mathcal{A}_{\mu,\varepsilon} := \mu_0\Lambda^{-1}_{\mu_0,\varepsilon_0}\left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon}\right)\left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1}\left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0}\right)\Lambda^{-1}_{\mu_0,\varepsilon_0}. \tag{5.3}$$

Proof. For a given incident field u_0, let $(\phi, \psi) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ are the density pair that solves (3.8). Then it follows from the jump conditions of the layer potentials in (3.5) that

$$\psi = \phi + \left(-\frac{1}{2}I + \mathcal{K}^*_{k_0,\Omega}\right)[\phi] + \frac{1}{\mu_0}\frac{\partial u_0}{\partial \nu} = \phi + \frac{\partial}{\partial \nu}\left(\mathcal{S}_{k_0}[\phi]\right)^- + \frac{1}{\mu_0}\frac{\partial u_0}{\partial \nu}, \tag{5.4}$$

$$\psi = \left(\frac{1}{2}I + \mathcal{K}^*_{k_0,\Omega}\right)[\phi] + \frac{1}{\mu_0}\frac{\partial u_0}{\partial \nu} = \frac{\partial}{\partial \nu}\left(\mathcal{S}_{k_0}[\phi]\right)^+ + \frac{1}{\mu_0}\frac{\partial u_0}{\partial \nu}. \tag{5.5}$$

By directly applying the interior and exterior NtD operators to (5.4) and (5.5), we obtain

$$\begin{cases} \Lambda^e_{\mu_0,\varepsilon_0}[\psi] & = \mu_0 \mathcal{S}_{k_0}[\phi] + \frac{1}{\mu_0}\Lambda^e_{\mu_0,\varepsilon_0}\left[\frac{\partial u_0}{\partial \nu}\right], \\ \Lambda_{\mu_0,\varepsilon_0}[\psi] & = \Lambda_{\mu_0,\varepsilon_0}[\phi] + \mu_0 \mathcal{S}_{k_0}[\phi] + u_0, \\ \Lambda_{\mu,\varepsilon}[\psi] & = u_0 + \mu_0 \mathcal{S}_{k_0}[\phi], \end{cases}$$

which combines to give

$$\begin{cases} (\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0})[\psi] = \frac{1}{\mu_0}(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0})\left[\frac{\partial u_0}{\partial \nu}\right] = (\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0})\Lambda^{-1}_{\mu_0,\varepsilon_0}[u_0], \\ (\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon})[\psi] = \Lambda_{\mu_0,\varepsilon_0}[\phi]. \end{cases} \tag{5.6}$$

Substituting the first equation in (5.6) into the second, we readily get

$$\begin{aligned} \phi &= \Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon})[\psi] \\ &= \Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon})(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1}(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0})\Lambda^{-1}_{\mu_0,\varepsilon_0}[u_0]. \end{aligned}$$

In particular, if $(\phi_m, \psi_m) \in L^2(\partial\Omega) \times L^2(\partial\Omega)$ are the density pair that satisfies (3.8) corresponding to the incident field $u_0(y) = (u_0)_m(y) := J_m(k_0|y|)e^{im\theta_y}$ as in (2.6), then ϕ_m satisfies

$$\begin{aligned} \phi_m &= \Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon})(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1}(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0})\Lambda^{-1}_{\mu_0,\varepsilon_0}[(u_0)_m] \\ &= \frac{1}{\mu_0}\mathcal{A}_{\mu,\varepsilon}(u_0)_m. \end{aligned} \tag{5.7}$$

Substituting (5.7) into (3.10), we conclude that

$$W_{nm} = \mu_0 \langle (u_0)_n, \phi_m \rangle_{L^2(\partial\Omega)} = \langle (u_0)_n, \mathcal{A}_{\mu,\varepsilon}(u_0)_m \rangle_{L^2(\partial\Omega)}. \qquad \square$$

With the representations (3.10) and (5.2), we can derive some special transformation rules for the scattering coefficients. Since some of these results are the well-known properties of the scattering amplitude via Lax–Phillips in disguise, in here we only present a rough sketch of the proof. For a more detailed argument, one may refer to [5]. Although these transformation rules are consistent with the properties of the scattering amplitude, they are presented from a different perspective, and reflect the very fact that the concept of scattering coefficients is an appropriate geometric entity for scatterer description.

COROLLARY 5.2. *The scattering coefficients* $\{W_{nm}\}_{n,m\in\mathbb{Z}}$ *in* (2.11) *meet the following transformation rules:*

1. $W_{nm}[\varepsilon,\mu,\omega,\Omega] = \overline{W_{mn}[\varepsilon,\mu,\omega,\Omega]}$;
2. $W_{nm}[\varepsilon,\mu,\omega,e^{i\theta}\Omega] = e^{i(m-n)\theta}W_{nm}[\varepsilon,\mu,\omega,\Omega]$ *for all* $\theta\in[0,2\pi]$;
3. $W_{nm}[\varepsilon,\mu,\omega,s\Omega] = W_{nm}[\varepsilon,\mu,s\omega,\Omega]$ *for all* $s>0$;
4. $W_{nm}[\varepsilon,\mu,\omega,\Omega+z] = \sum_{l,l\in\mathbb{Z}}\overline{(u_0)_p(z)}(u_0)_l(z)W_{n-p,m-l}[\varepsilon,\mu,\omega,\Omega]$ *for all* $z\in\mathbb{R}^2$,

where we identify the spaces before and after translation, rotation, and scaling by the natural isomorphism, e.g., $H^s(\partial\Omega)\cong H^s(e^{i\theta}\partial\Omega)$.

Proof. We start with the first result in Corollary 5.2. From representation (5.2) of W_{nm}, it suffices to show that the operator $\mathcal{A}_{\mu,\varepsilon}$ defined in (5.3) is self-adjoint. To do this, we utilize the following identity for any operators A, B, C such that $A-C$ and $B-C$ are invertible:

$$(A-C)^{-1}-(B-C)^{-1} = (A-C)^{-1}(B-A)(B-C)^{-1} = (B-C)^{-1}(B-A)(A-C)^{-1}. \tag{5.8}$$

Using this we can write

$$(\Lambda_{\mu_0,\varepsilon_0}-\Lambda^e_{\mu_0,\varepsilon_0})^{-1}-(\Lambda_{\mu,\varepsilon}-\Lambda^e_{\mu_0,\varepsilon_0})^{-1} = (\Lambda_{\mu,\varepsilon}-\Lambda^e_{\mu_0,\varepsilon_0})^{-1}(\Lambda_{\mu,\varepsilon}-\Lambda_{\mu_0,\varepsilon_0})(\Lambda_{\mu_0,\varepsilon_0}-\Lambda^e_{\mu_0,\varepsilon_0})^{-1}. \tag{5.9}$$

Substituting (5.9) into (5.3), we get

$$\mathcal{A}_{\mu,\varepsilon} = \mu_0\Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu_0,\varepsilon_0}-\Lambda_{\mu,\varepsilon})\Lambda^{-1}_{\mu_0,\varepsilon_0} + \mu_0\Lambda^{-1}_{\mu_0,\varepsilon_0}(\Lambda_{\mu,\varepsilon}-\Lambda_{\mu_0,\varepsilon_0})(\Lambda_{\mu,\varepsilon}-\Lambda^e_{\mu_0,\varepsilon_0})^{-1}(\Lambda_{\mu,\varepsilon}-\Lambda_{\mu_0,\varepsilon_0})\Lambda^{-1}_{\mu_0,\varepsilon_0}.$$

Now the self-adjointness of $\mathcal{A}_{\mu,\varepsilon}$ is a consequence of the self-adjointness of $\Lambda_{\mu_0,\varepsilon_0}$, $\Lambda_{\mu,\varepsilon}$, and $\Lambda^e_{\mu_0,\varepsilon_0}$.

To see the second result in Corollary 5.2, we consider the change of coordinates from $(|y|,\theta_y)$ to $(|\widetilde{y}|,\widetilde{\theta_y})$, with $\widetilde{\theta_y}+\theta=\theta_y$ and $|\widetilde{y}|=|y|$. It follows from definition (2.6) that $(u_0)_m(y)=(u_0)_m(\widetilde{y})e^{im\theta}$. With such a change of variable, we immediately obtain that the density pair (ψ_m,ϕ_m) satisfying (3.8) with incident field $u_0(y)=(u_0)_m(y)$ and electromagnetic parameters $\mu(y)$ and $\varepsilon(y)$ actually has the form $(\psi_m(y),\phi_m(y)) = (\widetilde{\phi}_m(\widetilde{y}),\widetilde{\phi}_m(\widetilde{y}))e^{im\theta}$, where $(\widetilde{\psi}_m,\widetilde{\phi}_m)$ satisfies (3.8) with incident field $(u_0)_m(\widetilde{y})$ and parameters $\mu(\widetilde{y})$, $\varepsilon(\widetilde{y})$. Then a direct substitution of expressions $(u_0)_m$ and ϕ_m into (3.10) and a change of variable immediately prove the second result in Corollary 5.2.

The derivation of the third result is similar to the second, except that we now consider the change of coordinates from $(|y|,\theta_y)$ to $(|\widetilde{y}|,\widetilde{\theta_y})$, with $\widetilde{\theta_y}=\theta_y$ and $s|\widetilde{y}|=|y|$. Then $(u_0)_m(y)=(u_0)_m(s\widetilde{y})$ and with (ψ_m,ϕ_m) satisfying (3.8) with incident field $u_0(y)=(u_0)_m(y)$ and parameters $\mu(y)$ and $\varepsilon(y)$ given by $(\psi_m(y),\phi_m(y)) = (\widetilde{\psi}_m(s\widetilde{y}),\widetilde{\phi}_m(s\widetilde{y}))/s$ with $(\widetilde{\psi}_m,\widetilde{\phi}_m)$ satisfying (3.8) with incident field $(u_0)_m(\widetilde{y})$ and parameters $\mu(\widetilde{y})$, $\varepsilon(\widetilde{y})$. Now the desired third result in Corollary 5.2 follows from the straightforward substitution of expressions $(u_0)_m$ and ϕ_m into (3.10) and the chain rule.

Finally we come to derive the last relation in Corollary 5.2. To do so, we consider the change of coordinates from $(|y|,\theta_y)$ to $(|\widetilde{y}|,\widetilde{\theta_y})$ that has point z as the origin. Then the definition of $(u_0)_m$ in (2.6) and the Graf's addition formula (2.5) allow us to write

$$(u_0)_m = J_m(k_0|y|)e^{im\theta_y} = \sum_{a\in\mathbb{Z}} J_a(k_0|z|)e^{im\theta_z}J_{m-a}(k_0|\widetilde{y}|)e^{i(m-a)\widetilde{\theta_y}}.$$

By the linearity of operator A in (3.7), the density pair (ψ_m, ϕ_m) satisfying (3.8) with the incident field $u_0(y) = (u_0)_m(y)$ can be expressed in the form $(\psi_m, \phi_m) = \sum_{a\in\mathbb{Z}} J_a(k_0|z|)e^{im\theta_z}(\widetilde{\psi}_{m-a}(\widetilde{y}), \widetilde{\phi}_{m-a}(\widetilde{y}))$, where $(\widetilde{\psi}_m, \widetilde{\phi}_m)$ satisfies (3.8) with the incident field $(u_0)_m(\widetilde{y})$. Then the last result in Corollary 5.2 follows readily from a direct substitution of the representations $(u_0)_m$ and ϕ_m into (3.10). ∎

We end this section with one more representation of W_{nm}.

LEMMA 5.3. *Let $(u_0)_m$ be defined as in* (2.6) *and u_m be the solution to* (1.1)–(1.3) *with the incident field $(u_0)_m$. Then the scattering coefficients in* (2.11) *admit the following representation for any n, $m \in \mathbb{Z}$:*

$$W_{nm} = \omega^2\mu_0 \int_\Omega (\varepsilon_0(y) - \varepsilon(y))\,\overline{(u_0)_n}(y)u_m(y)d\sigma(y) + \mu_0 \int_\Omega \left(\frac{1}{\mu(y)} - \frac{1}{\mu_0(y)}\right)\overline{\nabla(u_0)_n}(y)\nabla u_m(y)d\sigma(y). \tag{5.10}$$

Proof. Let (ψ_m, ϕ_m) be the density pair (ψ_m, ϕ_m) that satisfies (3.8) with the incident field $u_0(y) = (u_0)_m(y)$. Then it follows directly from (3.10), (3.5), and (3.8) that

$$W_{nm} = \mu_0 \int_{\partial\Omega} \overline{(u_0)_n}(y)\left(\psi_m(y) - \frac{1}{\mu_0}\frac{\partial (u_0)_m}{\partial\nu}(y)\right)d\sigma(y) - \mu_0 \int_{\partial\Omega} \overline{(u_0)_n}(y)\frac{\partial\left(\mathcal{S}_{k_0}[\phi_m]\right)^-}{\partial\nu}(y)d\sigma(y).$$

Using Green's identity and (3.8), we can further derive

$$\begin{aligned} W_{nm} &= \mu_0 \int_{\partial\Omega} \overline{(u_0)_n}(y)\left(\psi_m(y) - \frac{1}{\mu_0}\frac{\partial (u_0)_m}{\partial\nu}(y)\right)d\sigma(y) \\ &\quad - \mu_0 \int_{\partial\Omega} \frac{\partial\overline{(u_0)_n}}{\partial\nu}(y)\mathcal{S}_{k_0}[\phi_m](y)d\sigma(y) \\ &= \mu_0 \int_{\partial\Omega} \overline{(u_0)_n}(y)\psi_m(y)d\sigma(y) - \int_{\partial\Omega} \frac{\partial\overline{(u_0)_n}}{\partial\nu}(y)\Lambda_{\mu,\varepsilon}[\psi_m](y)d\sigma(y). \end{aligned}$$

Now the desired representation of W_{nm} follows from (3.6) and (3.2), the comparison of (5.6) with (2.11) and (3.10), and the Green's identity:

$$\begin{aligned} W_{nm} &= \mu_0 \int_{\partial\Omega} \overline{(u_0)_n}(y)\frac{1}{\mu}\frac{\partial u_m^-}{\partial\nu}d\sigma(y) - \int_{\partial\Omega} \frac{\partial\overline{(u_0)_n}}{\partial\nu}(y)u_m(y)d\sigma(y) \\ &= \omega^2\mu_0 \int_\Omega (\varepsilon_0 - \varepsilon)\,\overline{(u_0)_n}(y)u_m(y)d\sigma(y) \\ &\quad + \mu_0 \int_\Omega \left(\frac{1}{\mu} - \frac{1}{\mu_0}\right)\overline{\nabla(u_0)_n}(y)\nabla u_m(y)d\sigma(y). \end{aligned}$$ ∎

6. Sensitivity analysis.

In this section, we shall investigate the sensitivity of the scattering coefficients with respect to the changes in the permittivity and permeability distributions. This will provide us with perturbation formulas for evaluating the gradients that are needed in numerical minimization algorithms for reconstructing the permittivity and permeability distributions.

We study a perturbation of W_{nm} for $n, m \in \mathbb{Z}$ with respect to a change of (μ, ε). More specifically, we consider the difference $W^\delta_{nm} - W_{nm}$ between

$$W^\delta_{nm} := W_{nm}\left[\varepsilon^\delta, \mu^\delta, \omega, \Omega\right] \quad \text{and} \quad W_{nm} := W_{nm}[\varepsilon, \mu, \omega, \Omega] \tag{6.1}$$

in terms of the differences $\varepsilon^\delta - \varepsilon$ and $1/\mu^\delta - 1/\mu$, where (μ, ε) and $(\mu^\delta, \varepsilon^\delta)$ are two different sets of electromagnetic parameters. In the subsequent analysis, we shall often write

$$\widehat{\varepsilon} := \left\{ \|\varepsilon^\delta - \varepsilon\|^2_{L^\infty(\Omega)} + \left\| \frac{1}{\mu^\delta} - \frac{1}{\mu} \right\|^2_{L^\infty(\Omega)} \right\}^{1/2}. \tag{6.2}$$

We first note that if $\widehat{\varepsilon}$ is small enough, then the NtD map $\Lambda_{\mu^\delta,\varepsilon^\delta}$ is well-defined provided that $\Lambda_{\mu,\varepsilon}$ is well-defined. This follows from the theory of collectively compact operators; see [15, 10].

Next we show the following expression for the difference $W^\delta_{nm} - W_{nm}$.

LEMMA 6.1. *For all $n, m \in \mathbb{Z}$, the difference $W^\delta_{nm} - W_{nm}$ can be represented in terms of the interior and exterior NtD maps $\Lambda_{\mu,\varepsilon}$ and $\Lambda^e_{\mu_0,\varepsilon_0}$ as follows:*

$$W^\delta_{nm} - W_{nm} = \mu_0 \int_{\partial\Omega} \overline{\psi_n}(y) \left(\Lambda_{\mu,\varepsilon} - \Lambda_{\mu^\delta,\varepsilon^\delta}\right) [\psi^\delta_m](y) d\sigma(y), \tag{6.3}$$

where ψ_n and ψ^δ_m are given by

$$\psi_n = \left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0}\right) \Lambda^{-1}_{\mu_0,\varepsilon_0} (u_0)_n, \tag{6.4}$$

$$\psi^\delta_m = \left(\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0}\right) \Lambda^{-1}_{\mu_0,\varepsilon_0} (u_0)_m. \tag{6.5}$$

Proof. Using the identity (5.8) we can write

$$\begin{aligned}
&(\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1} - (\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1} \\
&\quad = (\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1} (\Lambda_{\mu,\varepsilon} - \Lambda_{\mu^\delta,\varepsilon^\delta}) (\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1},
\end{aligned} \tag{6.6}$$

which enables us to derive

$$\begin{aligned}
&\left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu^\delta,\varepsilon^\delta}\right) \left(\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} - \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon}\right) \left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \\
&= \left(\Lambda_{\mu,\varepsilon} - \Lambda_{\mu^\delta,\varepsilon^\delta}\right) \left(\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \\
&\quad + \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon}\right) \left[\left(\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} - \left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \right] \\
&= \left[I + \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon}\right) \left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \right] \left(\Lambda_{\mu,\varepsilon} - \Lambda_{\mu^\delta,\varepsilon^\delta}\right) (\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1} \\
&= \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0}\right) \left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} (\Lambda_{\mu,\varepsilon} - \Lambda_{\mu^\delta,\varepsilon^\delta}) (\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1}.
\end{aligned} \tag{6.7}$$

It follows directly from (6.7) and definition (5.3) for the operators $\mathcal{A}_{\mu,\varepsilon}$ and $\mathcal{A}_{\mu^\delta,\varepsilon^\delta}$ that

$$\begin{aligned}
&\mathcal{A}_{\mu^\delta,\varepsilon^\delta} - \mathcal{A}_{\mu,\varepsilon} \\
&\quad = \mu_0 \Lambda^{-1}_{\mu_0,\varepsilon_0} \Big\{ \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu^\delta,\varepsilon^\delta}\right) \left(\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \\
&\qquad - \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda_{\mu,\varepsilon}\right) \left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \Big\} \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0}\right) \Lambda^{-1}_{\mu_0,\varepsilon_0} \\
&\quad = \mu_0 \Lambda^{-1}_{\mu_0,\varepsilon_0} \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0}\right) \left(\Lambda_{\mu,\varepsilon} - \Lambda^e_{\mu_0,\varepsilon_0}\right)^{-1} \\
&\qquad \times (\Lambda_{\mu,\varepsilon} - \Lambda_{\mu^\delta,\varepsilon^\delta}) (\Lambda_{\mu^\delta,\varepsilon^\delta} - \Lambda^e_{\mu_0,\varepsilon_0})^{-1} \left(\Lambda_{\mu_0,\varepsilon_0} - \Lambda^e_{\mu_0,\varepsilon_0}\right) \Lambda^{-1}_{\mu_0,\varepsilon_0}.
\end{aligned}$$

Now identity (6.3) is a consequence of the above relation and the representation (5.2) for W_{nm} and W_{nm}^{δ},

$$\begin{aligned} W_{nm}^{\delta} - W_{nm} &= \langle (u_0)_n, \left(\mathcal{A}_{\mu^\delta,\varepsilon^\delta} - \mathcal{A}_{\mu,\varepsilon}\right)(u_0)_m\rangle_{L^2(\partial\Omega)} \\ &= \langle \psi_n, \left(\Lambda_{\mu,\varepsilon} - \Lambda_{\mu^\delta,\varepsilon^\delta}\right)[\psi_m^\delta]\rangle_{L^2(\partial\Omega)}, \end{aligned}$$

where $\langle , \rangle$ denotes the complex inner product on $L^2(\partial\Omega)$. ∎

The following identity will be useful for the subsequent analysis.

LEMMA 6.2. *For the solutions u_i $(i = 1, 2)$ to the two systems*

$$\nabla \cdot \left(\frac{1}{\mu_i}\nabla u_i\right) + \omega^2 \varepsilon_i u_i = 0 \text{ in } \Omega; \quad \frac{1}{\mu_i}\frac{\partial u_i}{\partial \nu} = g \text{ on } \partial\Omega, \tag{6.8}$$

the following identity holds:

$$\begin{aligned} &\int_{\partial\Omega} \overline{g}\left(\Lambda_{\mu_2,\varepsilon_2} - \Lambda_{\mu_1,\varepsilon_1}\right)[g] d\sigma \\ &= \frac{1}{2}\int_\Omega \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)\left(-|\nabla(u_1-u_2)|^2 + |\nabla u_1|^2 + |\nabla u_2|^2\right) \\ &\quad - \frac{1}{2}\omega^2 \int_\Omega (\varepsilon_1 - \varepsilon_2)\left(-|u_1-u_2|^2 + |u_1|^2 + |u_2|^2\right) dx. \end{aligned} \tag{6.9}$$

Proof. It follows easily from (6.8) and integration by parts that

$$\int_{\partial\Omega} \overline{g}(\Lambda_{\mu_1,\varepsilon_1})[g]\, d\sigma = \int_\Omega \left(\frac{1}{\mu_1}|\nabla u_1|^2 - \omega^2\varepsilon_1|u_1|^2\right) dx, \tag{6.10}$$

$$\int_{\partial\Omega} \overline{g}(\Lambda_{\mu_2,\varepsilon_2})[g]\, d\sigma = \int_\Omega \left(\frac{1}{\mu_1}\overline{\nabla u_1}\cdot\nabla u_2 - \omega^2\varepsilon_1\overline{u_1}u_2\right) dx, \tag{6.11}$$

$$\int_{\partial\Omega} \overline{g}(\Lambda_{\mu_2,\varepsilon_2})[g]\, d\sigma = \int_\Omega \left(\frac{1}{\mu_2}|\nabla u_2|^2 - \omega^2\varepsilon_2|u_2|^2\right) dx, \tag{6.12}$$

$$\int_{\partial\Omega} \frac{1}{\mu_1}\overline{\frac{\partial u_2}{\partial\nu}}u_2\, d\sigma = \int_\Omega \left(\frac{1}{\mu_1}|\nabla u_2|^2 - \omega^2\varepsilon_1|u_2|^2\right) dx. \tag{6.13}$$

Combining (6.10)–(6.13) yields

$$\begin{aligned} &\int_\Omega \frac{1}{\mu_1}|\nabla(u_2-u_1)|^2 dx - \omega^2\int_\Omega \varepsilon_1|u_2-u_1|^2 dx \\ &\quad + \int_\Omega \left(\frac{1}{\mu_2} - \frac{1}{\mu_1}\right)|\nabla u_2|^2 dx - \omega^2\int_\Omega (\varepsilon_2-\varepsilon_1)|u_2|^2 dx \\ &= \int_{\partial\Omega} \overline{g}(\Lambda_{\mu_1,\varepsilon_1})[g]\, d\sigma - 2\int_{\partial\Omega} \overline{g}(\Lambda_{\mu_2,\varepsilon_2})[g]\, d\sigma \\ &\quad + \int_{\partial\Omega} \frac{1}{\mu_1}\overline{\frac{\partial u_2}{\partial\nu}}u_2\, d\sigma + \int_{\partial\Omega} \overline{g}(\Lambda_{\mu_2,\varepsilon_2})[g]\, d\sigma - \int_{\partial\Omega} \frac{1}{\mu_1}\overline{\frac{\partial u_2}{\partial\nu}}u_2\, d\sigma \\ &= \int_{\partial\Omega} \overline{g}\left(\Lambda_{\mu_1,\varepsilon_1} - \Lambda_{\mu_2,\varepsilon_2}\right)[g]\, d\sigma, \end{aligned}$$

which gives the identity

$$\begin{aligned}\int_\Omega \frac{1}{\mu_1}|\nabla(u_2-u_1)|^2dx - \omega^2\int_\Omega \varepsilon_1|u_2-u_1|^2dx \\ + \int_\Omega\left(\frac{1}{\mu_2}-\frac{1}{\mu_1}\right)|\nabla u_2|^2dx - \omega^2\int_\Omega(\varepsilon_2-\varepsilon_1)|u_2|^2dx \\ = \int_{\partial\Omega}\overline{g}\left(\Lambda_{\mu_1,\varepsilon_1}-\Lambda_{\mu_2,\varepsilon_2}\right)[g]\,d\sigma.\end{aligned} \tag{6.14}$$

Swapping u_1 and u_2 in the above identity implies

$$\begin{aligned}\int_\Omega \frac{1}{\mu_2}|\nabla(u_1-u_2)|^2dx - \omega^2\int_\Omega \varepsilon_2|u_1-u_2|^2dx \\ + \int_\Omega\left(\frac{1}{\mu_1}-\frac{1}{\mu_2}\right)|\nabla u_1|^2dx - \omega^2\int_\Omega(\varepsilon_1-\varepsilon_2)|u_1|^2dx \\ = \int_{\partial\Omega}\overline{g}\left(\Lambda_{\mu_2,\varepsilon_2}-\Lambda_{\mu_1,\varepsilon_1}\right)[g]\,d\sigma.\end{aligned} \tag{6.15}$$

Now (6.9) follows by subtracting (6.14) from (6.15). ∎

By the same arguments as those in [1] (see also [8]), we can derive the following estimate.

LEMMA 6.3. *The difference between the interior NtD maps $\Lambda_{\mu,\varepsilon}$ and $\Lambda_{\mu^\delta,\varepsilon^\delta}$ can be represented in terms of the differences between two sets of electromagnetic parameters (μ,ε) and $(\mu^\delta,\varepsilon^\delta)$:*

$$\|\Lambda_{\mu^\delta,\varepsilon^\delta}-\Lambda_{\mu,\varepsilon}\| \le C\bigg(\|\varepsilon^\delta-\varepsilon\|_{L^\infty(\Omega)} + \left\|\frac{1}{\mu^\delta}-\frac{1}{\mu}\right\|_{L^\infty(\Omega)}\bigg). \tag{6.16}$$

Now we can further our analysis on the difference $W^\delta_{nm}-W_{nm}$ in terms of $\varepsilon^\delta-\varepsilon$ and $1/\mu^\delta-1/\mu$ using (6.3) and (6.16). Recalling ψ_n and ψ^δ_m from (6.4) and (6.5), we can define the solutions u_m, u^γ_m, u^δ_n, and $u^{\delta\gamma}_n$ to the following four systems:

$$\nabla\cdot\frac{1}{\mu}\nabla u_m + \varepsilon\omega^2u_m = 0 \text{ in } \Omega;\quad \frac{1}{\mu}\frac{\partial}{\partial\nu}u_m = \psi_m \text{ on } \partial\Omega; \tag{6.17}$$

$$\nabla\cdot\frac{1}{\mu^\delta}\nabla u^\gamma_m + \varepsilon^\delta\omega^2u^\gamma_m = 0 \text{ in } \Omega;\quad \frac{1}{\mu^\delta}\frac{\partial}{\partial\nu}u^\gamma_m = \psi_m \text{ on } \partial\Omega; \tag{6.18}$$

$$\nabla\cdot\frac{1}{\mu}\nabla u^{\delta\gamma}_n + \varepsilon\omega^2u^{\delta\gamma}_n = 0 \text{ in } \Omega;\quad \frac{1}{\mu}\frac{\partial}{\partial\nu}u^{\delta\gamma}_n = \psi^\delta_n \text{ on } \partial\Omega; \tag{6.19}$$

$$\nabla\cdot\frac{1}{\mu^\delta}\nabla u^\delta_n + \varepsilon^\delta\omega^2u^\delta_n = 0 \text{ in } \Omega;\quad \frac{1}{\mu^\delta}\frac{\partial}{\partial\nu}u^\delta_n = \psi^\delta_n \text{ on } \partial\Omega. \tag{6.20}$$

Noting from (3.6) that ψ_n and ψ^δ_m are the density functions in the Neumann potential along $\partial\Omega$ with coefficients (μ,ε) and $(\mu^\delta,\varepsilon^\delta)$, respectively, the solutions u_m and u^δ_n solve (1.1)–(1.3) with coefficients (μ,ε) and $(\mu^\delta,\varepsilon^\delta)$ and the incident field $(u_0)_m$ and $(u_0)_n$ defined as in (2.6). For convenience, we introduce a bilinear form

$$B(p,q) := \int_{\partial\Omega}\overline{p}\left(\Lambda_{\mu,\varepsilon}-\Lambda_{\mu^\delta,\varepsilon^\delta}\right)[q]d\sigma \quad \forall\, p,q\in H^{-\frac{1}{2}}(\partial\Omega). \tag{6.21}$$

Then (6.9) gives us an explicit expression of $B(g,g)$ for $g \in H^{-\frac{1}{2}}(\partial\Omega)$. By (6.3), the difference $W_{nm}^{\delta} - W_{nm}$ can be split using the bilinear form B as

$$
\begin{aligned}
W_{nm}^{\delta} - W_{nm} &= \mu_0 B(\psi_m, \psi_n^{\delta}) \\
&= \frac{\mu_0}{2}\left[B(\psi_m+\psi_n^{\delta}, \psi_m+\psi_n^{\delta}) - B(\psi_m,\psi_m) - B(\psi_n^{\delta},\psi_n^{\delta})\right] \\
&\quad + \frac{i\mu_0}{2}\left[B(\psi_m+i\psi_n^{\delta}, \psi_m+i\psi_n^{\delta}) - B(\psi_m,\psi_m) - B(\psi_n^{\delta},\psi_n^{\delta})\right] \\
&:= \frac{\mu_0}{2}(\mathrm{I}) + \frac{i\mu_0}{2}(\mathrm{II}),
\end{aligned}
\tag{6.22}
$$

where (I) and (II) are given by

$$
(\mathrm{I}) := B(\psi_m+\psi_n^{\delta}, \psi_m+\psi_n^{\delta}) - B(\psi_m,\psi_m) - B(\psi_n^{\delta},\psi_n^{\delta}), \tag{6.23}
$$

$$
(\mathrm{II}) := B(\psi_m+i\psi_n^{\delta}, \psi_m+i\psi_n^{\delta}) - B(\psi_m,\psi_m) - B(\psi_n^{\delta},\psi_n^{\delta}). \tag{6.24}
$$

By direct calculations, we get the following expression of the term (I):

$$
\begin{aligned}
(\mathrm{I}) &= B(\psi_m+\psi_n^{\delta}, \psi_m+\psi_n^{\delta}) - B(\psi_m,\psi_m) - B(\psi_n^{\delta},\psi_n^{\delta}) \\
&= \frac{1}{2}\int_{\Omega}\left(\frac{1}{\mu^{\delta}} - \frac{1}{\mu}\right)(-|\nabla(u_m + u_n^{\delta\gamma} - u_m^{\gamma} - u_n^{\delta})|^2 \\
&\quad + |\nabla(u_m + u_n^{\delta\gamma})|^2 + |\nabla(u_m^{\gamma} + u_n^{\delta})|^2)dx \\
&\quad - \frac{1}{2}\omega^2\int_{\Omega}\left(\varepsilon^{\delta} - \varepsilon\right)\left(-|u_m + u_n^{\delta\gamma} - u_m^{\gamma} - u_n^{\delta}|^2 + |u_m + u_n^{\delta\gamma}|^2 + |u_m^{\gamma} + u_n^{\delta}|^2\right)dx \\
&\quad - \left[\frac{1}{2}\int_{\Omega}\left(\frac{1}{\mu^{\delta}} - \frac{1}{\mu}\right)\left(-|\nabla(u_m - u_m^{\gamma})|^2 + |\nabla u_m|^2 + |\nabla u_m^{\gamma}|^2\right)dx\right. \\
&\quad \left. - \frac{1}{2}\omega^2\int_{\Omega}\left(\varepsilon^{\delta} - \varepsilon\right)\left(-|u_m - u_2|^2 + |u_m|^2 + |u_m^{\gamma}|^2\right)dx\right] \\
&\quad - \left[\frac{1}{2}\int_{\Omega}\left(\frac{1}{\mu^{\delta}} - \frac{1}{\mu}\right)\left(-|\nabla(u_n^{\delta\gamma} - u_n^{\delta})|^2 + |\nabla u_n^{\delta\gamma}|^2 + |\nabla u_n^{\delta}|^2\right)dx\right. \\
&\quad \left. - \frac{1}{2}\omega^2\int_{\Omega}\left(\varepsilon^{\delta} - \varepsilon\right)\left(-|u_n^{\delta\gamma} - u_n^{\delta}|^2 + |u_n^{\delta\gamma}|^2 + |u_n^{\delta}|^2\right)dx\right].
\end{aligned}
$$

From (6.16), we get

$$
\|u_m - u_m^{\gamma}\|_{H^1(\Omega)}^2 = O(\widehat{\varepsilon}) \quad \text{and} \quad \|u_n^{\delta\gamma} - u_n^{\delta}\|_{H^1(\Omega)}^2 = O(\widehat{\varepsilon}), \tag{6.25}
$$

where $\widehat{\varepsilon}$ is defined as in (6.2). Then using (6.25), we further the estimate of the term (I) as follows:

$$
\begin{aligned}
(\mathrm{I}) &= \int_{\Omega}\left(\frac{1}{\mu^{\delta}} - \frac{1}{\mu}\right)\left(|\nabla(u_m + u_n^{\delta})|^2 - |\nabla u_m|^2 - |\nabla u_n|^2\right)dx \\
&\quad - \omega^2\int_{\Omega}\left(\varepsilon^{\delta} - \varepsilon\right)\left(|u_m + u_n^{\delta}|^2 - |u_m|^2 - |u_n|^2\right)dx + O(\widehat{\varepsilon}^2) \\
&= 2\,\mathrm{Re}\left[\int_{\Omega}\left(\frac{1}{\mu^{\delta}} - \frac{1}{\mu}\right)\overline{\nabla u_n^{\delta}}\cdot\nabla u_m\,dx - \omega^2\int_{\Omega}\left(\varepsilon^{\delta} - \varepsilon\right)\overline{u_n^{\delta}}u_m\,dx\right] + O(\widehat{\varepsilon}^2).
\end{aligned}
\tag{6.26}
$$

Similarly, we can derive the following estimate for the term (II):

(6.27)
$$\begin{aligned}(\mathrm{II}) &= B(\psi_m + i\psi_n^\delta, \psi_m + i\psi_n^\delta) - B(\psi_m, \psi_m) - B(\psi_n^\delta, \psi_n^\delta)\\ &= 2\,\mathrm{Im}\left[\int_\Omega \left(\frac{1}{\mu^\delta} - \frac{1}{\mu}\right) \overline{\nabla u_n^\delta} \cdot \nabla u_m \, dx - \omega^2 \int_\Omega \left(\varepsilon^\delta - \varepsilon\right) \overline{u_n^\delta} u_m \, dx\right] + O(\widehat{\varepsilon}^2).\end{aligned}$$

Substituting (6.26) and (6.27) into (6.22) gives

$$\begin{aligned}W_{nm}^\delta - W_{nm} = \mu_0 &\int_\Omega \left(\frac{1}{\mu^\delta} - \frac{1}{\mu}\right) \overline{\nabla u_n^\delta} \cdot \nabla u_m \, dx\\ &- \mu_0\omega^2 \int_\Omega \left(\varepsilon^\delta - \varepsilon\right) \overline{u_n^\delta} u_m \, dx + O(\widehat{\varepsilon}^2).\end{aligned} \tag{6.28}$$

Furthermore, we have from (2.12) that for all $m \in \mathbb{Z}$,

$$\begin{aligned}u_m - (u_0)_m &= -\frac{i}{4} \sum_m H_n^{(1)}(k_0|x|) e^{in\theta_x} W_{nm},\\ u_m^\delta - (u_0)_m &= -\frac{i}{4} \sum_m H_n^{(1)}(k_0|x|) e^{in\theta_x} W_{nm}^\delta;\end{aligned} \tag{6.29}$$

subtracting the first one from the second in (6.29) gives

$$u_m^\delta = u_m - \frac{i}{4} \sum_m H_n^{(1)}(k_0|x|) e^{in\theta_x} \left(W_{nm}^\delta - W_{nm}\right). \tag{6.30}$$

Now replacing u_m^δ in (6.28) by (6.30), we arrive at the following theorem.

THEOREM 6.4. *Assume (μ, ε) and $(\mu^\delta, \varepsilon^\delta)$ are two different sets of electromagnetic parameters, and W_{nm} and W_{nm}^δ are defined as in (6.1). Let $\widehat{\varepsilon}$ be defined as in (6.2). For any $m \in \mathbb{Z}$, let u_m be the solution to (1.1)–(1.3) with the coefficients (μ, ε) and the incident field $(u_0)_m$ in (2.6). Then the following estimate holds for any n, $m \in \mathbb{Z}$:*

(6.31)
$$W_{nm}^\delta - W_{nm} = \mu_0 \int_\Omega \left(\frac{1}{\mu^\delta} - \frac{1}{\mu}\right) \overline{\nabla u_n} \cdot \nabla u_m - \omega^2\mu_0 \int_\Omega \left(\varepsilon^\delta - \varepsilon\right) \overline{u_n} u_m + O(\widehat{\varepsilon}^2).$$

The above formula provides a sensitivity analysis in terms of electromagnetic parameters (μ, ε) for arbitrary medium domains Ω. In order to derive reconstruction formulas for μ and ε from the scattering coefficients, we shall achieve more explicit and detailed sensitivity analysis and representation formulas for scattering coefficients W_{nm} when the medium domains are of some special geometry. This is our focus in the next section.

7. Explicit reconstruction formulas in the linearized case.

For a given $\widehat{\varepsilon} > 0$, consider μ, ε such that $(\|\varepsilon - \varepsilon_0\|^2_{L^\infty(\Omega)} + \left\|\mu^{-1} - \mu_0^{-1}\right\|^2_{L^\infty(\Omega)})^{1/2} = \widehat{\varepsilon}$. Then it follows from (5.10) and the definition of $(u_0)_m$ in (2.6) that

$$\begin{aligned}W_{nm} = \mu_0 &\int_\Omega \left(\frac{1}{\mu(y)} - \frac{1}{\mu_0}\right) \nabla(J_n(k_0|y|)e^{-in\theta_y}) \cdot \nabla(J_m(k_0|y|)e^{im\theta_y}) dy\\ &- \omega^2\mu_0 \int_\Omega \left(\varepsilon(y) - \varepsilon_0\right) J_n(k_0|y|) J_m(k_0|y|) e^{i(m-n)\theta_y} dy + O(\widehat{\varepsilon}^2).\end{aligned} \tag{7.1}$$

Now for all $n \neq 0$, we have by direct computing

$$\begin{aligned} \partial_{x_1}(J_n(kr)e^{in\theta}) &= \left(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta\right)(J_n(kr)e^{in\theta}) \\ &= \frac{k}{2}\cos\theta(J_{n-1}(kr) - J_{n+1}(kr))e^{in\theta} - \frac{in\sin\theta}{r}J_n(kr)e^{in\theta}, \end{aligned} \tag{7.2}$$

$$\begin{aligned} \partial_{x_2}(J_n(kr)e^{in\theta}) &= \left(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta\right)(J_n(kr)e^{in\theta}) \\ &= \frac{k}{2}\sin\theta(J_{n-1}(kr) - J_{n+1}(kr))e^{in\theta} + \frac{in\cos\theta}{r}J_n(kr)e^{in\theta}, \end{aligned} \tag{7.3}$$

which implies the explicit expression for the gradient term in (7.1) for n, $m \neq 0$:

$$\begin{aligned} &\nabla(J_n(k_0|y|)e^{-in\theta_y}) \cdot \nabla(J_m(k_0|y|)e^{im\theta_y}) \\ &\quad = \left[\frac{k_0^2}{4}\left(J_{n-1}(k_0|y|) - J_{n+1}(k_0|y|)\right)\left(J_{m-1}(k_0|y|) - J_{m+1}(k_0|y|)\right) \right. \\ &\qquad \left. + \frac{nm}{|y|^2}J_n(k_0|y|)J_m(k_0|y|)\right]e^{i(m-n)\theta_y}. \end{aligned} \tag{7.4}$$

For $n = 0$, we have $J_0' = -J_1$ and

$$\partial_{x_1}(J_0(kr)) = -k\cos\theta(J_1(kr)), \quad \partial_{x_2}(J_0(kr)) = -k\sin\theta(J_1(kr)), \tag{7.5}$$

which yields the following explicit expressions for the gradient term for $n = 0$ or $m = 0$:

$$\begin{aligned} &\nabla(J_0(k_0|y|)) \cdot \nabla(J_m(k_0|y|)e^{im\theta_y}) \\ &\quad = \left[-\frac{k_0^2}{2}\left(J_1(k_0|y|)\right)\left(J_{m-1}(k_0|y|) - J_{m+1}(k_0|y|)\right)\right]e^{im\theta_y}, \end{aligned} \tag{7.6}$$

$$\begin{aligned} &\nabla(J_n(k_0|y|)e^{-in\theta_y}) \cdot \nabla(J_0(k_0|y|)) \\ &\quad = \left[-\frac{k_0^2}{2}(J_{n-1}(k_0|y|) - J_{n+1}(k_0|y|))\left(J_1(k_0|y|)\right)\right]e^{-in\theta_y}, \end{aligned} \tag{7.7}$$

$$\nabla(J_0(k_0|y|)) \cdot \nabla(J_0(k_0|y|)) = k_0^2\left(J_1(k_0|y|)\right)^2. \tag{7.8}$$

These explicit formulas lead us to the following corollary.

Corollary 7.1. *Let (μ, ε) be a pair of electromagnetic parameters in Ω, and $\widehat{\varepsilon} = (\|\varepsilon - \varepsilon_0\|_{L^\infty}^2 + \left\|\mu^{-1} - \mu_0^{-1}\right\|_{L^\infty}^2)^{1/2}$. Then the scattering coefficients $W_{nm}[\varepsilon, \mu, \omega, \Omega]$ admit the following expansions:*

$$\begin{aligned} W_{nm} = &\frac{\mu_0 k_0^2}{4}\int_\Omega\left(\frac{1}{\mu(y)} - \frac{1}{\mu_0}\right)\left(J_{n-1}(k_0|y|)\right. \\ &\left. - J_{n+1}(k_0|y|)\right)\left(J_{m-1}(k_0|y|) - J_{m+1}(k_0|y|)\right)e^{i(m-n)\theta_y}\,dy \\ &+ \mu_0 nm\int_\Omega\left(\frac{1}{\mu(y)} - \frac{1}{\mu_0}\right)\frac{1}{|y|^2}J_n(k_0|y|)J_m(k_0|y|)e^{i(m-n)\theta_y}\,dy \\ &- \omega^2\mu_0\int_\Omega\left(\varepsilon(y) - \varepsilon_0\right)J_n(k_0|y|)J_m(k_0|y|)e^{i(m-n)\theta_y}dy + O(\widehat{\varepsilon}^2) \quad (\textit{for } n, m \neq 0), \end{aligned} \tag{7.9}$$

(7.10)
$$W_{00} = \mu_0 k_0^2 \int_\Omega \left(\frac{1}{\mu(y)} - \frac{1}{\mu_0}\right) (J_1(k_0|y|))^2 \, dy - \omega^2\mu_0 \int_\Omega (\varepsilon(y) - \varepsilon_0) (J_0(k_0|y|))^2 \, dy + O(\widehat{\varepsilon}^2),$$

(7.11)
$$W_{n0} = -\frac{\mu_0 k_0^2}{2} \int_\Omega \left(\frac{1}{\mu(y)} - \frac{1}{\mu_0}\right) (J_{n-1}(k_0|y|) - J_{n+1}(k_0|y|)) (J_1(k_0|y|)) e^{-in\theta_y} \, dy - \omega^2\mu_0 \int_\Omega (\varepsilon(y) - \varepsilon_0) J_n(k_0|y|) J_0(k_0|y|) e^{-in\theta_y} \, dy + O(\widehat{\varepsilon}^2) \quad (\textit{for} \;\; n \neq 0).$$

By means of the asymptotic behavior (3.12) and the estimates in Corollary 7.1, we obtain the following estimate for all n, $m \in \mathbb{Z}$:

$$(7.12) \qquad |W_{nm}| \leq \frac{\mu_0^2\varepsilon_0\omega^2}{4} \left\| \frac{1}{\mu} - \frac{1}{\mu_0} \right\|_{L^\infty(\Omega)} \frac{C^{|n|+|m|-2}}{(|n|-1)^{(|n|-1)}(|m|-1)^{(|m|-1)}} + \mu_0 \left\| \frac{1}{\mu} - \frac{1}{\mu_0} \right\|_{L^\infty(\Omega)} \frac{C^{|n|+|m|-2}}{|n|^{|n|-1}|m|^{|m|-1}} + \omega^2\mu_0 \|\varepsilon - \varepsilon_0\|_{L^\infty(\Omega)} \frac{C^{|n|+|m|}}{|n|^{|n|}|m|^{|m|}} + C\widehat{\varepsilon}^2.$$

Moreover, by comparing (7.9) with (3.12) for large m and n, we can see that the two integrals with the term $(\mu^{-1} - \mu_0^{-1})$ dominate. This suggests that we may separate the effect of $(\mu^{-1} - \mu_0^{-1})$ and $\varepsilon - \varepsilon_0$ on W_{nm} and recover μ and ε alternatively: First, use the scattering coefficients W_{nm} for large m, n to recover μ, then use the scattering coefficients W_{nm} for small m, n to recover ε. Furthermore, with the integral expression (7.9) we may work out each term explicitly for some special domains, e.g., $\Omega = B_R(0)$. For simplicity, we will present our detailed derivations and calculations for the special case with $\mu = \mu_0$ but $\varepsilon \neq \varepsilon_0$ in the remainder of this section, though most of the conclusions can be extended to the general case with $\mu \neq \mu_0$ and $\varepsilon \neq \varepsilon_0$. It is easy to see for the special domain $\Omega = B_R(0)$ and the special case with $\mu = \mu_0$ but $\varepsilon \neq \varepsilon_0$ that the W_{nm} are simplified to be

(7.13)
$$W_{nm} = -\omega^2\mu_0 \int_0^R \int_0^{2\pi} (\varepsilon(y) - \varepsilon_0) J_n(k_0 r_y) J_m(k_0 r_y) e^{i(m-n)\theta_y} r_y dr_y d\theta_y + O(\widehat{\varepsilon}^2),$$

where $y = (r_y, \theta_y)$ is the polar coordinate.

7.1. Radially symmetric case. In this subsection we derive formulas to recover the electromagnetic parameter ε from the scattering coefficients W_{nm} in the case with $\mu = \mu_0$, but $\varepsilon \neq \varepsilon_0$ with ε being radially symmetric in $\Omega = B_R(0)$. We shall write $\widehat{\varepsilon} := \|\varepsilon - \varepsilon_0\|_{L^\infty(\Omega)}$ and $\varepsilon(y) = \varepsilon(r_y)$. It is straightforward to see from (7.13) that

$$(7.14) \qquad W_{nm} = -2\pi\omega^2\mu_0 \int_0^R (\varepsilon(r_y) - \varepsilon_0) [J_n(k_0 r_y)]^2 r_y dr_y + O(\widehat{\varepsilon}^2) \;\; \text{for} \;\; m = n \;\; \textit{and} \;\; O(\widehat{\varepsilon}^2) \;\; \text{for} \;\; m \neq n.$$

It follows readily from (7.9), (7.11), and (7.10) that the same conclusion as in (7.14) for $m \neq n$ can be obtained for the more general case when $\mu \neq \mu_0$ and $\varepsilon \neq \varepsilon_0$, provided that both μ and ε are radially symmetric in $\Omega = B_R(0)$.

In the later part of this subsection, we shall establish an explicit formula for computing the electromagnetic parameter ε in terms of the scattering coefficients $W_{nn}(k) := W_{nn}[\varepsilon, \mu, \omega(k), \Omega]$, where $\omega(k) = k/\sqrt{\varepsilon_0\mu_0}$ is the frequency depending on $k \in \mathbb{R}^+$. For the sake of convenience, we define the following coefficient

$$\mathcal{H}_n^{(0)} := \int_0^\infty \frac{W_{nn}(k)}{k}\, dk. \tag{7.15}$$

Using the following orthogonality of the Bessel functions $\{J_n(rk)\}_{r>0}$ for a given $n \in \mathbb{Z}$,

$$\int_0^\infty J_n(rk)J_n(r'k)k\, dk = \frac{\delta(r-r')}{r} \quad \forall\, r, r' > 0, \tag{7.16}$$

we obtain from (7.13) and (7.1) that

$$\begin{aligned}\mathcal{H}_n^{(0)} &= \int_0^\infty \frac{W_{nn}(k)}{k}\, dk \\ &= -\frac{2\pi}{\varepsilon_0}\int_0^R (\varepsilon(r_y) - \varepsilon_0)\left(\int_0^\infty J_n(kr_y)J_n(kr_y)kdk\right) r_y dr_y + O(\hat{\varepsilon}^2) \\ &= -\frac{2\pi}{\varepsilon_0}\int_0^R (\varepsilon(r_y) - \varepsilon_0)\, dr_y + O(\hat{\varepsilon}^2),\end{aligned}$$

which gives the average of $\varepsilon(r_y) - \varepsilon_0$ along the radial direction. Next, we shall extend the above observation to obtain more information about ε. This motivates us with the following definition.

Definition 7.2. *For $n \in \mathbb{Z}$, let $W_{nn}(k) := W_{nn}[\varepsilon, \mu, \omega(k), \Omega]$ be defined as in* (2.11) *with $\omega(k) = k/\sqrt{\varepsilon_0\mu_0}$. For l, $n \in \mathbb{Z}$ and $l \geq 0$, let $g_n^{(l)}(k)$ be functions such that*

$$\int_0^\infty g_n^{(l)}(k)J_n(kr)J_n(kr)k^2\, dk = r^{l-1} \quad \forall\, r > 0. \tag{7.17}$$

Then we define the coefficients $\mathcal{H}_n^{(l)}$ by

$$\mathcal{H}_n^{(l)} := \int_0^\infty g_n^{(l)}(k)W_{nn}(k)\, dk \quad \forall\ l, n \in \mathbb{Z},\ l \geq 0. \tag{7.18}$$

We will show the existence of functions $g_n^{(l)}$ satisfying (7.17) and derive their explicit expressions in Appendix B.

We see from the orthogonality relation (7.16) that $g_n^{(0)}(k) = 1/k$. Thus the definition of $\mathcal{H}_n^{(l)}$ in (7.18) is consistent with (7.15) for $l = 0$. With this definition, we are able to recover the lth moment of $\varepsilon(r_y) - \varepsilon_0$ from the scattering coefficients $W_{nn}(k)$ measured at different wavenumber k but for one fixed $n \in \mathbb{Z}$. Putting (7.13), (7.17) into (7.18), we get

$$\begin{aligned}\mathcal{H}_n^{(l)} &= \int_0^\infty g_n^{(l)}(k) W_{nn}(k)\, dk \\ &= -\frac{2\pi}{\varepsilon_0} \int_0^R (\varepsilon(r_y) - \varepsilon_0) \left(\int_0^\infty g_n^{(l)}(k) J_n(kr_y) J_n(kr_y) k^2 dk \right) r_y dr_y + O(\widehat{\varepsilon}^2) \\ &= -\frac{2\pi}{\varepsilon_0} \int_0^R r_y^l\, (\varepsilon(r_y) - \varepsilon_0)\, dr_y + O(\widehat{\varepsilon}^2).\end{aligned}$$

By this relation, the electromagnetic coefficient ε can be reconstructed explicitly.

COROLLARY 7.3. *Let $\Omega = B_R(0)$ be the disk of center 0 and radius R. Let (μ, ε) be the pair of electromagnetic parameters in Ω and (μ_0, ε_0) be the parameters of the homogeneous background. Assume that the parameters satisfy $\mu = \mu_0$ and ε is radially symmetric, i.e., $\varepsilon(y) = \varepsilon(r_y)$, and $\widehat{\varepsilon} = \|\varepsilon - \varepsilon_0\|_{L^\infty(\Omega)}$. Then the coefficients $\mathcal{H}_n^{(l)}$ defined in (7.18) satisfy the following relationship for l, $n \in \mathbb{Z}$ and $l \geq 0$,*

$$\mathcal{H}_n^{(l)} = -\frac{2\pi}{\varepsilon_0} \int_0^R r_y^l\, (\varepsilon(r_y) - \varepsilon_0)\, dr_y + O(\widehat{\varepsilon}^2). \tag{7.19}$$

For $\alpha \in \mathbb{Z}$, the αth Fourier coefficient $\mathfrak{F}_{r_y}[\varepsilon(r_y) - \varepsilon_0](\alpha)$ of $\varepsilon(r_y) - \varepsilon_0$ can be written explicitly by

$$\mathfrak{F}_{r_y}[\varepsilon(r_y) - \varepsilon_0](\alpha) = -\frac{2\pi}{\varepsilon_0} \sum_{l=0}^{\infty} \frac{(-\frac{2\pi}{R} i\alpha)^l}{l!} \mathcal{H}_n^{(l)} + O(\widehat{\varepsilon}^2) \tag{7.20}$$

and the electromagnetic coefficient ε can be explicitly expressed as, for a fixed $n \in \mathbb{Z}$,

$$(\varepsilon - \varepsilon_0)(r_y) = -\frac{2\pi}{\varepsilon_0} \sum_{\alpha=-\infty}^{\infty} \sum_{l=0}^{\infty} e^{i\frac{2\pi}{R}\alpha r_y} \frac{(-\frac{2\pi}{R} i\alpha)^l}{l!} \mathcal{H}_n^{(l)} + O(\widehat{\varepsilon}^2). \tag{7.21}$$

We remark that, with (7.21), we are able to reconstruct ε from a set of scattering coefficients $\{W_{nn}(k) | k \in \mathbb{R}^+\}$ for all wavenumbers k but with only a fixed $n \in \mathbb{Z}$. Choosing n small yields a stable reconstruction of ε from far-field patterns at frequencies $k \in [0, k_{\max}]$ by approximating $\mathcal{H}_n^{(l)}$ with $\int_0^{k_{\max}} g_n^{(l)}(k) W_{nn}(k)\, dk$ and truncating the infinite sums in (7.21).

In what follows, we would like to have a brief discussion on how this choice of $k_{\max}$ may affect the approximate reconstruction of ε. Actually, in view of (4.7), for any given wavenumber k, the scattering coefficients and the far-field pattern shall provide the same information about the function $\varepsilon - \epsilon_0$. Therefore, in order to understand the effect of the truncation threshold $k_{\max}$, we would first like to obtain a representation of the far-field pattern in terms of the function $\varepsilon - \epsilon_0$. For a given wavenumber k and $\xi \in \mathbb{S}^1$, and an incidence field u_0 of the form $u_0 = e^{ik\xi \cdot x}$ as in section 4, we obtain from the well-known Lippmann–Schwinger equation and the Born approximation [20] the following representation of the scattered field:

$$u(x) - e^{ik\xi \cdot x} = -(\omega(k))^2 \int_\Omega (\varepsilon(y) - \epsilon_0) e^{ik\xi \cdot y} \Phi_k(|x - y|)\, dy + O\left(\hat{\varepsilon}^2\right), \tag{7.22}$$

where $\omega(k) = k/\sqrt{\varepsilon_0 \mu_0}$ and Φ_k is the fundamental solution defined as in (2.1) with k_0 replaced by k. Therefore, substituting (4.3) into the above equation and comparing it with (4.6), we directly infer the following representation of the far-field pattern,

$$A_\infty[\varepsilon,\mu,\omega(k)](\theta_\xi,\theta_x) = -\left(\omega(k)\right)^2 \int_\Omega (\varepsilon(y)-\epsilon_0)e^{ik(\xi-\hat{x})\cdot y}\,dy + O(\hat{\varepsilon}^2)$$
$$= -\frac{k^2}{\varepsilon_0\mu_0}\left[\mathfrak{F}_y\left(\varepsilon-\epsilon_0\right)\right](d_{\xi,x}) + O(\hat{\varepsilon}^2), \tag{7.23}$$

where $\hat{x} := x/|x| \in \mathbb{S}^1$, $d_{\xi,x} := k\,(\xi-\hat{x}) \in \overline{B_{2k}(0)}$, and for any $d \in \mathbb{R}^2$, the notation $\left[\mathfrak{F}_y\left(\varepsilon-\epsilon_0\right)\right](d)$ denotes the Fourier transform of $\varepsilon-\epsilon_0$ at d as follows:

$$\left[\mathfrak{F}_y\left(\varepsilon-\epsilon_0\right)\right](d) := \int_{\mathbb{R}^2}(\varepsilon(y)-\epsilon_0)e^{id\cdot y}\,dy = \int_\Omega(\varepsilon(y)-\epsilon_0)e^{id\cdot y}\,dy \quad \forall d\in\mathbb{R}^2. \tag{7.24}$$

Now, from (4.7), the scattering coefficients for a fixed wavenumber k provide the far-field pattern $A_\infty[\varepsilon,\mu,\omega(k)]$, which in turn provides the Fourier transform of $\varepsilon-\epsilon_0$ for $d \in \overline{B_{2k}(0)}$. Hence we can see that the larger the truncation threshold $k_{\max}$ is, the more information about the Fourier transform of $\varepsilon-\epsilon_0$ is available, and therefore the better is the resolution of the reconstructed image. But a larger $k_{\max}$ leads to more computational efforts.

7.2. Angularly symmetric case. In this subsection we would like to recover the electromagnetic parameter ε from the scattering coefficients W_{nm} for the special domain $\Omega = B_R(0)$ and the special case when $\mu = \mu_0$ and the electromagnetic coefficient ε only depends on θ_y, i.e., $\varepsilon(y) = \varepsilon(\theta_y)$. Directly from (7.13), we have, for n, $m \in \mathbb{Z}$,

(7.25)
$$W_{nm} = -\omega^2\mu_0\left(\int_0^{2\pi}\left(\varepsilon(\theta_y)-\varepsilon_0\right)e^{i(m-n)\theta_y}d\theta_y\right)\left(\int_0^R J_n(k_0r_y)J_m(k_0r_y)r_ydr_y\right)$$
$$+O(\widehat{\varepsilon}^2)$$
$$= -\omega^2\mu_0C_{k_0}(m,n)\int_0^{2\pi}\left(\varepsilon(\theta_y)-\varepsilon_0\right)e^{i(m-n)\theta_y}d\theta_y + O(\widehat{\varepsilon}^2),$$

where $C_{k_0}(m,n)$ is given by

$$C_{k_0}(m,n) := \int_0^R J_n(k_0r_y)J_m(k_0r_y)r_y\,dr_y, \quad n,m\in\mathbb{Z}. \tag{7.26}$$

We can see that for n, $m \in \mathbb{Z}$, $C_{k_0}(m,n)$ actually satisfies

$$C_{k_0}(m,n) := \int_0^R J_n(k_0r_y)J_m(k_0r_y)r_ydr_y$$
$$= \mathfrak{F}_{\theta,\phi}\left[\frac{Re^{ik_0R(\sin\theta+\sin\phi)}}{ik_0\left(\sin\theta+\sin\phi\right)} + \frac{e^{ik_0R(\sin\theta+\sin(\phi))}}{k_0^2\left(\sin\theta+\sin\phi\right)^2}\right](n,m), \tag{7.27}$$

where $\mathfrak{F}_{\theta,\phi}$ stands for the Fourier coefficient in both arguments θ and ϕ. Formula (7.27) indicates that the coefficients $C_{k_0}(m,n)$, m, $n\in\mathbb{Z}$, can be approximated via FFT or calculated explicitly. From (7.25), we can obtain the Fourier coefficients $\mathfrak{F}_{\theta_y}\left[\varepsilon(\theta_y)-\varepsilon_0\right]$ of $\varepsilon(\theta_y)-\varepsilon_0$ as follows:

$$\mathfrak{F}_{\theta_y}\left[\varepsilon(\theta_y)-\varepsilon_0\right](n-m) = -\frac{W_{nm}}{\omega^2\mu_0C_{k_0}(m,n)} + O(\varepsilon^2),$$

for all n, $m\in\mathbb{Z}$. Thus we have the following corollary.

COROLLARY 7.4. *Let $\Omega = B_R(0)$ and $\widehat{\varepsilon} := \|\varepsilon - \varepsilon_0\|_{L^\infty(\Omega)}$, and the same assumptions be assumed for (μ, ε) and (μ_0, ε_0) as in Corollary* 7.3*, except that the radial symmetry of ε is now replaced by the angular symmetry, i.e., $\varepsilon(y) = \varepsilon(\theta_y)$. Then for all n, $m \in \mathbb{Z}$, the scattering coefficients W_{nm} defined in* (2.11) *satisfy the following relationship with the Fourier coefficients of $\varepsilon(\theta_y) - \varepsilon_0$:*

$$\mathfrak{F}_{\theta_y}\left[\varepsilon(\theta_y) - \varepsilon_0\right](n-m) = -\frac{W_{nm}}{\omega^2 \mu_0 C_{k_0}(m,n)} + O(\varepsilon^2). \tag{7.28}$$

Let $\{(n_l, m_l)\}_{l\in\mathbb{Z}} \subset \mathbb{Z} \times \mathbb{Z}$ be such that $n_l - m_l = l$ for all $l \in \mathbb{Z}$. Then the electromagnetic coefficient ε can be explicitly expressed by

$$(\varepsilon - \varepsilon_0)(\theta_y) = -\sum_{l=-\infty}^{\infty} \frac{W_{n_l m_l}}{\omega^2 \mu_0 C_{k_0}(m_l, n_l)} e^{i2\pi l \theta_y} + O(\widehat{\varepsilon}^2). \tag{7.29}$$

We can see from (7.29) that in order to recover the electromagnetic coefficient ε in the angular symmetric case, we only need to know $\{W_{n_l m_l}\}_{l\in\mathbb{Z}}$, where $\{(n_l, m_l)\}_{l\in\mathbb{Z}} \subset \mathbb{Z} \times \mathbb{Z}$ is such that $n_l - m_l = l$ for $l \in \mathbb{Z}$. So we do not necessarily require all the scattering coefficients W_{nm} to recover ε, instead we may choose $\{W_{n_l m_l}\}_{l\in\mathbb{Z}}$ of any particular $\{(n_l, m_l)\}_{l\in\mathbb{Z}}$; for instance, we may fix $n_l = 0$. Truncating the sum in (7.29) up to N gives a stable reconstruction formula (for the low-frequency part) with an angular resolution limit depending on N. The higher N is, the better is the angular resolution. When W_{nm} are corrupted by noise, N can be computed as a function of the signal-to-noise ratio in the measurements.

7.3. General case. In this subsection, we try to derive formulas to recover the parameter ε from the set of scattering coefficients $\{W_{nm}(k) | n, m \in \mathbb{Z}, k \in \mathbb{R}^+\}$, where $W_{nm}(k) := W_{nm}[\varepsilon, \mu, \omega(k), \Omega]$ is defined in (2.11) with $\omega(k)$ satisfying (7.1) when $\Omega = B_R(0)$, $\mu = \mu_0$, without any assumption on the parameter ε. We would like to combine the ideas in the proofs of (7.21) and (7.29) to get a more general result. Now we start with a general ε which admits the Fourier expansion

$$\varepsilon(r_y, \theta_y) - \varepsilon_0 = \sum_{\alpha\in\mathbb{Z}} \mathfrak{F}_{\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](\alpha) e^{i\alpha\theta_y}, \tag{7.30}$$

where $\mathfrak{F}_{\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](\alpha)$ is the αth Fourier coefficient with respect to θ_y fixing r_y. Then we plug the expansion (7.30) into (7.13) to get

$$W_{nm} = -2\pi\omega^2\mu_0 \int_0^R \mathfrak{F}_{\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](n-m) J_n(k_0 r_y) J_m(k_0 r_y) r_y dr_y + O(\widehat{\varepsilon}^2). \tag{7.31}$$

Following the definition of $\mathcal{H}_n^{(l)}$ in (7.18), we define a generalized coefficient $\mathcal{H}_{nm}^{(l)}$ below.

DEFINITION 7.5. *For n, $m \in \mathbb{Z}$, let $W_{nm}(k) := W_{nm}[\varepsilon, \mu, \omega(k), \Omega]$ be defined as in* (2.11)*, where $\omega(k)$ is defined as in* (7.1)*. For l, n, $m \in \mathbb{Z}$ and $l \geq 0$, let $g_{nm}^{(l)}(k)$ be functions such that*

$$\int_0^\infty g_{nm}^{(l)}(k) J_n(kr) J_m(kr) k^2\, dk = r^{l-1} \tag{7.32}$$

for any $r > 0$. Then the coefficients $\mathcal{H}_{nm}^{(l)}$ are defined as, for l, n, $m \in \mathbb{Z}$ and $l \geq 0$,

$$\mathcal{H}_{nm}^{(l)} := \int_0^\infty g_{nm}^{(l)}(k) W_{nm}(k)\, dk. \tag{7.33}$$

We refer to Appendix C for the existence of functions $g_{nm}^{(l)}$ satisfying (7.32).

With this definition, we are able to recover, for all n, $m \in \mathbb{Z}$, the lth moment of the Fourier coefficients $\mathfrak{F}_{\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](n-m)$ with respect to r_y from the scattering coefficients $W_{nm}(k)$ measured at different frequencies k. Actually, we have, putting (7.1), (7.32), and (7.31) into (7.18),

$$\begin{aligned}
\mathcal{H}_{nm}^{(l)} &= \int_0^\infty g_{nm}^{(l)}(k) W_{nm}(k)\, dk \\
&= -\frac{2\pi}{\varepsilon_0} \int_0^R \mathfrak{F}_{\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](n-m) \\
&\quad \times \left(\int_0^\infty g_{nm}^{(l)}(k) J_n(kr_y) J_m(kr_y) k^2\, dk \right) r_y dr_y + O(\widehat{\varepsilon}^2) \\
&= -\frac{2\pi}{\varepsilon_0} \int_0^R r_y^l \mathfrak{F}_{\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](n-m) dr_y + O(\widehat{\varepsilon}^2)
\end{aligned}$$

for all n, $m \in \mathbb{Z}$. Therefore, similarly to (7.20), we get, for all n, m, $\alpha \in \mathbb{Z}$,

$$\mathfrak{F}_{r_y,\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](\alpha, n-m) = -\frac{2\pi}{\varepsilon_0} \sum_{l=0}^\infty \frac{(-\frac{2\pi}{R} i\alpha)^l}{l!} \mathcal{H}_{nm}^{(l)} + O(\widehat{\varepsilon}^2).$$

Fixing a set $\{(n_p, m_p)\}_{p\in\mathbb{Z}} \subset \mathbb{Z} \times \mathbb{Z}$ such that $n_p - m_p = p$ for $p \in \mathbb{Z}$, we are able to recover $\varepsilon - \varepsilon_0$ explicitly expressed as

$$\varepsilon - \varepsilon_0 = -\frac{2\pi}{\varepsilon_0} \sum_{\alpha=-\infty}^\infty \sum_{p=-\infty}^\infty \sum_{l=0}^\infty e^{i\left[l\theta_y + \frac{2\pi}{R}\alpha r_y\right]} \frac{(-\frac{2\pi}{R} i\alpha)^l}{l!} \mathcal{H}_{n_p m_p}^{(l)} + O(\widehat{\varepsilon}^2).$$

Corollary 7.6. *Let $\Omega = B_R(0)$ and $\widehat{\varepsilon} := \|\varepsilon - \varepsilon_0\|_{L^\infty(\Omega)}$, and the same assumptions be assumed for (μ, ε) and (μ_0, ε_0) as in Corollary 7.3, except that the radial symmetry of ε is now replaced by the Fourier expansion (7.30). Then for l, n, $m \in \mathbb{Z}$ and $l \geq 0$, the coefficients $\mathcal{H}_{nm}^{(l)}$ defined in (7.33) satisfy the following relationship:*

$$\mathcal{H}_{nm}^{(l)} = -\frac{2\pi}{\varepsilon_0} \int_0^R r_y^l \mathfrak{F}_{\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](n-m) dr_y + O(\widehat{\varepsilon}^2). \tag{7.34}$$

Moreover, for all n, m, $\alpha \in \mathbb{Z}$, the $(\alpha, n-m)$th Fourier coefficient of $\varepsilon - \varepsilon_0$ can be written explicitly by

$$\mathfrak{F}_{r_y,\theta_y}\left[\varepsilon(r_y, \theta_y) - \varepsilon_0\right](\alpha, n-m) = -\frac{2\pi}{\varepsilon_0} \sum_{l=0}^\infty \frac{(-\frac{2\pi}{R} i\alpha)^l}{l!} \mathcal{H}_{nm}^{(l)} + O(\widehat{\varepsilon}^2). \tag{7.35}$$

Let $\{(n_p, m_p)\}_{p\in\mathbb{Z}} \subset \mathbb{Z} \times \mathbb{Z}$ be such that $n_p - m_p = p$ for all $p \in \mathbb{Z}$, then the electromagnetic coefficient ε can be explicitly expressed by

$$(\varepsilon - \varepsilon_0)(r_y, \theta_y) = -\frac{2\pi}{\varepsilon_0} \sum_{\alpha=-\infty}^\infty \sum_{p=-\infty}^\infty \sum_{l=0}^\infty e^{i\left[p\theta_y + \frac{2\pi}{R}\alpha r_y\right]} \frac{(-\frac{2\pi}{R} i\alpha)^l}{l!} \mathcal{H}_{n_p m_p}^{(l)} + O(\widehat{\varepsilon}^2). \tag{7.36}$$

We remark that expression (7.36) generalizes (7.21) and (7.29). Moreover, similarly to observations in previous subsections, we can see that in order to recover the electromagnetic coefficient ε, we only need to know $\{W_{n_p m_p}(k) | p \in \mathbb{Z}, k \in \mathbb{R}^+\}$, where $\{(n_p, m_p)\}_{p\in\mathbb{Z}} \subset \mathbb{Z} \times \mathbb{Z}$ is such that $n_p - m_p = p$ for $p \in \mathbb{Z}$. Therefore, we may choose a particular choice $\{(n_p, m_p)\}_{p\in\mathbb{Z}}$, for instance, we can let $n_p = 0$. This tells us that we are able to recover ε with incomplete data for the scattering coefficients. As pointed out earlier, we may truncate the series in (7.36) and approximate $\mathcal{H}^{(l)}_{n_p m_p}$ by $\int_0^{k_{\max}} g^{(l)}_{n_p m_p}(k) W_{n_p m_p}(k)\, dk$.

8. Concluding remarks. In this paper we have introduced the concept of scattering coefficients for inverse medium scattering problems in heterogeneous media, and established important properties (such as symmetry and tensorial properties) of the scattering coefficients as well as their various representations in terms of the NtD maps. An important relationship between the scattering coefficients and the far-field pattern is also derived. Furthermore, the sensitivity of the scattering coefficients with respect to the changes in the permittivity and permeability distributions is explored, which enables us to derive explicit reconstruction formulas for the permittivity and permeability parameters in the linearized case. These formulas show on one hand the stability of the reconstruction from multifrequency measurements and, on the other hand, the exponential instability of the reconstruction from far-field measurements at a fixed frequency. The scattering-coefficient-based approach introduced in this work is a new promising direction for solving the long-standing inverse scattering problem with heterogeneous inclusions. It can be combined with some existing methods such as the continuation method [16, 17, 18, 19] to improve the stability and the resolution of the reconstructed images.

Appendix A. Construction of the Neumann function.

In this section we construct the Neumann function $N_{\mu,\varepsilon}$ associated with

$$-Lu := \nabla \cdot \frac{1}{\mu} \nabla u + \omega^2 \varepsilon u \tag{A.1}$$

in Ω, which is an open connected domain with $\mathcal{C}^2$ boundary in $\mathbb{R}^d$ for $d = 2, 3$. We shall also estimate its singularity. Again, we assume that 0 is not a Neumann eigenvalue of L on Ω.

Some results for the Neumann function and its singularity are known and can be found in [25]. We shall follow similar arguments to the ones in [1] and [11], with only a sketch of the proof. A more detailed analysis can be found in [5].

In order to show the existence of the Neumann function, we first consider the following problem: given $f \in \mathcal{C}^\infty_c(\Omega)$, find $u \in H^1(\Omega)$ such that

$$\nabla \cdot \frac{1}{\mu} \nabla u + \omega^2 \varepsilon u = f \text{ in } \Omega; \quad \frac{1}{\mu} \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{A.2}$$

Let $R > 0$ such that $B_R(0) \subset \Omega$. Then by the well-known De Giorgi–Nash–Moser theorem [26] for the L^∞ coefficient and Sobelov embedding, we have

$$\|u\|_{L^\infty(B_{R/2}(0))} \leq C\big(R^{1-\frac{d}{2}} \|u\|_{H^1(\Omega)} + R^2 \|f\|_{L^\infty(B_R(0))}\big). \tag{A.3}$$

On the other hand, one can prove using the same argument as in [1] that for any $f \in \mathcal{C}^\infty_c(\Omega)$, there exists a unique $u \in H^1(\Omega)$ satisfying (A.2) such that

$$\|u\|_{H^1(\Omega)} \leq C \|f\|_{H^1(\Omega)}. \tag{A.4}$$

Therefore, combining (A.3) and (A.4), we deduce

$$\|u\|_{L^\infty(B_{R/2}(0))} \le C\big(R^{1-\frac{d}{2}}\|f\|_{H^1(\Omega)} + R^2\|f\|_{L^\infty(B_R(0))}\big). \tag{A.5}$$

Now consider $f \in \mathcal{C}_c^\infty(\Omega)$ such that the support of f is contained in $B_R(0) \subset \Omega$ for some R. Then following the argument in [11] and using the following Poincaré-type inequality for all $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ and $\frac{\partial u}{\partial \nu} = 0$,

$$\|u\|^2_{H^1(\Omega)} \le C|\langle Lu, u\rangle_{L^2(\Omega)}|, \tag{A.6}$$

we can directly derive that

$$\|u\|_{H^1(\Omega)} \le CR^{\frac{d+2}{2}}\|f\|_{L^\infty(B_R(0))}. \tag{A.7}$$

Therefore, combining (A.3) and (A.7), we come to

$$\|u\|_{L^\infty(B_{R/2}(0))} \le CR^2\|f\|_{L^\infty(B_R(0))}. \tag{A.8}$$

This inequality is essential for the existence of the Neumann function and its estimates.

Now we are ready to construct a Neumann function for the system (A.1), following the technique in [11]. Fix a function $\varphi \in \mathcal{C}_c^\infty(B_1(0))$ and $0 \le \varphi \le 2$ such that $\int_{B_1(0)} \varphi dx = 1$. Let $y \in \Omega$ be fixed. For any $\varepsilon > 0$, we define

$$\varphi_{\varepsilon,y}(x) = \varepsilon^{-d}\varphi\left(\frac{x-y}{\varepsilon}\right). \tag{A.9}$$

Let $N^\varepsilon(\cdot, y) \in H^1(\Omega)$ be the "averaged Neumann function" such that it satisfies (A.2) with $f = \varphi_{\varepsilon,y}$, then we immediately have from (A.8) that for all $\varepsilon \le R/2$,

$$\|N^\varepsilon(\cdot, y)\|_{L^\infty(B_{R/2}(0))} \le CR^2\|\varphi_{\varepsilon,y}\|_{L^\infty(B_R(0))}. \tag{A.10}$$

This L^∞-estimate for $N^\varepsilon(\cdot, y)$ can be further improved later. It is worth mentioning that, following the same argument as in [11], we have the following H^1-estimate for $N^\varepsilon(\cdot, y)$ by the Hölder inequality, Sobelov embedding, and (A.6):

$$\|N^\varepsilon(\cdot, y)\|_{H^1(\Omega)} \le C\varepsilon^{\frac{2-d}{2}}, \tag{A.11}$$

as well as the following L^1-estimate for $N^\varepsilon(\cdot, y)$ from (A.8) and the duality argument, that for $\varepsilon \le \frac{R}{2}$ and $R \le d_y$,

$$\|N^\varepsilon(\cdot, y)\|_{L^1(B_R(0))} \le CR^2. \tag{A.12}$$

Now, we can use the De Giorgi–Nash–Moser theorem once again to get the following sharp L^∞-estimate for $N^\varepsilon(\cdot, y)$ from (A.12) following the routine idea in [11]: for any $x, y \in \Omega$ satisfying $0 < |x - y| < d_y/2$,

$$|N^\varepsilon(x, y)| \le C|x - y|^{2-d} \quad \forall\, \varepsilon < \frac{|x-y|}{3}. \tag{A.13}$$

Next, we would like to show the weak convergence of a subsequence of $N^\varepsilon(\cdot, y)$ in $W^{1,p}(B_r(y))$ and $H^1(\Omega\backslash B_r(y))$. For this purpose, we need to have a uniform bound of $N^\varepsilon(\cdot,y)$ in such norms with respect to ε. We shall proceed as in [11]. First for an $r \le d_y/2$, we get directly from (A.10) for $\varepsilon \ge r/6$ that

$$\|\nabla N^\varepsilon(\cdot,y)\|_{L^2(\Omega\backslash B_r(y))} \le \|\nabla N^\varepsilon(\cdot,y)\|_{L^2(\Omega)} \le C r^{\frac{2-d}{2}}. \tag{A.14}$$

For $\varepsilon < r/6$, we wish to control the gradient of $N^\varepsilon(\cdot,y)$ by $N^\varepsilon(\cdot,y)$ outside the ball $B_r(y)$ and establish an estimate similar to Caccioppoli's inequality inside the ball $B_r(y)$. To do so, we introduce a smooth function η on $\mathbb{R}^d$ satisfying

$$0 \le \eta \le 1, \quad |\nabla \eta| \le \frac{4}{r}, \quad \eta \equiv 1 \text{ in } \mathbb{R}^d\backslash B_r(y), \quad \eta \equiv 0 \text{ in } B_{\frac{r}{2}}(y). \tag{A.15}$$

Using (A.6), the properties (A.15), and some basic calculations (c.f. [5]), we can deduce

$$\|N^\varepsilon(\cdot,y)\eta\|^2_{H^1(\Omega)} \le C|\langle L(N^\varepsilon\eta), N^\varepsilon(\cdot,y)\eta\rangle_{L^2(\Omega)}| \le C\int_\Omega |\nabla\eta|^2\eta^2 dx.$$

From this and the Cauchy–Schwarz inequality it follows that

$$\|\nabla N^\varepsilon(\cdot,y)\|^2_{L^2(\Omega\backslash B_r(y))} \le C\left(\int_\Omega |\nabla\eta|^2\eta^2 + |N^\varepsilon(\cdot,y)|^2|\nabla\eta|^2 dx\right). \tag{A.16}$$

Now we have, after some basic calculations, from (A.13) and (A.15) that for $\varepsilon < \frac{r}{6}$,

$$\|\nabla N^\varepsilon(\cdot,y)\|^2_{L^2(\Omega\backslash B_r(y))} \le C\left(\int_{B_r(y)\backslash B_{\frac{r}{2}}(y)} |\nabla\eta|^2\eta^2 + |N^\varepsilon(\cdot,y)|^2|\nabla\eta|^2 dx\right) \le C r^{2-d}. \tag{A.17}$$

Combining (A.14) and (A.17), we have

$$\|\nabla N^\varepsilon(\cdot,y)\|_{L^2(\Omega\backslash B_r(y))} \le C r^{\frac{2-d}{2}} \quad \forall\, r \in \left(0, \frac{d_y}{2}\right), \ \varepsilon > 0. \tag{A.18}$$

Then for all $r \in (0, d_y)$, the argument to get the following estimate for the averaged Neumann function from (A.14) and (A.18) is routine and follows as in [11]:

$$\|N^\varepsilon(\cdot,y)\|_{L^{\frac{2d}{d-2}}(\Omega\backslash B_r(y))} + \|\nabla N^\varepsilon(\cdot,y)\|_{L^2(\Omega\backslash B_r(y))} \le C r^{\frac{2-d}{2}} \quad \forall \varepsilon > 0. \tag{A.19}$$

With this estimate, we can readily derive

$$\|N^\varepsilon(\cdot,y)\|_{L^p(B_r(y))} \le C r^{2-d+\frac{d}{p}} \quad \forall \varepsilon > 0, \quad \forall p \in \left[1, \frac{d}{d-2}\right), \tag{A.20}$$

$$\|\nabla N^\varepsilon(\cdot,y)\|_{L^p(B_r(y))} \le C r^{1-d+\frac{d}{p}} \quad \forall \varepsilon > 0, \quad \forall p \in \left[1, \frac{d}{d-1}\right). \tag{A.21}$$

Now the same argument as in [11] will ensure the existence of a sequence $\{\varepsilon_n\}_{n=1}^\infty$ going to zero and a function $N(\cdot,y)$ such that $N^{\varepsilon_n}(\cdot,y)$ converges to $N(\cdot,y)$ weakly in $W^{1,p}(B_r(y))$ for $1 < p < \frac{d}{d-1}$ and weakly in $H^1(\Omega\backslash B_r(y))$ for all $r \in (0,d_y)$. It is then routine [11] to derive from (A.19) for all $r \in (0, d_y)$ that

$$\|N(\cdot,y)\|_{L^{\frac{2d}{d-2}}(\Omega\backslash B_r(y))} + \|\nabla N(\cdot,y)\|_{L^2(\Omega\backslash B_r(y))} \le C r^{\frac{2-d}{2}}, \tag{A.22}$$

and from (A.20) and (A.21) that

$$\|N(\cdot,y)\|_{L^p(B_r(y))} \le C r^{2-d+\frac{d}{p}} \quad \forall p \in \left[1, \frac{d}{d-2}\right), \tag{A.23}$$

$$\|\nabla N(\cdot,y)\|_{L^p(B_r(y))} \le C r^{1-d+\frac{d}{p}} \quad \forall p \in \left[1, \frac{d}{d-1}\right). \tag{A.24}$$

Our section ends with the following pointwise estimate for $N(x,y)$ by using the De Giorgi–Nash–Moser theorem, (A.22), and the same technique as the one for (A.13):

$$|N(x,y)| \le C|x-y|^{2-d}. \tag{A.25}$$

This gives the estimate of the singularity type as x approaches y.

Appendix B. Existence of functions $g_n^{(l)}$. In this section we wish to show the existence of functions $g_n^{(l)}$ satisfying (7.17) for all l, $n \in \mathbb{N}$ and provide their explicit expressions. From the fact that

$$[J_n(kr)]^2 = \frac{2}{\pi}\int_0^{\pi/2} J_{2n}(2kr\sin\phi)d\phi \tag{B.1}$$

for all $n \in \mathbb{N}$, we substitute (B.1) into (7.17) to get, for all l, $n \in \mathbb{N}$,

$$\frac{2}{\pi}\int_0^{\pi/2}\int_0^{\infty} g_n^{(l)}(k) J_{2n}(2kr\sin\phi)k^2\, dk d\phi = r^{l-1} \quad \forall\, r>0. \tag{B.2}$$

Recall the following orthogonal relationship for Hankel functions

$$\int_0^{\infty} J_{2n}(kr)J_{2n}(k'r)r\, dr = \frac{\delta(k-k')}{k'} \tag{B.3}$$

for all k, $k' > 0$ and $n \in \mathbb{N}$,. Now, for l, $n \in \mathbb{N}$, consider the Hankel tranform of r^{l-1} of order $2n$ at $p>0$,

$$[\mathcal{H}_{2n}(r^{l-1})](p) := \int_0^{\infty} r^{l-1} J_{2n}(rp) r\, dr. \tag{B.4}$$

By a change of variables, we have

(B.5)

$$\begin{aligned}
[\mathcal{H}_{2n}(r^{l-1})](p) &= \int_0^{\infty} r^{l-1}J_{2n}(rp)r\, dr \\
&= \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\infty} g_n^{(l)}(k)\left(\int_0^{\infty} J_{2n}(2kr\sin\phi)J_{2n}(rp)r\, dr\right)k^2\, dk d\phi \\
&= \frac{1}{\pi}\int_0^{\infty}\int_{\phi=0}^{\phi=\pi/2} k g_n^{(l)}(k)\frac{\left(\int_0^{\infty} J_{2n}(2kr\sin\phi)J_{2n}(rp)r\, dr\right)}{\cos\phi}\, d(2k\sin\phi)\, dk \\
&= \frac{1}{\pi}\int_0^{\infty}\int_0^{2k} k g_n^{(l)}(k)\frac{\left(\int_0^{\infty} J_{2n}(rl)J_{2n}(rp)r\, dr\right)}{\sqrt{1-(\frac{l}{2k})^2}}\, dl\, dk.
\end{aligned}$$

From orthogonality relation (B.3), we get that from (B.5) that

$$\text{(B.6)}\qquad \begin{aligned}[\mathcal{H}_{2n}(r^{l-1})](p) &= \frac{1}{\pi}\int_0^\infty \chi_{\{p<2k\}}(k)\frac{kg_n^{(l)}(k)}{p\sqrt{1-(\frac{p}{2k})^2}}\,dk\\ &= \frac{1}{p\pi}\int_{\frac{p}{2}}^\infty \frac{k^2 g_n^{(l)}(k)}{\sqrt{k^2-(\frac{p}{2})^2}}\,dk.\end{aligned}$$

Therefore, for $p>0$, we have

$$\text{(B.7)}\qquad -2p\,[\mathcal{H}_{2n}(r^{l-1})](2p) = -\frac{1}{\pi}\int_p^\infty \frac{k^2 g_n^{(l)}(k)}{\sqrt{k^2-p^2}}\,dk.$$

Now we recall that the Abel transform of an integrable function $f(r)$ defined on $r\in(0,\infty)$ is as follows:

$$\text{(B.8)}\qquad F(y) := [\mathcal{A}(f)](y) := 2\int_y^\infty \frac{f(r)r}{\sqrt{r^2-y^2}}\,dr, \quad y\in(0,\infty),$$

whenever the above integral is well-defined. If $f(r)=O(\frac{1}{r})$ as $r\to\infty$, then its inverse Abel transform is well-defined and f satisfies the following:

$$\text{(B.9)}\qquad f(r) = [\mathcal{A}^{-1}(F)](r) := -\frac{1}{\pi}\int_r^\infty \frac{F'(y)}{\sqrt{y^2-r^2}}\,dy, \quad r\in(0,\infty).$$

Comparing (B.9) and (B.7), we can see that, for all l, $n\in\mathbb{N}$, the functions

$$\text{(B.10)}\qquad G_n^{(l)}(p) := -2p\,[\mathcal{H}_{2n}(r^{l-1})](2p), \quad p\in(0,\infty)$$

are nothing but the inverse Abel transform of a primitive function of $k^2 g_n^{(l)}(k)$. Therefore, applying the Abel transform to both sides of (B.7) and then differentiating with respect to the argument of the function, we get

$$\text{(B.11)}\qquad \frac{\partial}{\partial k}\big[\mathcal{A}\big(G_n^{(l)}\big)\big](k) = k^2 g_n^{(l)}(k).$$

Consequently, we have the following explicit expression for $g_n^{(l)}$:

$$\text{(B.12)}\qquad g_n^{(l)}(k) = \frac{1}{k^2}\frac{\partial}{\partial k}\big[\mathcal{A}\big(G_n^{(l)}\big)\big](k), \quad k\in(0,\infty),$$

where $G_n^{(l)}$ is defined as in (B.10). One can see by direct substitution of (B.12) back into (7.17) that the functions $g_n^{(l)}$ defined as (B.12) satisfy (7.17). Therefore, we have shown the existence of functions satisfying (7.17).

Appendix C. Existence of functions $g_{nm}^{(l)}$. In this section we show the existence of functions $g_{nm}^{(l)}$ for l, n, $m\in\mathbb{Z}$ and $l\geq 0$ which satisfy (7.32), namely, the integral equation

$$\text{(C.1)}\qquad \int_0^\infty g_{nm}^{(l)}(k)J_n(kr)J_m(kr)k^2\,dk = r^{l-1} \quad \forall\, r>0.$$

For this purpose, we would like to first investigate the following integral, which will be useful in the subsequent discussion. For $n,\, m \in \mathbb{N}$ and $p \in \mathbb{C}$ such that $m+n > Re(p) > 0$, we consider the following integral,

$$A_{nm}(p) := \int_0^\infty J_n(x)J_m(x)x^{-p}\,dx, \quad p \in \mathbb{C}, \quad m+n > Re(p) > 0. \tag{C.2}$$

We observe that the function $A_{nm} : \{p \in \mathbb{C} : m+n > Re(p) > 0\} \to \mathbb{C}$ is a holomorphic function on the strip $\{p \in \mathbb{C} : a < Re(p) < b\}$ for some $a,\, b \in \mathbb{R}$ such that $a < b$. This comes from the fact that for $n,\, m \in \mathbb{N}$ and $p \in \mathbb{C}$ such that $m+n > Re(p) > 0$, the integral $A_{nm}(p)$ defined in (C.2) can be expressed in the following form,

$$A_{nm}(p) = \int_0^\infty J_m(x)J_n(x)x^{-p}\,dx = \frac{2^{-p}\,\Gamma(p)\,\Gamma\left(\frac{1+m+n-p}{2}\right)}{\Gamma\left(\frac{1+m-n+p}{2}\right)\Gamma\left(\frac{1-m+n+p}{2}\right)\Gamma\left(\frac{1+m+n+p}{2}\right)}. \tag{C.3}$$

Now given $a,\, b \in \mathbb{R}$ and $s \in \mathbb{C}$ such that $a < Re(s) < b$, we recall the definition of the Mellin tranform of an integrable function $f(r)$ defined for $r \in (0,\infty)$:

$$[\mathcal{M}(f)](s) := \int_0^\infty r^{s-1} f(r)dr, \quad a < Re(s) < b \tag{C.4}$$

whenever the above integral is well-defined. With $a,\, b \in \mathbb{R}$, we write the function $\zeta_{a,b}$ as

$$\zeta_{a,b}(x) = x^{-a} \text{ for } 0 < x \leq 1, \qquad \text{and } x^{-b} \text{ for } 1 < x < \infty. \tag{C.5}$$

We define the linear space $\mu_{a,b}(0,\infty)$ as the space of all infinitely smooth compactly supported complex valued functions $\phi \in C_c^\infty(0,\infty)$ for which

$$\|\phi\|_{k,\zeta_{a,b},K} := \sup_K |\zeta_{a,b}(x)x^{k+1}D_x^k\phi(x)| \tag{C.6}$$

is finite for all $k \in \mathbb{N}$ and for any compact set $K \Subset (0,\infty)$. Consider an increasing sequence of compact sets $\{K_n \Subset (0,\infty)\}_{n\in\mathbb{N}}$ such that $\bigcup_{n\in\mathbb{N}} K_n = (0,\infty)$; the countable norms $\|\cdot\|_{k,\zeta_{a,b},K_n}, k, n \in \mathbb{N}$ give a topology on $\mu_{a,b}(0,\infty)$ such that $\mu_{a,b}(0,\infty)$ becomes a complete locally convex space. We define the dual of $\mu_{a,b}(0,\infty)$, $\mu'_{a,b}(0,\infty)$, and equip it with the weak topology. With these definitions at hand, the Mellin transform can be naturally extended to the space $\mu'_{a,b}(0,\infty)$; see [3, 28] for more details. We denote the generalized Mellin transform also as $\mathcal{M}$.

Now from (C.1) and (C.2), we have for all $l,\, n,\, m \in \mathbb{Z}$ with $l \geq 0$ and $p \in \mathbb{C}$ such that $Re(p) > l$,

$$\begin{aligned}
[\mathcal{M}(1)]\,(l-p) &= \int_0^\infty \int_0^\infty g_{nm}^{(l)}(k)J_n(kr)J_m(kr)r^{-p}k^2\,dk\,dr \\
&= \int_0^\infty g_{nm}^{(l)}(k)\left(\int_{r=0}^\infty J_n(kr)J_m(kr)(kr)^{-p}\,d(kr)\right)k^{p+1}\,dk \\
&= \int_0^\infty g_{nm}^{(l)}(k)\left(\int_0^\infty J_n(r)J_m(r)r^{-p}\,dr\right)k^{p+1}\,dk \\
&= A_{nm}(p)\big[\mathcal{M}(g_{nm}^{(l)})\big](p+2),
\end{aligned} \tag{C.7}$$

where $A_{nm}(p)$ is known explicitly as (C.3). Therefore we get, for all l, n, $m \in \mathbb{Z}$ with $l \geq 0$ and $p \in \mathbb{C}$ such that $Re(p) > l$,

$$\left[\mathcal{M}(g_{nm}^{(l)})\right](p+2) = \frac{\left[\mathcal{M}(1)\right](l-p)}{A_{nm}(p)}; \tag{C.8}$$

then the existence of $g_{nm}^{(l)}$ is ensured by the Mellin inverse transform.

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