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A discrete weighted Helmholtz decomposition and its application

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Abstract We propose a discrete weighted Helmholtz decomposition for edge element functions. The decomposition is orthogonal in a weighted L^2 inner product and stable uniformly with respect to the jumps in the discontinuous weight function. As an application, the new Helmholtz decomposition is applied to demonstrate the quasi-optimality of a preconditioned edge element system for solving a saddle-point Maxwell system in non-homogeneous media by a non-overlapping domain decomposition preconditioner, i.e., the condition number grows only as the logarithm of the dimension of the local subproblem associated with an individual subdomain, and more importantly, it is independent of the jumps of the physical coefficients across the interfaces between any two subdomains of different media. Numerical experiments are

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presented to validate the effectiveness of the non-overlapping domain decomposition preconditioner.

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1 Introduction

The numerical simulation of electromagnetic wave propagation often involves, at each time step, the solution of the following saddle-point Maxwell system [14,23,27–29]:

$$\mathbf{curl}(\alpha\, \mathbf{curl}\, \mathbf{u}) + \gamma_0 \beta \mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\mathrm{div}(\beta \mathbf{u}) = g \quad \text{in} \quad \Omega\,, \tag{1.2}$$

where Ω is a simply-connected open polyhedral domain in $\mathbf{R}^3$ with boundary $\partial\Omega$, occupied often by more than one physical medium. Coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are two physical parameters, which may have jumps (possibly very large) across the interface between any two neighboring different media in Ω. $\mathbf{f}$ and g are two source functions satisfying the compatibility condition $\gamma_0\, g = \mathrm{div}\mathbf{f}$. The coefficient γ_0 in (1.1) is a constant, taking either value 1 or 0, which is added here deliberately so that the system (1.1)–(1.2) covers more physical cases. System (1.1)–(1.2) with $\gamma_0 = 0$ appears in the Darwin model for Maxwell's equations [10,12] and the vector potential model for magneto static fields [3]. When $\gamma_0 = 0$ in (1.1) or when $\gamma_0 = 1$ but the coefficient β in the zero-th order term is much smaller in magnitude than the coefficient α in the higher order term, system (1.1)–(1.2) becomes more challenging numerically as the divergence equation in (1.2) must be explicitly reinforced in the discretization in order to avoid the spurious non-physical solutions. We shall complement the system (1.1)–(1.2) with the following boundary condition:

$$\mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal direction on $\partial\Omega$.

Efficient preconditioning-type solvers such as multigrid and domain decomposition methods have been well developed for second order elliptic problems in H^1-Sobolev space, in particular non-overlapping domain decomposition methods have proved also to be robust and efficient when the elliptic equations have large jumps in coefficients, see, e.g., [24,29,30]. However, the construction of such efficient solvers for elliptic equations in the H^1-space fails to work for the Maxwell equations (1.1)–(1.2) in the $H(\mathbf{curl})$-space, especially in three dimensions. One of the reasons for the failure is due to the type of finite element methods used in the discretizations. Contrary to the popularity of classical nodal elements in the discretization of elliptic equations, Nédélec edge elements have been more widely used for the discretization of the Maxwell system (1.1)–(1.2), see, e.g., [15,25] and the references therein. And the resulting algebraic systems arising from the discretization of the Maxwell system by edge element methods is of essentially different nature from the ones arising from the discretization of elliptic problems by standard nodal element methods. Another ingredient causing the failure comes from the fact that the curl operator involved in the Maxwell system has a much larger null space than the one for the gradient in elliptic problems. A fundamental tool, which may treat the larger null space and at the same time take the advantage of some existing methodologies in developing effective multigrid and domain decomposition methods for elliptic equations, is the Helmholtz-type decompositions (see, e.g., [2,15,25]). Based on these decompositions, many variants of efficient multigrid and domain decomposition methods have been constructed and analyzed for the edge element systems arising from the discretization of the Maxwell equations; see [2,14,15,20,21,28,30,31] and the references therein.

However, all the existing Helmholtz-type decompositions do not involve any coefficients in the Maxwell system (1.1)–(1.2), so they can not help analyze in general how the convergence of the existing methods depend on the coefficients or their jumps across interfaces between different media. In this work we shall establish a discrete weighted Helmholtz decomposition based on a decomposition of the global domain Ω into a set of nonoverlapping subdomains so that the Helmholtz decomposition is stable uniformly with respect to the discontinuous coefficients or their jumps across the interface between any two subdomains. To the best of our knowledge, this is the first discrete weighted Helmholtz decomposition of the kind in the literature. Considering the complexity of the construction of such a decomposition, one can imagine that the subsequent analysis is rather technical and delicate. The new (weighted) Helmholtz decomposition can be used to analyze convergence of various preconditioners for Maxwell's equations with large jumps in coefficients. As an example, we will show with the help of such a weighted Helmholtz decomposition that the substructuring preconditioner constructed in [21] converges not only nearly optimally in terms of the subdomain diameter and the finite element mesh size, but also independently of the jumps in the coefficients across the interfaces between any two subdomains of different media.

The outline of the paper is as follows. In Sect. 2 we describe the decomposition of the original domain into subdomains, the triangulation of the subdomains and some basic Sobolev and edge element spaces. The results on the new weighted Helmholtz decomposition and several variants of discrete Helmholtz decomposition are presented in Sects. 3–4. The new weighted Helmholtz decomposition is constructed for general edge element functions in Sect. 5 and analysed in Sect. 6. A direct application of the new discrete weighted Helmholtz decomposition is discussed in Sect. 7.

2 Domain decomposition, finite elements and subspaces

This section shall introduce some Soboleve spaces and edge elements, that are most frequently used for the discretization and analysis of the system (1.1)–(1.2), as well as subdomain decompositions and some fundamental edge element subspaces and concepts to be used in the construction and analysis of a discrete weighted Helmholtz decomposition.

We will need the following spaces associated with an open bounded domain $\mathcal{O}$ in $\mathbf{R}^3$:

$$\begin{aligned} H(\mathbf{curl};\mathcal{O}) &= \{\mathbf{v}\in L^2(\mathcal{O})^3;\ \mathbf{curl}\,\mathbf{v}\in L^2(\mathcal{O})^3\},\\ H_0(\mathbf{curl};\mathcal{O}) &= \{\mathbf{v}\in H(\mathbf{curl};\mathcal{O});\ \mathbf{v}\times\mathbf{n}=0 \text{ on } \partial\mathcal{O}\},\\ H(\mathrm{div};\mathcal{O}) &= \{\mathbf{v}\in L^2(\mathcal{O})^3;\ \mathrm{div}\,\mathbf{v}\in L^2(\mathcal{O})^3\},\\ H_0(\mathrm{div};\mathcal{O}) &= \{\mathbf{v}\in H(\mathrm{div};\mathcal{O});\ \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \partial\mathcal{O}\}. \end{aligned}$$

2.1 Subdomains and edge elements

The central aim of the work is to construct a discrete weighted Helmholtz decomposition based on a decomposition of the global domain Ω into a set of nonoverlapping subdomains so that the Helmholtz decomposition is stable uniformly with respect to a desired discontinuous weight function. For this purpose, we first decompose the entire domain Ω into subdomains based on the discontinuity of the weight function, which plays a role as the coefficient $\beta(\mathbf{x})$ of (1.2) in applications.

Domain decomposition based on the distribution of coefficients. Associated with the coefficient $\beta(\mathbf{x})$ in (1.2), we assume that the entire domain Ω can be decomposed into N_0 open convex polyhedral subdomains $\Omega_1^0, \Omega_2^0, \ldots, \Omega_{N_0}^0$ such that $\bar{\Omega} = \cup_{i=1}^{N_0}\bar{\Omega}_i^0$ and $\beta(\mathbf{x})$ is constant on each subdomain, namely for $r = 1, 2, \ldots, N_0$,

$$\beta(\mathbf{x}) = \beta_r \qquad \forall\,\mathbf{x}\in\Omega_r^0 \tag{2.1}$$

where each β_r is a positive constant. Such a convex decomposition is possible in many applications when Ω is formed by multiple media. In some cases when a medium forms an irregular nonconvex subregion in Ω, one may need to further split such nonconvex medium subregion into smaller convex subdomains. In this sense our assumption is not restrictive and does cover many practical cases.

Remark 2.1 The subdomains $\{\Omega_r^0\}_{r=1}^{N_0}$ are of different nature from those in the context of the standard domain decomposition methods: $\{\Omega_r^0\}_{r=1}^{N_0}$ is decomposed based only on the distribution of the jumps of the coefficient $\beta(x)$ (so N_0 is a fixed integer). Therefore the size of every such subdomain Ω_r^0 is basically irrelevant to the finite element mesh size or the subdomain size meant in the standard domain decomposition methods. When applying our results in this work to domain decomposition methods (see Sect. 7), each subdomain Ω_r^0 should be divided into several smaller subdomains.

Edge and nodal element spaces. Next, we further divide each Ω_r^0 into smaller tetrahedral elements of size h so that the restrictions of the triangulations from any two neighboring subdomains on their common face match each other. Let $\mathcal{T}_h$ be the resulting triangulation of the domain Ω, which we assume is quasi-uniform. By $\mathcal{E}_h$ and $\mathcal{N}_h$ we denote the set of edges of $\mathcal{T}_h$ and the set of nodes in $\mathcal{T}_h$ respectively. Then the Nédélec edge element space, of the lowest order, is a subspace of piecewise linear polynomials defined on $\mathcal{T}_h$:

$$V_h(\Omega) = \Big\{\mathbf{v}\in H_0(\mathbf{curl};\Omega);\ \mathbf{v}|_K\in R(K),\ \forall K\in\mathcal{T}_h\Big\},$$

where $R(K)$ is a subset of all linear polynomials on the element K of the form:

$$R(K) = \left\{ \mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}, \mathbf{b} \in \mathbf{R}^3,\ \mathbf{x} \in K \right\}.$$

It is known that for any $\mathbf{v} \in V_h(\Omega)$, its tangential components are continuous on all edges of each element in the triangulation $\mathcal{T}_h$, and $\mathbf{v}$ is uniquely determined by its moments on each edge e of $\mathcal{T}_h$:

$$M_h(\mathbf{v}) = \left\{ \lambda_e(\mathbf{v}) = \int_e \mathbf{v} \cdot \mathbf{t}_e ds;\ e \in \mathcal{E}_h \right\}$$

where $\mathbf{t}_e$ denotes the unit vector on edge e, and this convention will be used for any edge or union of edges, either from an element $K \in \mathcal{T}_h$ or from a subdomain. For a vector-valued function $\mathbf{v}$ with appropriate smoothness, we introduce its edge element interpolation $r_h\mathbf{v}$ such that $r_h\mathbf{v} \in V_h(\Omega)$, and $r_h\mathbf{v}$ and $\mathbf{v}$ have the same moments as in $M_h(\mathbf{v})$. The interpolation operator r_h will be needed in the construction of a stable decomposition for any function $\mathbf{v}_h \in V_h(\Omega)$ in Sect. 5.

As we will see, the edge element analysis involves also frequently the nodal element space. For this purpose we introduce $Z_h(\Omega)$ to be the standard continuous piecewise linear finite element space in $H_0^1(\Omega)$ associated with the triangulation $\mathcal{T}_h$.

2.2 Edge- and face-related finite element subspaces

For the subsequent analysis, we need the subspaces of the global edge element space $V_h(\Omega)$ restricted on a subdomain or the boundary or part of the boundary of Ω.

Let $\hat{\Omega}$ be any of the subdomains $\Omega_1^0, \ldots, \Omega_{N_0}^0$ of Ω. We will often use F, E and V to denote a general face, edge and vertex of $\hat{\Omega}$ respectively, but use e to denote a general edge of $\mathcal{T}_h$ lying on $\hat{\Gamma} = \partial\hat{\Omega}$. Associated with $\hat{\Omega}$, we write the natural restriction of $V_h(\Omega)$ on $\hat{\Omega}$ by $V_h(\hat{\Omega})$. Let G be either the entire boundary $\hat{\Gamma} = \partial\hat{\Omega}$ or a face F of $\hat{\Gamma}$, then we define the restrictions of the tangential components of functions in $V_h(\Omega)$ on G as

$$V_h(G) = \left\{ \psi \in L^2(G)^3;\ \psi = \mathbf{v} \times \mathbf{n} \ \text{ on } \ G \ \text{ for some } \ \mathbf{v} \in V_h(\Omega) \right\}.$$

The following local subspaces of $V_h(\hat{\Omega})$ and $V_h(\mathrm{F})$ will be important to our analysis:

$$V_h^0(\hat{\Omega}) = \left\{ \mathbf{v} \in V_h(\hat{\Omega});\ \mathbf{v} \times \mathbf{n} = 0 \ \text{ on } \ \hat{\Gamma} \right\},$$
$$V_h^0(\mathrm{F}) = \left\{ \Phi = \mathbf{v} \times \mathbf{n} \in V_h(\mathrm{F});\ \lambda_e(\mathbf{v}) = 0, \ \ \forall\, e \subset \partial\mathrm{F} \cap \mathcal{E}_h \right\}.$$

Similarly, the restrictions of $Z_h(\Omega)$ in subdomain $\hat{\Omega}$, on its boundary $\hat{\Gamma}$ and on a face F, are written as $Z_h(\hat{\Omega})$, $Z_h(\hat{\Gamma})$, and $Z_h(\mathrm{F})$, respectively. For a subset G of $\hat{\Gamma}$, we define a "local" subspace

$$Z_h^0(G) = \{v \in Z_h(\hat{\Gamma});\ v = 0 \text{ at all nodes on } \hat{\Gamma}\backslash G\}.$$

Finally we introduce the discrete **curl curl**-extension operator $\hat{\mathbf{R}}_h : V_h(\hat{\Gamma}) \to V_h(\hat{\Omega})$. We define $\hat{\mathbf{R}}_h$ as follows: for any $\Phi \in V_h(\hat{\Gamma})$, $\hat{\mathbf{R}}_h\Phi \in V_h(\hat{\Omega})$ satisfies $\hat{\mathbf{R}}_h\Phi \times \mathbf{n} = \Phi$ on $\hat{\Gamma}$ and

$$(\mathbf{curl}\,\hat{\mathbf{R}}_h\Phi, \mathbf{curl}\,\mathbf{v}_h) + (\hat{\mathbf{R}}_h\Phi, \mathbf{v}_h) = 0, \quad \forall\, \mathbf{v}_h \in V_h^0(\hat{\Omega}).$$

3 A stable weighted Helmholtz-type decomposition

As is well known, the Helmholtz decomposition plays an essential role in the convergence analysis of the multigrid and non-overlapping domain decomposition methods for solving the Maxwell system (1.1)–(1.2) by edge element methods; see, e.g., [2,14,15,20,21]. Any edge element function $\mathbf{v}_h$ from $V_h(\Omega)$ admits a Helmholtz decomposition of the form

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h \tag{3.1}$$

for some $p_h \in Z_h(\Omega)$ and $\mathbf{w}_h \in V_h(\Omega)$, and p_h and $\mathbf{w}_h$ are orthogonal in the inner product of $L^2(\Omega)$, namely $(\mathbf{w}_h, \nabla p_h) = 0$, and have the following stability estimates

$$\|\nabla p_h\|_{0,\Omega} \le C\|\mathbf{v}_h\|_{0,\Omega}\,, \quad \|\mathbf{w}_h\|_{0,\Omega} \le C\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}. \tag{3.2}$$

But in order to effectively deal with the divergence constraint in (1.2), one needs the decomposition (3.1) to be orthogonal with respect to the weight function β, namely $(\beta\mathbf{w}_h, \nabla p_h) = 0$. This can be done naturally, with the stability estimates (3.2) holding. But unfortunately, it is unclear how the coefficient C appearing in the two stability estimates in (3.2) depends on the coefficient β, especially for the practically important case where β is discontinuous in Ω and may have large jumps across the interface between any two different physical media. For this reason, although there are many multigrid or domain decomposition methods available in the literature for the Maxwell system (1.1)–(1.2), with optimal or nearly optimal convergence in terms of the mesh size and subdomain size, it is still unclear how their convergence depend on the jumps of the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ in (1.1)–(1.2).

The aim of this work is to fill in this gap and construct a discrete weighted Helmholtz-type decomposition, that is stable uniformly with respect to the jumps of the weight coefficient $\beta(\mathbf{x})$. The new (weighted) Helmholtz decomposition can be used to analyze convergence of various preconditioners for Maxwell's equations with large jumps in coefficients. For an application, we will show in Sect. 7 with the help of such a weighted Helmholtz decomposition that the substructuring preconditioner constructed in [21] converges not only nearly optimally in terms of the subdomain diameter and the finite element mesh size, but also independently of the jumps in the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ in (1.1)–(1.2).

From now on, we shall frequently use the notations $\lesssim$ and $\eqsim$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \le Cy$ for some constant C independent of mesh size h, subdomain size d and the possible large jumps of some related coefficient functions across the interface between any two subdomains. $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

We need to introduce a few concepts in order to describe the relation between different subdomains from $\{\Omega_r^0\}_{r=1}^{N_0}$, which are described in Sect. 2.1, based on the distribution of the discontinuity of the coefficient function $\beta(\mathbf{x})$ in (1.2).

Definition 3.1 For a subdomain Ω_r^0, another subdomain $\Omega_{r'}^0$ is called a "child" of Ω_r^0 if $\bar{\Omega}_{r'}^0 \cap \bar{\Omega}_r^0 \neq \emptyset$ and $\beta_{r'} < \beta_r$. In this case, the subdomain Ω_r^0 is called a "parent" of $\Omega_{r'}^0$.

Now we make an assumption on the coefficient $\beta(\mathbf{x})$. From now on, when we say two subdomains Ω_r^0 and $\Omega_{r'}^0$ do not intersect if $\bar{\Omega}_r^0 \cap \bar{\Omega}_{r'}^0 = \emptyset$; otherwise we say the two subdomains intersect each other. So based on this definition, two subdomains sharing only a common vertex are also said to intersect each other. For any subdomain Ω_r^0 $(1 \leq r \leq N_0)$, we assume that it satisfies *one* of the following two conditions:

Condition A. At most two " parent " subdomains of Ω_r^0 do not intersect each other. Here a " parent " subdomain may be the union of all parent subdomains of Ω_r^0 on which $\beta(\mathbf{x})$ takes the same value.

Condition B. The intersection of Ω_r^0 with the union of all parent subdomains of Ω_r^0 is a connected set.

In many applications, one may encounter only two or three different media involved in the entire physical domain, and in this case Condition A or B should be fulfilled naturally. In general, these two conditions are also mild and reasonable, and cover a lot of practical applications with complicated multiple medium cases; see Fig. 1 for an example with 7 to 12 media, where each block with a different number is a different medium, and the relation $i > j$ means that the physical coefficients in the two blocks satisfy $\beta_i > \beta_j$, so the medium domain i is a parent of medium j if they intersect each other. One can readily check that the left medium example in Fig. 1 satisfies Condition A, while the right one satisfies Condition B. Clearly all the cubic blocks can be of curved shape as well.

The following theorem provides an auxiliary result which is essential to the derivation of the desired weighted Helmholtz decomposition. The proof of the theorem is delayed to Sect. 6.

Theorem 3.1 *Assume that either Condition A or Condition B holds for each subdomain* $\Omega_r^0 (1 \leq r \leq N_0)$. *Then for any edge element function* $\mathbf{w}_h \in V_h(\Omega)$ *satisfying*

$$(\beta \mathbf{w}_h, \nabla q_h) = 0, \quad \forall q_h \in Z_h(\Omega)\,, \tag{3.3}$$

we have the following estimate

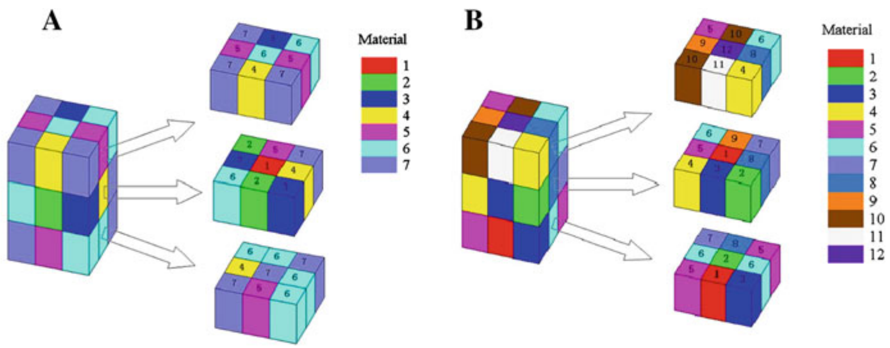

Fig. 1 A domain Ω with multiple media satisfying Condition A (*left*) or B (*right*)

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}^2 \le C\log^{m+1}(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\mathbf{w}_h\|_{0,\Omega}^2, \tag{3.4}$$

where constants m and C are independent of h and the jumps of the coefficient β.

The following theorem presents the main result of this paper.

Theorem 3.2 *Assume that either Condition A or Condition B holds for each subdomain* Ω_r^0 $(1 \le r \le N_0)$. *Then any* $\mathbf{v}_h \in V_h(\Omega)$ *admits a decomposition of the form*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h \tag{3.5}$$

for some $p_h \in Z_h(\Omega)$ *and* $\mathbf{w}_h \in V_h(\Omega)$, *and* $\mathbf{w}_h$ *satisfies*

$$(\beta\mathbf{w}_h, \nabla q_h) = 0, \quad \forall q_h \in Z_h(\Omega). \tag{3.6}$$

Moreover, p_h *and* $\mathbf{w}_h$ *have the estimates*

$$\|\beta^{\frac{1}{2}}\nabla p_h\|_{0,\Omega}^2 \le \|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}^2, \quad \|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}^2 \le C\log^{m+1}(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\mathbf{v}_h\|_{0,\Omega}^2 \tag{3.7}$$

where constants m and C are independent of h and the jumps of the coefficient β.

Proof For any $\mathbf{v}_h \in V_h(\Omega)$, let $p_h \in Z_h(\Omega)$ be the solution of the problem

$$(\beta\nabla p_h, \nabla q_h) = (\beta\mathbf{v}_h, \nabla q_h), \quad \forall q_h \in Z_h(\Omega).$$

Then the first estimate in (3.7) follows directly from the above definition of p_h and the Cauchy–Schwarz inequality. Now setting $\mathbf{w}_h = \mathbf{v}_h - \nabla p_h$, then relations (3.5) and (3.6) follow immediately also from the definition of p_h, while the second estimate in (3.7) is a direct consequence of (3.6) and Theorem 3.1. □

The remaining part of this work is devoted to the demonstration of Theorem 3.1 and the application of the new discrete weighted Helmholtz decomposition. To this end, we need to prepare quite a few technical tools and results.

4 Several variants of the Helmholtz decomposition

This section is a preparatory section for the establishment of a stable discrete Helmholtz decomposition as stated in Theorem 3.2. Throughout this subsection, we shall consider a convex polyhedron $\hat{\Omega}$ with its diameter of size $O(1)$ (see Remark 2.1), which represents a generic convex polyhedron from the medium subdomains $\Omega_1^0, \Omega_2^0, \ldots, \Omega_{N_0}^0$.

Let $Z_h(\hat{\Omega})$ and $V_h(\hat{\Omega})$ be the standard nodal and Nédélec finite element space on $\hat{\Omega}$ respectively as defined in Sect. 2.1.

Lemma 4.1 *Let $\hat{\Gamma}$ be either an empty set or a (closed) face of $\hat{\Omega}$ or the union of several faces of $\hat{\Omega}$, and $\mathbf{v}_h$ be a function in $V_h(\hat{\Omega})$ satisfying $\mathbf{v}_h \times \mathbf{n} = \mathbf{0}$ on $\hat{\Gamma}$. Then $\mathbf{v}_h$ admits a decomposition $\mathbf{v}_h = \nabla p_h + \mathbf{w}_h$ for some $p_h \in Z_h(\hat{\Omega})$ and $\mathbf{w}_h \in V_h(\hat{\Omega})$ such that $p_h = 0$, $\mathbf{w}_h \times \mathbf{n} = \mathbf{0}$ on $\hat{\Gamma}$, and $\mathbf{w}_h$ satisfies $\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\hat{\Omega}}$.*

Proof As $\mathbf{v}_h \in V_h(\hat{\Omega})$, we have $\mathbf{v}_h \cdot \mathbf{n} \in L^2(\partial\hat{\Omega})$. Consider $p \in H^1(\hat{\Omega})$ satisfying

$$\triangle p = \operatorname{div} \mathbf{v}_h \quad \text{in } \hat{\Omega}, \tag{4.1}$$

$$p = 0 \quad \text{on } \hat{\Gamma}, \tag{4.2}$$

$$\frac{\partial p}{\partial \mathbf{n}} = \mathbf{v}_h \cdot \mathbf{n} \quad \text{on } \partial\hat{\Omega}\backslash\hat{\Gamma} \tag{4.3}$$

and $\mathbf{w} = \mathbf{v}_h - \nabla p$. Then we know $\mathbf{w} \in \mathbf{H}(\mathbf{curl}; \hat{\Omega}) \cap \mathbf{H}(\mathrm{div}; \hat{\Omega})$, and $\mathbf{w}$ satisfies

$$\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\mathbf{v}_h \quad \text{in } \hat{\Omega}, \tag{4.4}$$

$$\mathbf{w} \times \mathbf{n} = \mathbf{0} \quad \text{on } \hat{\Gamma}, \tag{4.5}$$

$$\mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on } \partial\hat{\Omega}\backslash\hat{\Gamma}. \tag{4.6}$$

As in the proof of Theorem 4.3 in [2], we can verify, with some natural modifications, that

$$\|\mathbf{w}\|_{\delta,\hat{\Omega}} \lesssim \|\mathbf{curl}\mathbf{w}\|_{0,\hat{\Omega}} = \|\mathbf{curl}\mathbf{v}_h\|_{0,\hat{\Omega}}, \tag{4.7}$$

where $\delta \in (\frac{1}{2}, 1]$ depends on the geometric shape of $\hat{\Omega}$ only. Now by applying the edge element interpolation r_h on both sides of the decomposition $\mathbf{v}_h = \nabla p + \mathbf{w}$, we know how to take the desired functions p_h and $\mathbf{w}_h$ in the lemma, i.e., $\mathbf{w}_h = r_h\mathbf{w}$ and $\nabla p_h = r_h \nabla p$. Indeed, we have by the error estimate of the operator r_h (cf. [2,11]) and (4.7) that

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} = \|r_h\mathbf{w}\|_{0,\hat{\Omega}} \lesssim \|\mathbf{w}\|_{0,\hat{\Omega}} + \|r_h\mathbf{w} - \mathbf{w}\|_{0,\hat{\Omega}} \lesssim \|\mathbf{w}\|_{0,\hat{\Omega}} + h^\delta \|\mathbf{w}\|_{\delta,\hat{\Omega}} \lesssim \|\mathbf{curl}\mathbf{v}_h\|_{0,\hat{\Omega}}.$$

□

Lemma 4.2 *For any face F of $\hat{\Omega}$, assume that $\mathbf{v}_h \in V_h(\hat{\Omega})$ satisfies $\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on $\partial\mathrm{F}$. Then there exist $p_h \in Z_h(\hat{\Omega})$ and $\mathbf{w}_h \in V_h(\hat{\Omega})$ such that $p_h = 0$, $\mathbf{w}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on $\partial\mathrm{F}$,* *and*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h, \tag{4.8}$$

with the following estimate

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\mathbf{w}_h\|_{0,\hat{\Omega}}. \tag{4.9}$$

The conclusion is also valid for the case when F *is replaced by a union of some faces.*

Proof We separate the proof into two steps.
Step 1: Establish the desired decomposition. We first establish a Hodge-type decomposition on the given two-dimensional face F. To do so, we introduce a space $W_h(\mathrm{F})$ on F, consisting of tangential vectors:

$$W_h(\mathrm{F}) = \{\mathbf{n} \times (\mathbf{v}_h \times \mathbf{n})|_{\mathrm{F}};\quad \mathbf{v}_h \in V_h(\Omega)\},$$

and define a function $\mathbf{v}_{h,\mathrm{F}} \in W_h(\mathrm{F})$ such that

$$\mathbf{v}_{h,\mathrm{F}} = \mathbf{n} \times (\mathbf{v}_h \times \mathbf{n}) \quad \text{on} \ \mathrm{F};\quad \mathbf{v}_{h,\mathrm{F}} = \mathbf{0} \quad \text{on} \ \partial\hat{\Omega}\backslash\mathrm{F}.$$

Then there exist $p_{h,\mathrm{F}} \in Z_h(\hat{\Omega})$ and $\mathbf{w}_{h,\mathrm{F}} \in V_h(\hat{\Omega})$ by Lemma 7.12 of [29] such that

$$\mathbf{v}_{h,\mathrm{F}} = \nabla_S\, p_{h,\mathrm{F}} + \mathbf{w}_{h,\mathrm{F}} \quad \text{on} \ \mathrm{F},$$

where ∇_S is the two-dimensional surface gradient, $p_{h,\mathrm{F}}$ and $\mathbf{w}_{h,\mathrm{F}}$ satisfy $p_{h,\mathrm{F}} = 0$, $\mathbf{w}_{h,\mathrm{F}} = 0$ on $\partial\hat{\Omega}\backslash\mathrm{F}$, and have the estimate

$$\|\mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} + \|\mathbf{curl}\ \mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} \lesssim \|curl_S\mathbf{v}_{h,\mathrm{F}}\|_{-\frac{1}{2},\mathrm{F}}, \tag{4.10}$$

where $curl_S$ is the so-called surface $curl$; see [29] for its definition. Note that the surface $curl$ is just the tangential divergence, i.e., $curl_S\mathbf{v}_{h,\mathrm{F}} = \mathrm{div}_\tau(\mathbf{n} \times \mathbf{v}_{h,\mathrm{F}})$; see [1,2,20].

Then we define

$$\hat{\mathbf{v}}_{h,\mathrm{F}} = \mathbf{v}_h - (\nabla p_{h,\mathrm{F}} + \mathbf{w}_{h,\mathrm{F}}). \tag{4.11}$$

We can check that $\hat{\mathbf{v}}_{h,\mathrm{F}} \times \mathbf{n} = \mathbf{0}$ on F. By Lemma 4.1 $\hat{\mathbf{v}}_{h,\mathrm{F}}$ admits the decomposition

$$\hat{\mathbf{v}}_{h,\mathrm{F}} = \nabla\hat{p}_h + \hat{\mathbf{w}}_h \tag{4.12}$$

for some $\hat{p}_h \in Z_h(\hat{\Omega})$ and $\hat{\mathbf{w}}_h \in V_h(\hat{\Omega})$ such that $\hat{p}_h = 0$, $\hat{\mathbf{w}}_h \times \mathbf{n} = 0$ on F, and $\hat{\mathbf{w}}_h$ satisfies

$$\|\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}} \lesssim \|\mathbf{curl}\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}}. \tag{4.13}$$

Now by defining

$$p_h = p_{h,\mathrm{F}} + \hat{p}_h \ \text{ and } \ \mathbf{w}_h = \mathbf{w}_{h,\mathrm{F}} + \hat{\mathbf{w}}_h,$$

we get the expected decomposition

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h \tag{4.14}$$

where p_h and $\mathbf{w}_h$ satisfy $p_h = 0$, $\mathbf{w}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on $\partial\mathrm{F}$.
Step 2: Verify the desired estimate (4.9) for the decomposition (4.14).

By the definition of $\mathbf{w}_h$ and the triangle inequality, we have

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \|\mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} + \|\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}}.$$

This, along with (4.11), (4.12) and (4.13), leads to

$$\|\mathbf{w}_h\|_{0,\ \hat{\Omega}} \lesssim \|\mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} + \|\mathbf{curl}\mathbf{w}_{h,\mathrm{F}}\|_{0,\hat{\Omega}} + \|\mathbf{curl}\mathbf{v}_h\|_{0,\hat{\Omega}}.$$

Then, we further get from (4.10) that

$$\|\mathbf{w}_h\|_{0,\ \hat{\Omega}} \lesssim \|curl_S\mathbf{v}_h\|_{-\frac{1}{2},\mathrm{F}} + \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}}. \tag{4.15}$$

On the other hand, using the known face $H^{-\frac{1}{2}}$-extension (cf. [17,29]) and the trace theorem, we obtain

$$\|curl_S\mathbf{v}_h\|_{-\frac{1}{2},\mathrm{F}} \lesssim \log(1/h)\|curl_S\mathbf{v}_h\|_{-\frac{1}{2},\partial\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}}.$$

Substituting this into (4.15), yields the desired result (4.9). □

Remark 4.1 The face $H^{-\frac{1}{2}}$-extension used in the proof of Lemma 4.2 brings in an a logarithmic factor in the estimate, thus an extra logarithmic factor in the main estimate of Theorem 3.2. This face $H^{-\frac{1}{2}}$-extension, which seems to be sharp, can be regarded as a dual result of the well-known face $H^{\frac{1}{2}}$-extension (see, e.g., [32]). To our knowledge, this kind of face $H^{-\frac{1}{2}}$-extensions was first estimated in [17].

Lemma 4.3 *Let* E *be a (closed) edge of* $\hat{\Omega}$*, and* $\mathbf{v}_h$ *be a finite element function in* $V_h(\hat{\Omega})$ *such that* $\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}} = 0$ *on* E*. Then* $\mathbf{v}_h$ *admits a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h$$

for some $p_h \in Z_h(\hat{\Omega})$ *and* $\mathbf{w}_h \in V_h(\hat{\Omega})$ *such that* $p_h = \mathbf{w}_h \cdot \mathbf{t}_{\mathrm{E}} = 0$ *on* E *and*

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{w}_h\|_{0,\hat{\Omega}}. \tag{4.16}$$

Proof We separate the proof into three steps.
Step 1: Establish an edge-related decomposition.

Let F be a face containing the edge E. We first consider a decomposition of the tangential component $\mathbf{v}_h \cdot \mathbf{t}_{\partial F}$ of $\mathbf{v}_h$ on ∂F. For convenience, we write $E^c = \partial F \backslash E$.

Let s be the arclength along E^c, taking values from 0 to l_0, where l_0 is the total length of E^c. In terms of s, $\mathbf{v}_h \cdot \mathbf{t}_{E^c}$ is piecewise linear on the interval $[0, l_0]$, denoted by $\hat{v}(s)$. Then we define

$$C_E = \frac{1}{l_0}\int_0^{l_0} \hat{v}(s)\, ds, \quad \phi_E(t) = \int_0^t (\hat{v}(s) - C_E) ds\,, \quad \forall t \in [0, l_0].$$

Clearly we see $\phi_E(t)$ vanishes at $t = 0$ and l_0. Now we can extend ϕ_E and C_E naturally by zero onto E, then extend by zero into $\partial\hat{\Omega}$ and $\hat{\Omega}$ such that their extensions $\tilde{\phi}_E \in Z_h(\hat{\Omega})$ and $\tilde{\mathbf{C}}_E \in V_h(\hat{\Omega})$. One can verify that (cf. [29]) that

$$\mathbf{v}_h \cdot \mathbf{t}_{\partial F} = (\nabla\tilde{\phi}_E) \cdot \mathbf{t}_{\partial F} + \tilde{\mathbf{C}}_E \cdot \mathbf{t}_{\partial F}. \tag{4.17}$$

Step 2: Construct the desired decomposition in Lemma 4.3. For the purpose, we set

$$\hat{\mathbf{v}}_{h,E} = \mathbf{v}_h - (\nabla\tilde{\phi}_E + \tilde{\mathbf{C}}_E). \tag{4.18}$$

By (4.17) we know $\hat{\mathbf{v}}_{h,E} \cdot \mathbf{t}_{\partial F} = 0$ on ∂F. For function $\hat{\mathbf{v}}_{h,E}$ in (4.18), following the same way as it was done in the proof of Lemma 4.2 one can find two functions $\hat{p}_h \in Z_h(\hat{\Omega})$ and $\hat{\mathbf{w}}_h \in V_h(\hat{\Omega})$ such that $\hat{p}_h = 0$, $\hat{\mathbf{w}}_h \cdot \mathbf{t}_{\partial F} = 0$ on ∂F, and (see (4.14))

$$\hat{\mathbf{v}}_{h,E} = \nabla\hat{p}_h + \hat{\mathbf{w}}_h,$$

with the following estimate (see (4.15))

$$\|\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}} \lesssim \|\mathbf{curl}\,\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}} + \|curl_S\hat{\mathbf{v}}_{h,E}\|_{-\frac{1}{2},F}. \tag{4.19}$$

Now by defining

$$p_h = \tilde{\phi}_E + \hat{p}_h \;\text{ and }\; \mathbf{w}_h = \tilde{\mathbf{C}}_E + \hat{\mathbf{w}}_h\,,$$

we get the final decomposition

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h \tag{4.20}$$

such that $p_h = 0$, $\mathbf{w}_h \cdot \mathbf{t}_E = 0$ on E.
Step 3: Derive the desired estimate in Lemma 4.3 for the decomposition (4.20).

Noting that $\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}} = 0$ on E, so $\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ on E, we have by the Green's formula on F and change of variables (cf. [29]) that (with l being the total arclength of ∂F)

$$C_{\mathrm{E}} = \frac{1}{l_0}\int_0^l \hat{v}(s)ds = \frac{1}{l_0}\int_{\mathrm{F}} curl_S \mathbf{v}_h \cdot 1ds. \tag{4.21}$$

Let $I_{\mathrm{F}}^0 1$ be the face interpolant of 1, namely $I_{\mathrm{F}}^0 1 \in Z_h(\hat{\Omega})$ and takes value 1 at those nodes in F, and zero at all other nodes on $\partial\hat{\Omega}$ and in $\hat{\Omega}$. Similarly we define the edge interpolant $I_{\partial\mathrm{F}}^0 1$. As in the analysis of the face $H^{-1/2}$-extension (cf. [17,29])), we can show

$$\|I_{\mathrm{F}}^0 1\|_{\frac{1}{2},\partial\hat{\Omega}} \lesssim \log^{\frac{1}{2}}(1/h), \quad \|curl_S \mathbf{v}_h\|_{0,\mathrm{F}} \lesssim h^{-\frac{1}{2}}\|curl_S \mathbf{v}_h\|_{-\frac{1}{2},\partial\hat{\Omega}}, \quad \|I_{\partial\mathrm{F}}^0 1\|_{0,\mathrm{F}} \lesssim h^{\frac{1}{2}},$$

thus obtaining

$$\Big|\int_{\mathrm{F}} curl_S \mathbf{v}_h \cdot 1ds\Big| \lesssim \log^{\frac{1}{2}}(1/h)\|curl_S \mathbf{v}_h\|_{-\frac{1}{2},\,\partial\hat{\Omega}} \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}}.$$

This, along with (4.21), leads to

$$|C_{\mathrm{E}}| \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}}.$$

By the definition of C_{E}, we further obtain

$$\|\tilde{\mathbf{C}}_{\mathrm{E}} \cdot \mathbf{t}_{\partial\mathrm{F}}\|_{0,\partial\mathrm{F}} \lesssim |C_{\mathrm{E}}| \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\hat{\Omega}}.$$

Using this estimate and the definition of $\tilde{\mathbf{C}}_{\mathrm{E}}$ we obtain

$$\|curl_S \tilde{\mathbf{C}}_{\mathrm{E}}\|_{-\frac{1}{2},\mathrm{F}} \lesssim \log^{\frac{1}{2}}(1/h)\|\tilde{\mathbf{C}}_{\mathrm{E}} \cdot \mathbf{t}_{\partial\mathrm{F}}\|_{0,\partial\mathrm{F}} \lesssim \log(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\hat{\Omega}}, \tag{4.22}$$

$$\|\tilde{\mathbf{C}}_{\mathrm{E}}\|_{0,\hat{\Omega}} + \|\mathbf{curl}\,\tilde{\mathbf{C}}_{\mathrm{E}}\|_{0,\hat{\Omega}} \lesssim \|\tilde{\mathbf{C}}_{\mathrm{E}} \cdot \mathbf{t}_{\partial\mathrm{F}}\|_{0,\partial\mathrm{F}} \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\hat{\Omega}}, \tag{4.23}$$

where we have used Lemma 6.8 in [21] for the derivation of the first inequality in (4.22). By the definition of $\hat{\mathbf{v}}_{h,\mathrm{E}}$, combining the $H^{-\frac{1}{2}}$-extension with (4.22), yields

$$\begin{aligned}\|curl_S \hat{\mathbf{v}}_{h,\mathrm{E}}\|_{-\frac{1}{2},\mathrm{F}} &\lesssim \|curl_S \mathbf{v}_h\|_{-\frac{1}{2},\mathrm{F}} + \|curl_S \tilde{\mathbf{C}}_{\mathrm{E}}\|_{-\frac{1}{2},\mathrm{F}}\\ &\lesssim \log(1/h)\|curl_S \mathbf{v}_h\|_{-\frac{1}{2},\partial\hat{\Omega}} + \log(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\,\hat{\Omega}}\\ &\lesssim \log(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\,\hat{\Omega}}.\end{aligned} \tag{4.24}$$

Now by the triangle inequality, we have

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \|\tilde{\mathbf{C}}_{\mathrm{E}}\|_{0,\hat{\Omega}} + \|\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}},$$

which, together with (4.23), (4.19) and (4.24), leads to

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\hat{\Omega}} + \|\mathbf{curl}\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}}. \tag{4.25}$$

Noting that

$$\mathbf{curl}\,\hat{\mathbf{w}}_h = \mathbf{curl}\,\mathbf{w}_h - \mathbf{curl}\,\tilde{\mathbf{C}}_{\mathrm{E}} = \mathbf{curl}\,\mathbf{v}_h - \mathbf{curl}\,\tilde{\mathbf{C}}_{\mathrm{E}},$$

we obtain by using (4.23) that

$$\|\mathbf{curl}\,\hat{\mathbf{w}}_h\|_{0,\hat{\Omega}} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\hat{\Omega}} + \|\mathbf{curl}\,\tilde{\mathbf{C}}_{\mathrm{E}}\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\hat{\Omega}}.$$

Combining this with (4.25), we get the desired estimate (4.16). □

Lemma 4.4 *Let* V *be a vertex of* $\hat{\Omega}$ *and* $\mathbf{v}_h$ *a function in* $V_h(\hat{\Omega})$*. Then we can write* $\mathbf{v}_h$ *as*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h$$

for some $p_h \in Z_h(\hat{\Omega})$ *and* $\mathbf{w}_h \in V_h(\hat{\Omega})$ *satisfying* $p_h(\mathrm{V}) = 0$ *and*

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{w}_h\|_{0,\hat{\Omega}}.$$

Proof Consider a face F containing V as a vertex, and let $\phi_{\partial\mathrm{F}}$ be a function that is linear on each edge of F and continuous on $\partial\mathrm{F}$ such that $\phi_{\partial\mathrm{F}}(\mathrm{V}) = 0$. Then as in the proof of Lemma 4.3, we can follow [29] to decompose $\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}}$ on $\partial\mathrm{F}$ and build the desired decomposition for $\mathbf{v}_h$. □

Lemma 4.5 *Let* E *be a (closed) edge of* $\hat{\Omega}$*, and* V *be a vertex of* $\hat{\Omega}$ *but* $\mathrm{V} \notin \mathrm{E}$*. Assume that* $\mathbf{v}_h \in V_h(\hat{\Omega})$ *satisfies* $\lambda_e(\mathbf{v}_h) = \mathbf{0}$ *for all* $e \subset \mathrm{E}$*. Then* $\mathbf{v}_h$ *can be decomposed as*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h$$

for some $p_h \in Z_h(\hat{\Omega})$ *and* $\mathbf{w}_h \in V_h(\hat{\Omega})$ *such that*

$$p_h(\mathrm{V}) = 0\,, \quad and \quad p_h = 0 \quad on \quad \mathrm{E}, \quad \lambda_e(\mathbf{w}_h) = \mathbf{0} \qquad \forall\, e \subset \mathrm{E},$$

and $\mathbf{w}_h$ *has the following estimate*

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\mathbf{w}_h\|_{0,\hat{\Omega}}. \tag{4.26}$$

Proof Let F be a (closed) face, which has V as one of its vertices, but does not have E as one of its edges. Let $C_{\partial\mathrm{F}}$ be the average of $\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}}$ over $\partial\mathrm{F}$, then we can split $\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}}$ into the sum $\phi'_{\partial\mathrm{F}} + C_{\partial\mathrm{F}}$ on $\partial\mathrm{F}$ such that $\phi_{\partial\mathrm{F}}$ is continuous on $\partial\mathrm{F}$, and piecewise linear on each edge of F and satisfies $\phi_{\partial\mathrm{F}}(\mathrm{V}) = 0$. Then we extend $\phi_{\partial\mathrm{F}}$ and $C_{\partial\mathrm{F}}$ naturally by zero onto $\hat{\Omega}$ such that their extensions $\tilde{\phi}_{\partial\mathrm{F}} \in Z_h(\hat{\Omega})$ and $\tilde{\mathbf{C}}_{\partial\mathrm{F}} \in V_h(\hat{\Omega})$.

We will treat the problem separately according to two different cases.

(i) There is a (closed) face F' such that $\mathrm{F}' \cap \mathrm{F} = \emptyset$ and F' has E as one of its edges. It is the case when $\hat{\Omega}$ is a hexahedron.

In this case, we can directly decompose $\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}^c}$ into the sum $\phi_{\mathrm{E}}' + C_{\mathrm{E}}$ on $\mathrm{E}^c = \partial\mathrm{F}' \backslash \mathrm{E}$ as in Lemma 4.3, then extend ϕ_{E} and C_{E} naturally by zero such that their extensions $\tilde{\phi}_{\mathrm{E}} \in Z_h(\hat{\Omega})$ and $\tilde{\mathbf{C}}_{\mathrm{E}} \in V_h(\hat{\Omega})$. Then we define

$$\hat{\mathbf{v}}_h = \mathbf{v}_h - (\nabla\tilde{\phi}_{\partial\mathrm{F}} + \nabla\tilde{\phi}_{\mathrm{E}} + \tilde{\mathbf{C}}_{\partial\mathrm{F}} + \tilde{\mathbf{C}}_{\mathrm{E}}).$$

It is clear to see $(\hat{\mathbf{v}}_h \cdot \mathbf{t}_{\partial\mathrm{F}})|_{\partial\mathrm{F}} = (\hat{\mathbf{v}}_h \cdot \mathbf{t}_{\partial\mathrm{F}'})|_{\partial\mathrm{F}'} = 0$. Now applying Lemma 4.2 for $\hat{\mathbf{v}}_h$, one can get a decomposition of $\hat{\mathbf{v}}_h$ based on the two faces F and F', and further construct the desired decomposition of $\mathbf{v}_h$.

(ii) The edge E has a common vertex with a (closed) edge E' of F. This is the case when $\hat{\Omega}$ is a tetrahedron. Then we set

$$\tilde{\mathbf{v}}_h = \mathbf{v}_h - (\nabla\tilde{\phi}_{\partial\mathrm{F}} + \tilde{\mathbf{C}}_{\partial\mathrm{F}}).$$

By the assumption, we know $\tilde{\mathbf{v}}_h \cdot \mathbf{t}_\Gamma = 0$ on $\Gamma = \mathrm{E} \cup \mathrm{E}'$. Let F' be the face with E and E' as two of its neighboring edges, and set $\Gamma^c = \partial\mathrm{F}' \backslash \Gamma$. As in Lemma 4.3, we can build a decomposition of $\tilde{\mathbf{v}}_h \cdot \mathbf{t}_{\Gamma^c}$ as follows:

$$\tilde{\mathbf{v}}_h \cdot \mathbf{t}_{\Gamma^c} = \phi_\Gamma' + C_\Gamma \quad \text{on } \Gamma^c,$$

where ϕ_Γ vanishes at the two endpoints of Γ^c. Let $\tilde{\phi}_\Gamma \in Z_h(\hat{\Omega})$ and $\tilde{\mathbf{C}}_\Gamma \in V_h(\hat{\Omega})$ be the natural extensions of ϕ_Γ and C_Γ by zero, respectively, and set

$$\hat{\mathbf{v}}_h = \mathbf{v}_h - (\nabla\tilde{\phi}_{\partial\mathrm{F}} + \nabla\tilde{\phi}_\Gamma + \tilde{\mathbf{C}}_{\partial\mathrm{F}} + \tilde{\mathbf{C}}_\Gamma),$$

one can easily check that $(\hat{\mathbf{v}}_h \cdot \mathbf{t}_{\partial\mathrm{F}})|_{\partial\mathrm{F}} = (\hat{\mathbf{v}}_h \cdot \mathbf{t}_{\partial\mathrm{F}'})|_{\partial\mathrm{F}'} = 0$. Now applying Lemma 4.2 for $\hat{\mathbf{v}}_h$, one can get a decomposition of $\hat{\mathbf{v}}_h$ based on the two faces F and F', and further get the desired decomposition of $\mathbf{v}_h$ as in Lemma 4.3. □

Following the same arguments as the ones in the proof of Lemma 4.5, we can show

Lemma 4.6 *Let $\hat{\Gamma}$ be the union of a set of neighboring (closed) edges and (closed) faces of $\hat{\Omega}$ such that it is connected and $\mathbf{v}_h$ be a function in $V_h(\hat{\Omega})$ satisfying $\lambda_e(\mathbf{v}_h) = \mathbf{0}$ for all $e \subset \hat{\Gamma}$. Then $\mathbf{v}_h$ admits a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h$$

for some $p_h \in Z_h(\hat{\Omega})$ and $\mathbf{w}_h \in V_h(\hat{\Omega})$ such that $p_h = 0$ on $\hat{\Gamma}$ and $\lambda_e(\mathbf{w}_h) = 0$ for $e \subset \hat{\Gamma}$, and $\mathbf{w}_h$ satisfies the estimate

$$\|\mathbf{w}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\mathbf{w}_h\|_{0,\hat{\Omega}}.$$

5 A stable decomposition for any function $\mathbf{v}_h$ in $V_h(\Omega)$

With the help of the preliminary results from Sect. 4, we are now ready to address the central task of this work, namely to construct a discrete weighted Helmholz-type decomposition for any function $\mathbf{v}_h$ in $V_h(\Omega)$. For the purpose, we start with a classification of all the polyhedra $\{\Omega_r^0\}_{r=1}^{N_0}$ based on the values of $\beta(\mathbf{x})$ in (2.1).

Let Σ_1 be the set of all polyhedral subdomains Ω_r^0 which do not have a parent subdomain. Namely, $\Omega_r^0 \in \Sigma_1$ if and only if it holds that for any subdomain $\Omega_{r'}^0$ with $r' \neq r$, either $\bar{\Omega}_{r'}^0 \cap \bar{\Omega}_r^0 = \emptyset$ or $\beta_{r'} \leq \beta_r$. Clearly Σ_1 is not empty, as it contains at least all the subdomains Ω_r^0 where it holds that $\beta_r = \max_{1\leq k\leq N_0} \beta_k$.

Let Σ_2 denote a subset of the children of all polyhedra belonging to Σ_1 such that each polyhedron in Σ_2 has no parent subdomain in $\{\Omega_r^0\}_{r=1}^{N_0}\backslash\Sigma_1$. If $\Sigma_2 = \emptyset$, then there are no subdomains where $\beta(\mathbf{x})$ takes values less than its value in Σ_1, and we stop the process.

Similarly, if $\Sigma_2 \neq \emptyset$ we let Σ_3 be a subset of the children of all polyhedra belonging to $\Sigma_1 \cup \Sigma_2$ such that each polyhedron in Σ_3 has no parent subdomain in $\{\Omega_r^0\}_{r=1}^{N_0}\backslash(\Sigma_1 \cup \Sigma_2)$. If $\Sigma_3 = \emptyset$, there are no subdomains where $\beta(\mathbf{x})$ takes values less than its values in Σ_1 and Σ_2, then we stop the process.

We continue this procedure to classify a sequence of non-empty sets, $\Sigma_1, \Sigma_2, \ldots$, till we have $\Sigma_{m+1} = \emptyset$ for some $m \geq 1$, that is, there are no subdomains where $\beta(\mathbf{x})$ takes values less than its value in Σ_m. Clearly such integer m exists and $m \leq N_0$.

We can see from the above classifying process that the sequence $\Sigma_1, \ldots, \Sigma_m$ satisfy the following conditions: (1) $\Sigma_l \neq \emptyset$ for $1 \leq l \leq m$; (2) Σ_l consists of some children of polyhedra belonging to $\cup_{i=1}^{l-1}\Sigma_i$; (3) each polyhedron in Σ_l has no parent subdomain in $\{\Omega_r^0\}_{r=1}^{N_0}\backslash(\cup_{i=1}^{l-1}\Sigma_i)$; (4) any two polyhedra in Σ_l either do not intersect each other or coefficient $\beta(\mathbf{x})$ takes the same value on both polyhedra; (5) $\{\Omega_1^0, \Omega_2^0, \ldots, \Omega_{N_0}^0\} = \{\Sigma_1, \Sigma_2, \ldots, \Sigma_m\}$.

Next, we set $n_0 = 0$. Without loss of generality, we assume that for $l = 1, \ldots, m$,

$$\Sigma_l = \left\{\Omega_{n_{l-1}+1}^0,\ \Omega_{n_{l-1}+2}^0, \ldots, \Omega_{n_l}^0\right\}$$

and $n_l > n_{l-1}$. Clearly, we see $n_m = N_0$ and that Σ_l contains $(n_l - n_{l-1})$ polyhedra.

We are now ready to construct a desired decomposition for any $\mathbf{v}_h$ in $V_h(\Omega)$, and try and achieve this by three steps.

Step 1: Decompose $\mathbf{v}_h$ on all the polyhedra in Σ_1.

We shall write $\mathbf{v}_{h,r} = \mathbf{v}_h|_{\Omega_r^0}$. For $r = 1, 2, \ldots, n_1$, we can follow the arguments of Lemma 4.1 to decompose $\mathbf{v}_{h,r}$ as follows:

$$\mathbf{v}_{h,r} = \nabla p_r + \mathbf{w}_r = \nabla p_{h,r} + r_h\mathbf{w}_r := \nabla p_{h,r} + \mathbf{w}_{h,r}\,, \tag{5.1}$$

where $p_r \in H^1(\Omega_r^0)$, and $\mathbf{w}_r \in \mathbf{H}(\mathbf{curl}; \Omega_r^0) \cap \mathbf{H}_0(\mathrm{div}; \Omega_r^0)$ and $\mathrm{div}\,\mathbf{w}_r = 0$ in Ω_r^0. Moreover,

$$\|\mathbf{w}_{h,r}\|_{0,\Omega_r^0} + \|\mathbf{curl}\,\mathbf{w}_{h,r}\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r^0} \qquad \forall r = 1, \ldots, n_1. \tag{5.2}$$

Let $\tilde{p}_{h,r} \in Z_h(\Omega)$ be the standard extensions of $p_{h,r}$ by zero onto Ω, and $\tilde{\mathbf{w}}_{h,r} \in V_h(\Omega)$ be the discrete **curl curl**-extension of $\mathbf{w}_{h,r}$ in each Ω_l^0 such that $\lambda_e(\tilde{\mathbf{w}}_{h,r}) = 0$ for every $e \subset \partial\Omega_l^0 \backslash \partial\Omega_r^0$ for all $l \neq r$. Then we define

$$\tilde{\mathbf{v}}_{h,r} = \nabla \tilde{p}_{h,r} + \tilde{\mathbf{w}}_{h,r} \;\text{ for all } r \text{ such that } \Omega_r^0 \in \Sigma_1. \tag{5.3}$$

We remark that if a subdomain Ω_r^0 in Σ_1 intersects one or more than one other subdomains in Σ_1, then $\beta(\mathbf{x})$ must take the same values in all these subdomains. In this case, we should take the union of all these subdomains to replace Ω_r^0 when we do the extensions for $\tilde{p}_{h,r}$ and $\tilde{\mathbf{w}}_{h,r}$ above.

Step 2: Decompose $\mathbf{v}_h$ on all the polyhedra in Σ_2.

Consider a subdomain Ω_r^0 from Σ_2. By the assumption of Theorem 3.2, Ω_r^0 satisfies either Condition A or Condition B. For the sake of exposition, we treat only one case in each step, namely Condition A in this step, and Condition B in Step 3. The other case in each step can be handled in a similar manner. As Ω_r^0 satisfies Condition A, it has at most two parent subdomains in Σ_1, which do not intersect each other. Without loss of generality, assume that Ω_r^0 has two parent subdomains in Σ_1, say $\Omega_{r_1}^0$ and $\Omega_{r_2}^0$, and $\bar{\Omega}_{r_1}^0 \cap \bar{\Omega}_{r_2}^0 = \emptyset$, while $\bar{\Omega}_r^0 \cap \bar{\Omega}_{r_1}^0 = \mathrm{V}$ (a vertex) and $\bar{\Omega}_r^0 \cap \bar{\Omega}_{r_2}^0 = \mathrm{E}$ (an edge). Set

$$\mathbf{v}_{h,r}^* = \mathbf{v}_{h,r} - (\tilde{\mathbf{v}}_{h,r_1} + \tilde{\mathbf{v}}_{h,r_2}) \quad \text{ on } \Omega_r^0.$$

It is easy to see that $\lambda_e(\mathbf{v}_{h,r}^*) = 0$ for $e \subset \mathrm{E}$. Then by Lemma 4.5, there exist $p_{h,r}^* \in Z_h(\Omega_r^0)$ and $\mathbf{w}_{h,r}^* \in V_h(\Omega_r^0)$ such that

$$\mathbf{v}_{h,r}^* = \nabla p_{h,r}^* + \mathbf{w}_{h,r}^* \quad \text{ on } \;\Omega_r^0\,, \tag{5.4}$$

and

$$p_{h,r}^*(\mathrm{V}) = 0\,,\;\; p_{h,r}^* = 0 \;\text{ on }\; \mathrm{E}\,,\;\; \text{and} \;\; \lambda_e(\mathbf{w}_{h,r}^*) = 0 \;\text{ for all }\; e \subset \mathrm{E}. \tag{5.5}$$

Moreover, for $r = n_1 + 1, \ldots, n_2$, i.e., for all indices r such that $\Omega_r^0 \in \Sigma_2$, it follows from (5.4) and (5.3) that

$$\begin{aligned}\|\mathbf{curl}\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} &= \|\mathbf{curl}\mathbf{v}_{h,r}^*\|_{0,\Omega_r^0} = \|\mathbf{curl}(\mathbf{v}_{h,r} - (\tilde{\mathbf{w}}_{h,r_1} + \tilde{\mathbf{w}}_{h,r_2}))\|_{0,\Omega_r^0}\\ &\lesssim \|\mathbf{curl}\mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{l=1}^{2} \|\mathbf{curl}\tilde{\mathbf{w}}_{h,r_l}\|_{0,\Omega_r^0}.\end{aligned} \tag{5.6}$$

We further get by (4.26) that

$$\begin{aligned}\|\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} &\lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0}\\ &\lesssim \log(1/h)\left(\|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{l=1}^{2} \|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,r_l}\|_{0,\Omega_r^0}\right).\end{aligned} \tag{5.7}$$

Now we can define the decomposition of $\mathbf{v}_h$ on $\Omega_r^0 \in \Sigma_2$ as

$$\mathbf{v}_{h,r} = \nabla\left(p_{h,r}^* + \sum_{l=1}^{2} \tilde{p}_{h,r_l}\right) + \mathbf{w}_{h,r}^* + \sum_{l=1}^{2} \tilde{\mathbf{w}}_{h,r_l}\,, \tag{5.8}$$

where $\tilde{\mathbf{w}}_{h,r_1} = 0$ on Ω_r^0, by noting that $\bar{\Omega}_r^0 \cap \bar{\Omega}_{r_1}^0 = \mathrm{v}$, with v being a common vertex.

For functions $p_{h,r}^*$ and $\mathbf{w}_{h,r}^*$ in (5.4), we shall extend them onto the entire domain Ω. Let $\tilde{p}_{h,r}^* \in Z_h(\Omega)$ be the standard extension of $p_{h,r}^*$ by zero onto Ω, and $\tilde{\mathbf{w}}_{h,r}^* \in V_h(\Omega)$ be an extension of $\mathbf{w}_{h,r}^*$ such that $\lambda_e(\tilde{\mathbf{w}}_{h,r}^*) = 0$ for every $e \subset \partial\Omega_l^0 \backslash \partial\Omega_r^0$ with all l's such that $l \neq r$, and $\tilde{\mathbf{w}}_{h,r}^*$ is the discrete **curl curl**-extension on each Ω_l^0. Then we set

$$\tilde{\mathbf{v}}_{h,r}^* = \nabla \tilde{p}_{h,r}^* + \tilde{\mathbf{w}}_{h,r}^* \quad \text{for all } r \text{ such that } \Omega_r^0 \in \Sigma_2. \tag{5.9}$$

We remark that if a subdomain Ω_r^0 in Σ_2 intersects one or more than one other subdomains in Σ_2, then $\beta(\mathbf{x})$ must take the same value in all these subdomains. In this case, we should take the union of all these subdomains to replace Ω_r^0 when we do the extensions for $\tilde{p}_{h,r}^*$ and $\tilde{\mathbf{w}}_{h,r}^*$ above.

Step 3: Obtain the final desired decomposition of $\mathbf{v}_h$.

We now consider the index $l \geq 3$, and assume that the decompositions of $\mathbf{v}_h$ on all polyhedra belonging to Σ_1 $\Sigma_2, \ldots, \Sigma_{l-1}$ are done as in Steps 1 and 2. Next, we will build up a decomposition of $\mathbf{v}_h$ in all subdomains $\Omega_r^0 \in \Sigma_l$.

Without loss of generality, we assume that Ω_r^0 satisfies Condition B; see the remark at the first part of Step 2. Then by Condition B, we use Γ_r to denote the corresponding connected set, which is the union of some edges and faces. For the ease of notation, we introduce two index sets:

$$\Lambda_r^1 = \{\, i\,;\ 1 \leq i \leq n_1 \text{ such that } \partial\Omega_i^0 \cap \partial\Omega_r^0 \neq \emptyset\},$$
$$\Lambda_r^{l-1} = \{\, i\,;\ n_1 + 1 \leq i \leq n_{l-1} \text{ such that } \partial\Omega_i^0 \cap \partial\Omega_r^0 \neq \emptyset\}.$$

Define

$$\mathbf{v}_{h,r}^* = \mathbf{v}_{h,r} - \sum_{i\in\Lambda_r^1} \tilde{\mathbf{v}}_{h,i} - \sum_{i\in\Lambda_r^{l-1}} \tilde{\mathbf{v}}_{h,i}^* \quad \text{on } \Omega_r^0. \tag{5.10}$$

By the definitions of $\tilde{\mathbf{v}}_{h,i}$ and $\tilde{\mathbf{v}}_{h,i}^*$, we know $\lambda_e(\mathbf{v}_{h,r}^*) = 0$ for all $e \subset \Gamma_r$. So by Lemma 4.6, one can find $p_{h,r}^* \in Z_h(\Omega_r^0)$ and $\mathbf{w}_{h,r}^* \in V_h(\Omega_r^0)$ such that

$$\mathbf{v}_{h,r}^* = \nabla p_{h,r}^* + \mathbf{w}_{h,r}^* \quad \text{on } \Omega_r^0, \tag{5.11}$$

and

$$p_{h,r}^* = 0 \ \text{ on } \Gamma_r \quad \text{and} \quad \lambda_e(\mathbf{w}_{h,r}^*) = 0 \ \text{ for all } e \subset \Gamma_r. \tag{5.12}$$

Using (5.10) and (5.11), we have the following decomposition for $\mathbf{v}_h$ on each $\Omega_r^0 \in \Sigma_l$:

$$\mathbf{v}_{h,r} = \nabla \left(p_{h,r}^* + \sum_{i \in \Lambda_r^1} \tilde{p}_{h,i} + \sum_{i \in \Lambda_r^{l-1}} \tilde{p}_{h,i}^* \right) + \mathbf{w}_{h,r}^* + \sum_{i \in \Lambda_r^1} \tilde{\mathbf{w}}_{h,i} + \sum_{i \in \Lambda_r^{l-1}} \tilde{\mathbf{w}}_{h,i}^* \quad \text{on } \Omega_r^0. \tag{5.13}$$

By (5.11) and the estimate for $\mathbf{w}_{h,r}^*$ in Lemma 4.6, one can verify for all $\Omega_r^0 \in \Sigma_l$ that (see Step 2)

$$\|\mathbf{curl}\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{i \in \Lambda_r^1} \|\mathbf{curl}\tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0} + \sum_{i \in \Lambda_r^{l-1}} \|\mathbf{curl}\tilde{\mathbf{w}}_{h,i}^*\|_{0,\Omega_r^0} \tag{5.14}$$

$$\|\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} \lesssim \log(1/h) \left(\|\mathbf{curl}\, \mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{i \in \Lambda_r^1} \|\mathbf{curl}\, \tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0} + \sum_{i \in \Lambda_r^{l-1}} \|\mathbf{curl}\, \tilde{\mathbf{w}}_{h,i}^*\|_{0,\Omega_r^0} \right). \tag{5.15}$$

As it was done in Steps 1 and 2, we can extend $p_{h,r}^*$ and $\mathbf{w}_{h,r}^*$ by zero onto the entire domain Ω to get $\tilde{p}_{h,r}^*$ and $\tilde{\mathbf{w}}_{h,r}^*$. Then we define

$$\tilde{\mathbf{v}}_{h,r}^* = \nabla \tilde{p}_{h,r}^* + \tilde{\mathbf{w}}_{h,r}^* \quad \text{for all } r \text{ such that } \Omega_r^0 \in \Sigma_l. \tag{5.16}$$

By the definition of $\tilde{\mathbf{v}}_{h,r}^*$ and the property (5.12), we know $\lambda_e(\tilde{\mathbf{v}}_{h,r}^*) = 0$ for all $e \in \Gamma_r$.

Continuing with the above procedure for all l's till $l = m$, we will have built up the decomposition of $\mathbf{v}_h$ over all the subdomains $\Omega_1^0, \Omega_2^0, \ldots, \Omega_{N_0}^0$ such that

$$\mathbf{v}_h = \sum_{r=1}^{n_1} \tilde{\mathbf{v}}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{\mathbf{v}}_{h,r}^* = \nabla p_h + \mathbf{w}_h \tag{5.17}$$

where $p_h \in Z_h(\Omega)$ and $\mathbf{w}_h \in V_h(\Omega)$ are given by

$$p_h = \sum_{r=1}^{n_1} \tilde{p}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{p}_{h,r}^* \quad \text{and} \quad \mathbf{w}_h = \sum_{r=1}^{n_1} \tilde{\mathbf{w}}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{\mathbf{w}}_{h,r}^*. \tag{5.18}$$

6 Proof of the key auxiliary result

This section is devoted to the proof of the key auxiliary result of this paper, Theorem 3.1. For this purpose a few important concepts about the relation between different subdomains are first introduced. It is reminded that all the subdomains $\Omega_1^0, \Omega_2^0, \ldots, \Omega_{N_0}^0$ below are the same as the ones described in Sect. 3.

Definition 6.1 A parent of subdomain Ω_r^0 is called a level-1 ancestor of Ω_r^0, and a parent of a level-1 ancestor of Ω_r^0 is called a level-2 ancestor of Ω_r^0. In general, a parent of a level-j ancestor of Ω_r^0 is called a level-$(j+1)$ ancestor of Ω_r^0.

Definition 6.2 A child of Ω_r^0 is called a level-1 offspring of Ω_r^0, and a child of a level-1 offspring of Ω_r^0 is called a level-2 offspring of Ω_r^0. In general, a child of a level-l offspring of Ω_r^0 is called a level-$(l+1)$ offspring of Ω_r^0.

By $\Lambda_r^{(j)}(a)$ we shall denote the set of all level-j ancestors of Ω_r^0, and $L_r(a)$ the number of all the levels of the ancestors of Ω_r^0. By $\Lambda_r^{(l)}(o)$ we shall denote the set of all l-level offsprings of Ω_r^0, and $L_r(o)$ the number of all the levels of the offsprings of Ω_r^0.

The following auxiliary estimate is needed in the proof of Theorem 3.1.

Lemma 6.1 *For any subdomain Ω_r^0 from $\Sigma_l (l \geq 2)$, let $\mathbf{w}_{h,r}^*$ be defined as in Steps 2 and 3 for the construction of the decomposition of any $\mathbf{v}_h \in V_h(\Omega)$ in Sect. 5. Then $\mathbf{w}_{h,r}^*$ admits the following estimate*

$$\|\mathbf{curl}\,\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r^0} + \sum_{j=1}^{L_r(a)} \log^j(1/h) \sum_{i\in\Lambda_r^{(j)}(a)} \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i^0}. \tag{6.1}$$

Proof We prove by induction, and start with the case of $l=2$. It follows from (5.6) that

$$\|\mathbf{curl}\,\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{l=1}^{2} \|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,r_l}\|_{0,\Omega_r^0}. \tag{6.2}$$

As $\tilde{\mathbf{w}}_{h,r_2}$ is the discrete **curl curl**-extension in Ω_r^0, we have (cf. Lemmata 4.5 and 6.10, [20])

$$\begin{aligned}\|\mathbf{curl}\tilde{\mathbf{w}}_{h,r_2}\|_{0,\Omega_r^0} &\lesssim \log^{1/2}(1/h)\|\tilde{\mathbf{w}}_{h,r_2}\times\mathbf{n}\|_{0,\mathrm{E}} \lesssim \log(1/h)\|\mathbf{curl}\mathbf{w}_{h,r_2}\|_{0,\Omega_{r_2}^0}\\ &= \log(1/h)\|\mathbf{curl}\mathbf{v}_h\|_{0,\Omega_{r_2}^0},\end{aligned} \tag{6.3}$$

where E denotes the common edge of Ω_r^0 and $\Omega_{r_2}^0$ or the union of these common edges. This, combining with (6.2) and the fact that $\tilde{\mathbf{w}}_{h,r_1}=0$ on Ω_r^0, yields

$$\begin{aligned}\|\mathbf{curl}\,\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} &\lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r^0} + \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_{r_2}^0}\\ &\lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r^0} + \log(1/h)\sum_{i\in\Lambda_r^{(1)}(a)}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i^0}.\end{aligned} \tag{6.4}$$

So (6.1) is verified for all the subdomains Ω_r^0 in Σ_2.

Now, assume that (6.1) is true for all subdomains $\Omega_r^0 \in \Sigma_l$ with $l \le n$. Then we need to verify (6.1) for all subdomains $\Omega_r^0 \in \Sigma_{n+1}$. It follows from (5.14) that

$$\|\mathbf{curl}\,\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r^0} + \sum_{i\in\Lambda_r^1} \|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0} + \sum_{i\in\Lambda_r^n} \|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}^*\|_{0,\Omega_r^0}. \tag{6.5}$$

Similarly as (6.3) was derived, one can check that for each $i \in \Lambda_r^n$,

$$\begin{aligned}
\|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}\|_{0,\Omega_r^0} &\lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{w}_{h,i}\|_{0,\Omega_i^0} = \log(1/h)\|\mathbf{curl}\,\mathbf{v}_{h,i}\|_{0,\Omega_i^0},\\
\|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,i}^*\|_{0,\Omega_r^0} &\lesssim \log^{1/2}(1/h)\|\tilde{\mathbf{w}}_{h,i}^* \times \mathbf{n}\|_{0,\mathrm{E}} = \log^{1/2}(1/h)\|\mathbf{w}_{h,i}^* \times \mathbf{n}\|_{0,\mathrm{E}}\\
&\lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{w}_{h,i}^*\|_{0,\Omega_i^0},
\end{aligned}$$

where E denotes the common edge of Ω_r^0 and Ω_i^0 or the union of these common edges. Combining these estimates with (6.5) gives

$$\begin{aligned}
\|\mathbf{curl}\,\mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r^0} &+ \log(1/h)\sum_{i\in\Lambda_r^1} \|\mathbf{curl}\,\mathbf{v}_{h,i}\|_{0,\Omega_i^0}\\
&+ \log(1/h)\sum_{i\in\Lambda_r^n} \|\mathbf{curl}\,\mathbf{w}_{h,i}^*\|_{0,\Omega_i^0}.
\end{aligned} \tag{6.6}$$

Noting that for $i \in \Lambda_r^n$, we have $\Omega_i^0 \in \Sigma_l$ for some $l \le n$. Thus by the inductive assumption,

$$\begin{aligned}
\sum_{i\in\Lambda_r^n} \|\mathbf{curl}\,\mathbf{w}_{h,i}^*\|_{0,\Omega_i^0} \lesssim \sum_{i\in\Lambda_r^n} &\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i^0}\\
&+ \sum_{i\in\Lambda_r^n}\sum_{j=1}^{L_i(a)} \log^j(1/h) \sum_{k\in\Lambda_i^{(j)}(a)} \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_k^0}.
\end{aligned} \tag{6.7}$$

But for all subdomains $\Omega_r^0 \in \Sigma_{n+1}$ and $i \in \Lambda_r^n$, we know $L_i(a) \le L_r(a)$ and $\Lambda_i^{(j)}(a) = \emptyset$ for $j > L_i(a)$ by definition, so we have the relation

$$\sum_{i\in\Lambda_r^n}\sum_{j=1}^{L_i(a)}\sum_{k\in\Lambda_i^{(j)}(a)} = \sum_{j=1}^{L_r(a)}\sum_{i\in\Lambda_r^n}\sum_{k\in\Lambda_i^{(j)}(a)} = \sum_{j=1}^{L_r(a)}\sum_{k\in\Lambda_r^{(j+1)}(a)}. \tag{6.8}$$

Combining this with the fact that $\Lambda_r^{(j+1)}(a) = \emptyset$ for $j \ge L_r(a)$, we get

$$\sum_{i\in\Lambda_r^n}\sum_{j=1}^{L_i(a)}\sum_{k\in\Lambda_i^{(j)}(a)} = \sum_{j=1}^{L_r(a)-1}\sum_{k\in\Lambda_r^{(j+1)}(a)}.$$

From this identity and (6.7) it follows that

$$\sum_{i\in\Lambda_r^n} \|\mathbf{curl}\, \mathbf{w}_{h,i}^*\|_{0,\Omega_i^0} \lesssim \sum_{i\in\Lambda_r^n} \|\mathbf{curl}\, \mathbf{v}_h\|_{0,\Omega_i^0} + \sum_{j=2}^{L_r(a)} \log^j(1/h) \sum_{k\in\Lambda_r^{(j)}(a)} \|\mathbf{curl}\, \mathbf{v}_h\|_{0,\Omega_k^0}. \tag{6.9}$$

Substituting this into (6.6), and using the identity

$$\sum_{i\in\Lambda_r^1} + \sum_{i\in\Lambda_r^n} = \sum_{i\in\Lambda_r^{(1)}(a)} \quad \text{for } \Omega_r^0 \in \Sigma_{n+1},$$

we can immediately derive that

$$\|\mathbf{curl}\, \mathbf{w}_{h,r}^*\|_{0,\Omega_r^0} \lesssim \|\mathbf{curl}\, v_h\|_{0,\Omega_r^0} + \sum_{j=1}^{L_r(a)} \log^j(1/h) \sum_{k\in\Lambda_r^{(j)}(a)} \|\mathbf{curl}\, \mathbf{v}_h\|_{0,\Omega_k^0}. \tag{6.10}$$

This proves (6.1) for all subdomains $\Omega_r^0 \in \Sigma_{n+1}$, thus completes the proof of Lemma 6.1 by the mathematical induction. □

Proof of Theorem 3.1 We are now ready to show Theorem 3.1. Let $\mathbf{v}_h \in V_h(\Omega)$ satisfy the orthogonality (3.3), then we can have the decomposition (5.17) for $\mathbf{v}_h$.

By means of (5.17) and the orthogonality (3.3), we first see

$$\begin{aligned}(\beta\mathbf{v}_h, \mathbf{v}_h) &= (\beta\nabla p_h, \nabla p_h) + (\beta\mathbf{w}_h, \mathbf{w}_h) + 2(\beta\nabla p_h, \mathbf{w}_h)\\ &= (\beta\nabla p_h, \nabla p_h) + (\beta\mathbf{w}_h, \mathbf{w}_h) + 2(\beta\nabla p_h, \mathbf{v}_h - \nabla p_h)\\ &= (\beta\mathbf{w}_h, \mathbf{w}_h) - (\beta\nabla p_h, \nabla p_h) \le (\beta\mathbf{w}_h, \mathbf{w}_h),\end{aligned}$$

which implies

$$(\beta\mathbf{v}_h, \mathbf{v}_h) \le \sum_{r=1}^{N_0} \|\beta^{\frac{1}{2}}\mathbf{w}_h\|^2_{0,\Omega_r^0}. \tag{6.11}$$

So it remains to estimate $\|\beta^{\frac{1}{2}}\mathbf{w}_h\|^2_{0,\Omega_r^0}$ for each subdomain Ω_r^0.

We start with the estimate of $\|\beta^{\frac{1}{2}}\mathbf{w}_h\|^2_{0,\Omega_r^0}$ for each subdomain Ω_r^0 in Σ_1, i.e., $1 \le r \le n_1$.

By the definition of $\tilde{\mathbf{w}}_{h,i}^*$ in (5.18), we have $\lambda_e(\tilde{\mathbf{w}}_{h,i}^*) = 0$ for $e \in \partial\Omega_r^0$. Moreover, any two of the subdomains $\Omega_1^0, \ldots, \Omega_{n_1}^0$ do not intersect, so we have

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|^2_{0,\Omega_r^0} = \|\beta^{\frac{1}{2}}\tilde{\mathbf{w}}_{h,r}\|^2_{0,\Omega_r^0} = \beta_r\|\mathbf{w}_{h,r}\|^2_{0,\Omega_r^0}.$$

This, along with (5.2), yields the following estimate for $r = 1, \ldots, n_1$,

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|^2_{0,\Omega^0_r} \lesssim \beta_r \|\mathbf{curl}\,\mathbf{v}_{h,r}\|^2_{0,\Omega^0_r} = \|\beta_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega^0_r}. \tag{6.12}$$

Next, we consider all the subdomains Ω^0_r in Σ_2. As in Step 2 of the construction of the stable decomposition for $\mathbf{v}_h$, we assume that Ω^0_r satisfies Condition A and has just two parent subdomains in Σ_1, $\Omega^0_{r_1}$ and $\Omega^0_{r_2}$, which satisfy that $\bar{\Omega}^0_r \cap \bar{\Omega}^0_{r_1} = \mathrm{V}$ (a vertex) and $\bar{\Omega}^0_r \cap \bar{\Omega}^0_{r_2} = \mathrm{E}$ (an edge). Then we have

$$\mathbf{w}_h|_{\Omega^0_r} = \mathbf{w}^*_{h,r} + \tilde{\mathbf{w}}_{h,r_2}|_{\Omega^0_r}.$$

By the triangle inequality,

$$\|\mathbf{w}_h\|_{0,\Omega^0_r} \lesssim \|\mathbf{w}^*_{h,r}\|_{0,\Omega^0_r} + \|\tilde{\mathbf{w}}_{h,r_2}\|_{0,\Omega^0_r}. \tag{6.13}$$

Noting that $\tilde{\mathbf{w}}_{h,r_2}$ is the discrete **curl curl**-extension in Ω^0_r, we can deduce by using Lemmata 4.5 and 6.10 in [20] that

$$\|\tilde{\mathbf{w}}_{h,r_2}\|_{0,\Omega^0_r} \lesssim \log^{\frac{1}{2}}(1/h)\|\tilde{\mathbf{w}}_{h,r_2} \times \mathbf{n}\|_{0,\mathrm{E}} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega^0_{r_2}}.$$

Using this estimate, (5.7) and (6.4) we derive from (6.13) that

$$\|\mathbf{w}_h\|_{0,\Omega^0_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega^0_r} + \log^2(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega^0_{r_2}}. \tag{6.14}$$

Then by inserting the coefficient β, we readily have for all subdomains $\Omega^0_r \in \Sigma_2$ that

$$\begin{aligned}\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega^0_r} &\lesssim \log(1/h)\|\beta_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega^0_r} + \log^2(1/h)\|\beta_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega^0_{r_2}}\\ &\lesssim \log(1/h)\|\beta_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega^0_r} + \log^2(1/h)\sqrt{\frac{\beta_r}{\beta_{r_2}}}\|\beta_{r_2}^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega^0_{r_2}}.\end{aligned} \tag{6.15}$$

Finally we consider all the subdomains Ω^0_r from the general class Σ_l with $l \geq 3$. By the definition of $\mathbf{w}_h$, we can establish the same decomposition for $\mathbf{w}_h|_{\Omega^0_r}$ as we did for $\mathbf{v}_{h,r} = \mathbf{v}_h|_{\Omega^0_r}$ in Sect. 5; see (5.10), which leads to the following decomposition in Ω^0_r for $\mathbf{w}_h$:

$$\mathbf{w}_h = \mathbf{w}^*_{h,r} + \sum_{i\in\Lambda^1_r}\tilde{\mathbf{w}}_{h,i} + \sum_{i\in\Lambda^{l-1}_r}\tilde{\mathbf{w}}^*_{h,i}.$$

In an analogous way as deriving (6.14), one can verify by using (5.15) that

$$\|\mathbf{w}_h\|_{0,\Omega_r^0} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0} + \log^2(1/h)\sum_{i\in\Lambda_r^1}\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}$$
$$+ \log^2(1/h)\sum_{i\in\Lambda_r^{l-1}}\|\mathbf{curl}\ \mathbf{w}_{h,i}^*\|_{0,\Omega_i^0}. \quad (6.16)$$

But it follows from Lemma 6.1 that

$$\|\mathbf{curl}\ \mathbf{w}_{h,i}^*\|_{0,\Omega_i^0} \lesssim \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0} + \sum_{j=1}^{L_i(a)}\log^j(1/h)\sum_{k\in\Lambda_r^{(j)}(a)}\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_k^0}.$$

Then we further deduce from (6.16) that

$$\|\mathbf{w}_h\|_{0,\Omega_r^0} \lesssim \log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0} + \sum_{j=1}^{L_r(a)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}.$$

Inserting the coefficient β gives

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega_r^0} \lesssim \log(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}$$
$$+ \sum_{j=1}^{L_r(a)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}\sqrt{\frac{\beta_r}{\beta_i}}\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}.$$

Summing up this estimate with the ones in (6.12) and (6.15), we come to

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}^2 \lesssim \log(1/h)\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega}^2$$
$$+ \sum_{r=n_1+1}^{N_0}\sum_{j=1}^{L_r(a)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}\frac{\beta_r}{\beta_i}\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}^2. \quad (6.17)$$

By the definitions of $L_r(a)$, $\Lambda_r^{(j)}(a)$ and $\Lambda_r^{(j)}(o)$, we can verify that

$$\sum_{r=n_1+1}^{N_0}\sum_{j=1}^{L_r(a)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(a)}\frac{\beta_r}{\beta_i}\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i^0}^2$$
$$= \sum_{r=1}^{N_0}\left(\beta_r^{-1}\sum_{j=1}^{L_r(o)}\log^{j+1}(1/h)\sum_{i\in\Lambda_r^{(j)}(o)}\beta_i\right)\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}^2$$
$$\leq \log^{m+1}(1/h)\sum_{r=1}^{N_0}C_r\|\beta^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r^0}^2, \quad (6.18)$$

where $m = \max_{1\le r\le N_0} L_r(o)$ and C_r is a constant given by

$$C_r = \beta_r^{-1}\sum_{j=1}^{L_r(o)}\sum_{i\in\Lambda_r^{(j)}(o)}\beta_i .$$

Noting the facts that $\beta_i < \beta_r$ for all $i \in \Lambda_r^{(j)}(o)$, $L_r(o)$ is a finite number and the set $\Lambda_r^{(j)}(o)$ contains only a few elements, the constant C_r must be uniformly bounded for all r's. Applying (6.18) to (6.17), we obtain

$$\|\beta^{\frac12}\mathbf{w}_h\|_{0,\Omega}^2 \lesssim C\log^{m+1}(1/h)\|\beta^{\frac12}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}^2$$

where C is a constant given by $C = \max_{1\le r\le N_0} C_r$. This completes the proof of Theorem 3.1. □

7 Application

In this section we shall apply the discrete weighted Helmholtz decomposition (cf. Theorem 3.2) developed in Sect. 3 to analyse how the condition number of the preconditioned edge element system by the non-overlapping domain decomposition preconditioner proposed in [21] depends on the jumps of the coefficients in (1.1) and (1.2) across the interfaces between any two subdomains of different media. We will adopt the same notations below as the ones in Sect. 2.1.

Associated with the Maxwell system (1.1)–(1.2), we may consider (cf. [21]) the following variational saddle-point formulation: Find $(\mathbf{u}, p) \in H_0(\mathbf{curl};\Omega)\times H_0^1(\Omega)$ such that

$$\begin{cases}(\alpha\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + \gamma_0(\beta\mathbf{u}, \mathbf{v}) + (\nabla p, \beta\mathbf{v}) = (\mathbf{f}, \mathbf{v}), & \forall\,\mathbf{v}\in H_0(\mathbf{curl};\Omega)\\ (\beta\mathbf{u}, \nabla q) = (g, q), \quad \forall\, q\in H_0^1(\Omega)\end{cases} \tag{7.1}$$

and its edge element approximation: Find $(\mathbf{u}_h, p_h) \in V_h(\Omega)\times Z_h(\Omega)$ such that

$$\begin{cases}(\alpha\mathbf{curl}\,\mathbf{u}_h, \mathbf{curl}\,\mathbf{v}_h) + \gamma_0(\beta\mathbf{u}_h, \mathbf{v}_h) + (\nabla p_h, \beta\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), & \forall\,\mathbf{v}_h\in V_h(\Omega)\\ (\beta\mathbf{u}_h, \nabla q_h) = (g, q_h), \quad \forall\, q_h\in Z_h(\Omega).\end{cases} \tag{7.2}$$

It is well known (cf. [11,15,25]) that the system (7.2) can be simplified to a symmetric and positive definite one, namely we can set the Lagrange multiplier $p_h = 0$ and remove the second equation, when $\gamma_0 = 1$ or the zeroth order term is present in the Maxwell equation (1.1). The most challenging case in the numerical solution of system (7.2) is the real saddle-point case when $\gamma_0 = 0$, where we have to keep p_h and the second equation there. Still no efficient iterative methods have been proposed in the literature for this saddle-point system by using non-overlapping domain decomposition preconditioners, except the one developed in [21], in combination with a preconditioned iterative Uzawa method. The preconditioned system using the substructuring preconditioner from [21] for the whole saddle-point system (7.2) was shown to be nearly optimal in the sense that its condition number grows only as the logarithm of the ratio between the subdomain diameter and the finite element mesh size, but no conclusion was achieved in [21] about how the condition number of the global preconditioned system depends on the jumps of the coefficients α and β in (7.2). We are able to show in the rest of this section, with the help of the novel stable discrete weighted Helmholtz decomposition developed in Sect. 3, that this condition number is indeed also independent of the jumps of the coefficients.

7.1 Augmented saddle-point system and Uzawa methods

In this and next subsections, we shall recall the non-overlapping domain decomposition preconditioner developed in [21] for the saddle-point system (7.2). We first write the system into an equivalent operator form by introducing the operators $\bar{A} : V_h(\Omega) \to V_h(\Omega)$ and $B : Z_h(\Omega) \to V_h(\Omega)$ by

$$(\bar{A}\mathbf{u}_h, \mathbf{v}_h) = (\alpha\, \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v}_h)\,, \quad (Bp_h, \mathbf{v}_h) = (\nabla p_h, \beta \mathbf{v}_h)$$

for all $\mathbf{u}_h, \mathbf{v}_h \in V_h(\Omega)$ and $p_h \in Z_h(\Omega)$, and the dual operator $B^t : V_h(\Omega) \to Z_h(\Omega)$ of B by

$$(B^t\mathbf{u}_h, q_h) = (\beta\, \mathbf{u}_h, \nabla q_h), \quad \forall q_h \in Z_h(\Omega). \tag{7.3}$$

Let $\bar{\mathbf{f}}_h \in V_h(\Omega)$, $g_h \in Z_h(\Omega)$ be the L^2-projections of $\mathbf{f}$ and g. Then we can rewrite the system (7.2) into

$$(\bar{A} + \gamma_0\, \beta\, I)\mathbf{u}_h + Bp_h = \bar{\mathbf{f}}_h\,, \quad B^t\mathbf{u}_h = g_h. \tag{7.4}$$

Noting that $\bar{A}$ is singular, we can transform the system (7.4) into the following equivalent augmented saddle-point problem:

$$A\mathbf{u}_h + Bp_h = \mathbf{f}_h\,, \qquad B^t\mathbf{u}_h = g_h. \tag{7.5}$$

where A and $\mathbf{f}_h$ are given by [21]

$$A = \bar{A} + \gamma_0\, \beta\, I + B\hat{C}^{-1}B^t \quad \text{and} \quad \mathbf{f}_h = \bar{\mathbf{f}}_h + B\hat{C}^{-1}g_h \tag{7.6}$$

and $\hat{C} : Z_h(\Omega) \to Z_h(\Omega)$ will be chosen to be a symmetric and positive definite preconditioner for the discrete Laplace operator on $Z_h(\Omega)$. Let $\hat{A}$ be a preconditioner for operator A. Then the system (7.5) can be solved by many existing preconditioned iterative methods, e.g., the nonlinear preconditioned Uzawa-type algorithm developed in [19]. As shown in [19], the efficiency of the Uzawa-type algorithm is completely determined by the condition numbers $\kappa(\hat{A}^{-1}A)$ and $\kappa(\hat{C}^{-1}B^tA^{-1}B)$. Two efficient preconditioners $\hat{A}$ and $\hat{C}$ were developed in [21], based on a domain decomposition method. And it was shown that $\kappa(\hat{A}^{-1}A)$ and $\kappa(\hat{C}^{-1}B^tA^{-1}B)$ are nearly optimal, i.e.

nearly independent of the subdomain size d and fine mesh size h, but it is unclear how they depend on the jumps of the coefficients in (1.1)–(1.2). The remaining part of this work will clarify this important issue. We will propose an improved variant of $\hat{A}$ introduced in [21], and demonstrate rigorously that the condition numbers of the preconditioned systems is independent of the jumps in the coefficients.

7.2 Construction of a preconditioner for A

In this section we present a substructuring preconditioner for A that improves the one proposed in [21]. First we recall the decomposition of the global domain Ω in Sect. 2.1 into a set of medium subdomains $\Omega_1^0, \Omega_2^0, \ldots, \Omega_{N_0}^0$, based on the distribution of the coefficient $\beta(\mathbf{x})$. Then we further decompose each medium subdomain Ω_r^0 $(1 \le r \le N_0)$ into a set of smaller polyhedral subdomains of size d (see [4,32]), thus leading to a domain decomposition of the global domain Ω: $\Omega_1, \Omega_1, \ldots, \Omega_N$. We assume that each Ω_k $(1 \le k \le N)$ is formed by a set of fine elements of the triangulation $\mathcal{T}_h$ over Ω.

We will write the common face of two subdomains Ω_i and Ω_j by Γ_{ij}, and set

$$\Gamma = \cup_{ij}\Gamma_{ij}\,, \quad \Gamma_i = \Gamma \cap \partial\Omega_i\,, \quad \Omega_{ij} = \Omega_i \cup \Omega_j \cup \Gamma_{ij}.$$

Γ is called *the interface* associated with the domain decomposition $\Omega_1, \Omega_1, \ldots, \Omega_N$. For the definiteness, a unique unit normal direction $\mathbf{n}$ is assigned to each face F of Γ. On each subdomain Ω_k $(k = 1, \ldots, N)$ let $V_h(\Omega_k)$ be the restriction of $V_h(\Omega)$ on Ω_k. Then we define operator $A_k : V_h(\Omega_k) \to V_h(\Omega_k)$ by

$$(A_k\mathbf{v}, \mathbf{w})_{\Omega_k} = (\alpha\, \mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v})_{\Omega_k} + (\beta\, \mathbf{u}, \mathbf{v})_{\Omega_k} \quad \forall\, \mathbf{u}, \mathbf{v} \in V_h(\Omega_k)\,,$$

and a local subspace

$$V^k(\Omega) = \{\mathbf{v} \in V_h(\Omega);\;\; \lambda_e(\mathbf{v}) = 0 \;\text{ for each }\; e \in \Omega\backslash\Omega_k\}.$$

We now introduce a subspace which is defined globally in Ω but is discrete A_k-harmonic in each subdomain:

$$V^H(\Omega) = \Big\{\mathbf{v} \in V_h(\Omega);\;\, \mathbf{v} \text{ is the discrete } A_k\text{-extension of } (\mathbf{v} \times \mathbf{n})|_{\partial\Omega_k} \text{ in each } \Omega_k\Big\}.$$

Let $\tilde{A} : V_h(\Omega) \to V_h(\Omega)$ be the self-adjoint operator defined by

$$(\tilde{A}\mathbf{u}, \mathbf{v}) = (\alpha\, \mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v}) + (\beta\, \mathbf{u}, \mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in V_h(\Omega),$$

then one can easily see that $V_h(\Omega)$ has the following orthogonal decomposition with respect to the inner product $(\tilde{A}\cdot, \cdot)$:

$$V_h(\Omega) = \sum_{k=1}^{N} V^k(\Omega) \oplus V^H(\Omega). \tag{7.7}$$

For any face F of Ω_i, we use F_b to denote the union of all $\mathcal{T}_h$-induced (closed) triangles on F, which have either one single vertex or one edge lying on $\partial\mathrm{F}$, and F_∂ to denote the open set $\mathrm{F}\backslash\mathrm{F}_b$. Furthermore, we define two subspaces of $V^H(\Omega)$:

$$V^{ij}(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \lambda_e(\mathbf{v}) = 0 \ \text{ for each } \ e \in \Omega\backslash\Omega_{ij}\Big\},$$
$$V^0(\Omega) = \Big\{\mathbf{v} \in V^H(\Omega);\ \lambda_e(\mathbf{v}) = 0 \text{ for each } \ e \in \mathrm{F}_\partial \ \text{ with } \mathrm{F} \subset \Gamma\Big\}.$$

The space $V^0(\Omega)$ is called the *coarse* subspace.

It is easy to see that the space $V_h(\Omega)$ has the following decomposition (not a direct sum):

$$V_h(\Omega) = \sum_{k=1}^{N} V^k(\Omega) \oplus (V^0(\Omega) + \sum_{\Gamma_{ij}} V^{ij}(\Omega)). \tag{7.8}$$

Next, we introduce a substructuring preconditioner for operator A, that improves the one proposed in [21]. Corresponding to the decomposition (7.8), we will define the local solvers and global coarse solver respectively on the subspaces $V^k(\Omega)$, $V^{ij}(\Omega)$ and $V^0(\Omega)$.

On each subdomain Ω_k and each face Γ_{ij}, the local solver $\hat{A}_k : V^k(\Omega) \to V^k(\Omega)$ and $\hat{A}_{ij} : V^{ij}(\Omega) \to V^{ij}(\Omega)$ can be naturally defined such that

$$(\hat{A}_k\mathbf{v}, \mathbf{v}) \eqsim (A_k\mathbf{v}, \mathbf{v})_{\Omega_k} \ \ \forall \mathbf{v} \in V^k(\Omega)\,; \quad (\hat{A}_{ij}\mathbf{v}, \mathbf{v}) \eqsim (A_i\mathbf{v}, \mathbf{v})_{\Omega_i}$$
$$+(A_j\mathbf{v}, \mathbf{v})_{\Omega_j} \ \ \forall \mathbf{v} \in V^{ij}(\Omega).$$

But the definition of an efficient global coarse solver on $V^0(\Omega)$ is much more tricky. For the sake of exposition, we assume that the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ in (1.2) are piecewise constant with respect to the medium domain decomposition $\{\Omega_r^0\}_{r=1}^{N_0}$, namely $\alpha(x) = \alpha_r$ and $\beta(x) = \beta_r$ for $x \in \Omega_r^0$. Then we set $\alpha_k^* = \alpha|_{\Omega_k}$ and $\beta_k^* = \beta|_{\Omega_k}$. Clearly we have $\alpha_k^* = \alpha_l$ and $\beta_k^* = \beta_l$ for $\Omega_k \subset \Omega_l^0$. For any subdomain Ω_k ($k = 1, \ldots, N$), we introduce an important set on its boundary:

$$\Delta_k = \bigcup_{\mathrm{F}\subset\Gamma_k} \mathrm{F}_b.$$

Then we define the global coarse solver $\hat{A}_0$ on $V^0(\Omega)$ as follows: for any $\mathbf{v}, \mathbf{w} \in V^0(\Omega)$,

$$(\hat{A}_0\mathbf{v}, \mathbf{w}) = h[1 + \log(d/h)] \sum_{k=1}^{N} \Big\{\alpha_k^* \langle \mathrm{div}_\tau(\mathbf{v}\times\mathbf{n})|_{\Gamma_k}, \mathrm{div}_\tau(\mathbf{w}\times\mathbf{n})|_{\Gamma_k}\rangle_{\Delta_k}$$
$$+\beta_k^* \langle \mathbf{v}\times\mathbf{n}, \mathbf{w}\times\mathbf{n}\rangle_{\Delta_k}\Big\}.$$

This is an improved coarse solver compared to the one proposed in [21], where β_k^* is taken to be the same as α_k^*. We note that the entries of the stiffness matrix associated with $\hat{A}_0$ can be easily computed [21].

With the above preparations, we are ready to define our new preconditioner $\hat{A}$ for A:

$$\hat{A}^{-1} = \sum_{k=1}^{N} \hat{A}_k^{-1} Q_k + \hat{A}_0^{-1} Q_0 + \sum_{\Gamma_{ij}} \hat{A}_{ij}^{-1} Q_{ij} \tag{7.9}$$

where Q_k, Q_0 and Q_{ij} are the L^2-projections from $V_h(\Omega)$ onto $V^k(\Omega)$, $V^0(\Omega)$ and $V^{ij}(\Omega)$ respectively.

In the subsequent analysis, we assume the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ satisfy that

$$1 \lesssim \frac{\beta_i^*}{\alpha_i^*} / \frac{\beta_j^*}{\alpha_j^*} \lesssim 1 \quad \text{for each face} \quad \Gamma_{ij} \quad \text{and} \quad \beta_k^* \lesssim \alpha_k^* \quad \text{for each } \Omega_k\,, \tag{7.10}$$

and for convenience, we introduce an operator $J : Z_h(\Omega) \to Z_h(\Omega)$ by

$$(J\phi_h, \psi_h) = (\beta\nabla\phi_h, \nabla\psi_h), \quad \forall \phi, \ \psi \in Z_h(\Omega). \tag{7.11}$$

We will show the following estimate for the preconditioner $\hat{A}$ defined in (7.9).

Theorem 7.1 *Let $G(\cdot) \geq 1$ be some given function, and the operator $\hat{C}$ satisfy*

$$(J\phi, \phi) \lesssim (\hat{C}\phi, \phi) \lesssim G(d/h)(J\phi, \phi), \quad \forall \phi \in Z_h(\Omega),$$

or equivalently,

$$(\hat{C}^{-1}\phi, \phi) \lesssim (J^{-1}\phi, \phi) \lesssim G(d/h)(\hat{C}^{-1}\phi, \phi), \quad \forall \phi \in Z_h(\Omega)\,, \tag{7.12}$$

then we have the following estimate (with the integer m from Theorem 3.2)

$$\operatorname{cond}(\hat{A}^{-1}A) \lesssim [G(d/h) + \log^{m+1}(1/h)][1 + \log(d/h)]^2. \tag{7.13}$$

A similar preconditioner to $\hat{A}$ in (7.9) was constructed in [21] and proved to be nearly optimal: the condition number of the resulting preconditioned system grows only as the logarithm of the dimension of the local subproblem associated with an individual subdomain, but possibly depends on the jumps of the coefficients $\alpha(x)$ and $\beta(x)$ in (1.1)–(1.2). In the next subsection, we can show that the estimate (7.13) is independent of possible large jumps in the coefficients $\alpha(x)$ and $\beta(x)$.

Now we introduce a preconditioner for the Schur complement $B^t A^{-1} B$ associated with the saddle-point system (7.5). Assume that $\hat{C}$ is a preconditioner for the discrete Laplacian and satisfies the condition (7.12), then we have

Theorem 7.2 *The condition number of the preconditioned Schur complement system can be estimated by*

$$\mathrm{cond}(\hat{C}^{-1}B^tA^{-1}B) \lesssim G(d/h)[G(d/h)+\log^{m+1}(1/h)]. \tag{7.14}$$

Proof Using (7.12) we can show [21] that for any $q \in Z_h(\Omega)$,

$$(B^tA^{-1}Bq,q) \eqsim \sup_{\mathbf{v}_h\in V_h(\Omega)} \frac{(\beta\mathbf{v}_h,\nabla q)^2}{(A\mathbf{v}_h,\mathbf{v}_h)}, \tag{7.15}$$

$$(B^tA^{-1}Bq,q) \geq \frac{(\beta\nabla q,\nabla q)^2}{(B\hat{C}^{-1}B^t(\nabla q),\nabla q)} \gtrsim (\beta\nabla q,\nabla q). \tag{7.16}$$

On the other hand, by means of Theorem 3.2 and (7.12) we can verify that (see (7.29) and (7.31) in Sect. 7.3)

$$\begin{aligned}(\beta\mathbf{v}_h,\mathbf{v}_h) &\lesssim (\beta\nabla p_h,p_h)+(\beta\mathbf{w}_h,\mathbf{w}_h) \lesssim [G(d/h)\\ &+\log^{(m+1)}(1/h)](A\mathbf{v}_h,\mathbf{v}_h), \quad \forall\mathbf{v}_h\in V_h(\Omega).\end{aligned}$$

Now it follows from (7.15), the above estimate and the Cauchy–Schwarz inequality that

$$\begin{aligned}(B^tA^{-1}Bq,q) &\lesssim [G(d/h)+\log^{m+1}(1/h)] \sup_{\mathbf{v}_h\in V_h(\Omega)} \frac{\|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}^2\,\|\beta^{\frac{1}{2}}\nabla q\|_{0,\Omega}^2}{(\beta\mathbf{v}_h,\mathbf{v}_h)}\\ &\lesssim [G(d/h)+\log^{m+1}(1/h)](\beta\nabla q,\nabla q), \quad \forall q\in Z_h(\Omega).\end{aligned}$$

The desired estimate is now a consequence of this estimate, (7.16) and (7.12). □

Remark 7.1 When $\hat{C}$ is chosen as the usual multigrid preconditioner and the substructuring preconditioner (cf. [4,32]) for the Laplacian operator, the function $G(d/h)$ in (7.12) can be taken to be 1 and $[1+\log(d/h)]^2$ respectively. For these two standard choices of $\hat{C}$, the estimates (7.13) and (7.14) can be simplified respectively as (note that $m \geq 1$)

$$\mathrm{cond}(\hat{A}^{-1}A) \lesssim \log^{m+1}(1/h)[1+\log(d/h)]^2$$

and

$$\mathrm{cond}(\hat{C}^{-1}B^tA^{-1}B) \lesssim G(d/h)\log^{m+1}(1/h).$$

7.3 Proof of Theorem 7.1

We devote this section to the proof of our main theorem of this paper, Theorem 7.1. We first present some important auxiliary results. On each subdomain Ω_k, we define a norm on its boundary Γ_k:

$$\|\Phi\|_{\mathcal{X}_{\Gamma_k}} = \left(\alpha_k^* \|\mathrm{div}_\tau \Phi\|^2_{-\frac{1}{2},\, \Gamma_k} + \beta_k^* \|\Phi\|^2_{-\frac{1}{2},\, \Gamma_k}\right)^{\frac{1}{2}} \quad \forall\, \Phi \in V_h(\Gamma_k).$$

The proof of the next lemma 7.1 follows the one of lemma 4.9 in [21] by means of the assumption (7.10) and the inverse estimates for $\|\mathrm{div}_\tau \Phi\|^2_{0,\Gamma_k}$ and $\|\Phi\|^2_{0,\Gamma_k}$, while the proof of lemma 7.2 is a consequence of the Green formulae and the assumption (7.10) (e.g., see [1]).

Lemma 7.1 *For any $\Phi \in V_h(\Gamma_k)$, there exists an extension $\mathbf{R}_k\Phi \in V_h(\Omega_k)$, such that*

$$\alpha_k^* \|\mathbf{curl}(\mathbf{R}_k\Phi)\|^2_{0,\Omega_k} + \beta_k^* \|\mathbf{R}_k\Phi\|^2_{0,\Omega_k} \lesssim \|\Phi\|^2_{\mathcal{X}_{\Gamma_k}}. \tag{7.17}$$

Lemma 7.2 *For any $\mathbf{v} \in V_h(\Omega_k)$, we have*

$$\|\mathbf{v} \times \mathbf{n}\|^2_{\mathcal{X}_{\Gamma_k}} \lesssim \alpha_k^* \|\mathbf{curl}\, \mathbf{v}\|^2_{0,\, \Omega_k} + \beta_k^* \|\mathbf{v}\|^2_{0,\, \Omega_k}. \tag{7.18}$$

The following lemma can be proved in nearly the same manner as it was done in [21] (see pp. 52–56), with the help of condition (7.10) and Lemmas 7.1, 7.2.

Lemma 7.3 *For any $\mathbf{w}_h \in V_h(\Omega)$, we can write*

$$\mathbf{w}_h = \mathbf{w}_h^0 + \sum_{k=1}^{N} \mathbf{w}_h^k + \sum_{\Gamma_{ij}} \mathbf{w}_h^{ij} \tag{7.19}$$

for some $\mathbf{w}_h^0 \in V^0(\Omega)$, $\mathbf{w}_h^k \in V^k(\Omega)$ and $\mathbf{v}_h^{ij} \in V^{ij}(\Omega)$ such that

$$(\hat{A}_0\mathbf{w}_h^0, \mathbf{w}_h^0) + \sum_{k=1}^{N} (\hat{A}_k\mathbf{w}_h^k, \mathbf{w}_h^k) + \sum_{\Gamma_{ij}} (\hat{A}_{ij}\mathbf{w}_h^{ij}, \mathbf{w}_h^{ij}) \lesssim [1 + \log(d/h)]^2 (\tilde{A}\mathbf{w}_h, \mathbf{w}_h). \tag{7.20}$$

Throughout this section we will use F to denote a face of some subdomain Ω_k, and $\mathcal{W}$ the set of the (*coarse*) edges of all subdomains Ω_k. For any given subset G of Γ and a function $\varphi \in L^2(G)$, we use $\gamma_G(\varphi)$ to denote the average value of φ on G. Then for any function $\varphi \in Z_h(\Gamma)$, we define $\pi^0\varphi \in Z_h(\Gamma)$ as follows:

$$\pi^0\varphi(\mathbf{x}) = \begin{cases} \varphi(\mathbf{x}), & \text{for} \quad \mathbf{x} \in \mathcal{W} \cap \mathcal{N}_h, \\ \gamma_{\mathrm{F}}(\varphi), & \text{for} \quad \mathbf{x} \in \mathrm{F} \cap \mathcal{N}_h \ (\mathrm{F} \subset \Gamma). \end{cases} \tag{7.21}$$

For any $p_h \in Z_h(\Omega)$, define $p_h^0 \in Z_h(\Omega)$ such that $p_h^0 = \pi^0(p_h|_\Gamma)$ on Γ and is discrete harmonic in each subdomain Ω_k. One can check that $\nabla p_h^0 \in V^0(\Omega)$, and

p_h^0 meets the following estimate which can be proved using the definition of $\hat{A}_0$ (see (5.17)–(5.18), [21]):

$$(\hat{A}_0(\nabla p_h^0), \nabla p_h^0) \lesssim [1+\log(d/h)]^2 \sum_{k=1}^{N} \beta_k^* |p_h|_{1,\Omega_k}^2. \tag{7.22}$$

Furthermore, for each Γ_{ij} we define $p_h^{ij} \in Z_h(\Omega)$ such that $p_h^{ij} = p_h - p_h^0$ on Γ_{ij}, $p_h^{ij} = 0$ on $\Gamma \backslash \Gamma_{ij}$, and p_h^{ij} is discrete harmonic in each Ω_j $(1 \le j \le N)$. Also, for each subdomain Ω_k we define $p_h^k \in Z_h(\Omega)$: $p_h^k = (p_h - p_h^0 - \sum_{ij} p_h^{ij})|_{\Omega_k}$. Clearly $\nabla p_h^{ij} \in V^{ij}(\Omega)$ and $\nabla p_h^k \in V^k(\Omega)$. And by the standard arguments (see, e.g., [32]) we can show that p_h^{ij} and p_h^k satisfy

$$\sum_{\Gamma_{ij}} (\beta \nabla p_h^{ij}, \nabla p_h^{ij}) + \sum_{k=1}^{N} (\beta \nabla p_h^k, \nabla p_h^k) \lesssim [1+\log(d/h)]^2 \sum_{k=1}^{N} \beta_k^* |p_h|_{1,\Omega_k}^2. \tag{7.23}$$

Now for any $\mathbf{v}_h^0 \in V_h^0(\Omega)$, we can verify (cf. (4.47), [21]) that

$$\begin{aligned} \|\mathbf{v}_h^0 \times \mathbf{n}\|_{\mathcal{X}_{\Gamma_k}}^2 \lesssim\ & h[1+\log(d/h)](\beta_k^* \|\mathbf{v}_h^0 \times \mathbf{n}\|_{0,\Gamma_k}^2 \\ & + \alpha_k^* \|\mathrm{div}_\tau(\mathbf{v}_h^0 \times \mathbf{n})\|_{0,\Gamma_k}^2), \quad \forall \mathbf{v}_h^0 \in V_h^0(\Omega), \end{aligned}$$

which implies

$$\sum_{k=1}^{N} \|\mathbf{v}_h^0 \times \mathbf{n}\|_{\mathcal{X}_{\Gamma_k}}^2 \lesssim (\hat{A}_0 \mathbf{v}_h^0, \mathbf{v}_h^0). \tag{7.24}$$

Using the above preparations, we can now build up a stable decomposition for any $\mathbf{v}_h \in V_h(\Omega)$. First by Theorem 3.2 we have the following Helmholtz-type decomposition:

$$\mathbf{v}_h = \nabla p_h + \mathbf{w}_h \tag{7.25}$$

for some $p_h \in Z_h(\Omega)$ and $\mathbf{w}_h \in V_h(\Omega)$, and they are orthogonal in the sense of (3.6) and satisfies the a priori estimates (3.7). For p_h and $\mathbf{w}_h$ in (7.25), let p_h^0, p_h^{ij}, p_h^k be defined as above and $\mathbf{w}_h^0$, $\mathbf{w}_h^{ij}$, $\mathbf{w}_h^k$ be the functions as those given in (7.19). Then we set

$$\mathbf{v}_h^0 = \nabla p_h^0 + \mathbf{w}_h^0 \in V^0(\Omega), \quad \mathbf{v}_h^k = \nabla p_h^k + \mathbf{w}_h^k \in V^k(\Omega), \quad \mathbf{v}_h^{ij} = \nabla p_h^{ij} + \mathbf{w}_h^{ij} \in V^{ij}(\Omega).$$

We can easily verify that

$$\mathbf{v}_h = \mathbf{v}_h^0 + \sum_{k=1}^{N} \mathbf{v}_h^k + \sum_{\Gamma_{ij}} \mathbf{v}_h^{ij}.$$

Next we show the stability of this decomposition:

$$(\hat{A}_0 \mathbf{v}_h^0, \mathbf{v}_h^0) + \sum_{k=1}^{N} (\hat{A}_k \mathbf{v}_h^k, \mathbf{v}_h^k) + \sum_{\Gamma_{ij}} (\hat{A}_{ij} \mathbf{v}_h^{ij}, \mathbf{v}_h^{ij})$$
$$\lesssim [G(d/h) + \log^{m+1}(1/h)][1 + \log(d/h)]^2 (A\mathbf{v}_h, \mathbf{v}_h). \tag{7.26}$$

But we obtain readily from (7.20) and (7.22)–(7.23) that

$$(\hat{A}_0 \mathbf{v}_h^0, \mathbf{v}_h^0) + \sum_{k=1}^{N} (\hat{A}_k \mathbf{v}_h^k, \mathbf{v}_h^k) + \sum_{\Gamma_{ij}} (\hat{A}_{ij} \mathbf{v}_h^{ij}, \mathbf{v}_h^{ij})$$
$$\lesssim [1 + \log(d/h)]^2 (\beta \nabla p_h,\ p_h) + [1 + \log(d/h)]^2 (\tilde{A} \mathbf{w}_h,\ \mathbf{w}_h). \tag{7.27}$$

Then (7.26) will follow if we can show

$$(\beta \nabla p_h,\ p_h) \lesssim G(d/h)(A\mathbf{v}_h, \mathbf{v}_h) \ \text{ and } \ (\tilde{A}\mathbf{w}_h,\ \mathbf{w}_h) \lesssim \log^{m+1}(1/h)(A\mathbf{v}_h, \mathbf{v}_h). \tag{7.28}$$

The second estimate in (7.28) follows immediately from (7.25), (3.7) and (7.10):

$$\begin{aligned}(\tilde{A}\mathbf{w}_h, \mathbf{w}_h) &= (\alpha \mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h) + (\beta \mathbf{w}_h, \mathbf{w}_h) \\ &\lesssim (\alpha \mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h) + \log^{m+1}(1/h)(\beta \mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h) \\ &\lesssim \log^{m+1}(1/h)(A\mathbf{v}_h, \mathbf{v}_h).\end{aligned} \tag{7.29}$$

Using (7.25) and the definitions of operators J in (7.11) and B^t in (7.3), we can write

$$\begin{aligned}(\beta \nabla p_h, \nabla p_h) &= (\beta \mathbf{v}_h, \nabla p_h) = (B^t \mathbf{v}_h, p_h) \\ &= (J p_h, J^{-1} B^t \mathbf{v}_h) = (\beta \nabla p_h, \nabla (J^{-1} B^t \mathbf{v}_h)) \\ &= (\beta \mathbf{v}_h, \nabla (J^{-1} B^t \mathbf{v}_h)) = (B^t \mathbf{v}_h, J^{-1} B^t \mathbf{v}_h).\end{aligned}$$

Using this relation and the estimate (3.7), we derive

$$(B J^{-1} B^t \mathbf{v}_h, \mathbf{v}_h) = (\beta \nabla p_h, \nabla p_h) \le (\beta \mathbf{v}_h, \mathbf{v}_h). \tag{7.30}$$

Combining this relation with (7.12) gives

$$(\beta \nabla p_h,\ \nabla p_h) \lesssim G(d/h)(B \hat{C}^{-1} B^t \mathbf{v}_h, \mathbf{v}_h) \le G(d/h)(A\mathbf{v}_h, \mathbf{v}_h), \tag{7.31}$$

so the first estimate in (7.28) is proved. Now we readily get the estimate of the smallest eigenvalue of the preconditioned system $\hat{A}^{-1}A$ from (7.27):

$$\lambda_{\min}(\hat{A}^{-1}A) \gtrsim 1/([G(d/h) + \log^{m+1}(1/h)][1 + \log(d/h)]^2).$$

To estimate the largest eigenvalue, we use (7.12) and (7.30) to obtain that

$$\begin{aligned}(A\mathbf{v}_h, \mathbf{v}_h) &\lesssim (\alpha\,\mathbf{curl}\ \mathbf{v}_h, \mathbf{curl}\ \mathbf{v}_h) + \gamma_0(\beta\mathbf{v}_h, \mathbf{v}_h) + (BJ^{-1}B^t\mathbf{v}_h, \mathbf{v}_h)\\ &\le (\alpha\mathbf{curl}\mathbf{v}_h, \mathbf{curl}\mathbf{v}_h) + (\beta\mathbf{v}_h, \mathbf{v}_h) + (\beta\mathbf{v}_h, \mathbf{v}_h)\\ &\lesssim (\tilde{A}\mathbf{v}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h(\Omega). \end{aligned} \tag{7.32}$$

Let $\mathbf{v}_h^0 \in V^0(\Omega)$. Since $\mathbf{v}_h^0$ is discrete A_k-harmonic in Ω_k for each k, it possesses the minimum energy property on each Ω_k. Then it follows from (7.32), Lemma 7.1 and (7.24) that

$$\begin{aligned}(A\mathbf{v}_h^0, \mathbf{v}_h^0) &\lesssim (\tilde{A}\mathbf{v}_h^0, \mathbf{v}_h^0) = \sum_{k=1}^{N}(\alpha_k^*\|\mathbf{curl}\mathbf{v}_h^0\|_{0,\Omega_k}^2 + \beta_k^*\|\mathbf{v}_h^0\|_{0,\Omega_k}^2)\\ &\lesssim \sum_{k=1}^{N}(\alpha_k^*\|\mathbf{curl}(\mathbf{R}_k(\mathbf{v}_h^0\times\mathbf{n}|_{\Gamma_k}))\|_{0,\Omega_k}^2 + \beta_k^*\|\mathbf{R}_k(\mathbf{v}_h^0\times\mathbf{n}|_{\Gamma_k})\|_{0,\Omega_k}^2)\\ &\lesssim (\hat{A}_0\mathbf{v}_h^0,\ \mathbf{v}_h^0), \quad \forall \mathbf{v}_h^0 \in V^0(\Omega).\end{aligned}$$

This, along with the following bounds directly from the definitions of $\hat{A}_k$ and $\hat{A}_{ij}$,

$$(A\mathbf{v}_h^k, \mathbf{v}_h^k) \lesssim (\hat{A}_k\mathbf{v}_h^k,\ \mathbf{v}_h^k) \quad \forall \mathbf{v}_h^k \in V^k(\Omega); \quad (A\mathbf{v}_h^{ij}, \mathbf{v}_h^{ij}) \lesssim (\hat{A}_{ij}\mathbf{v}_h^{ij},\ \mathbf{v}_h^{ij}) \quad \forall \mathbf{v}_h^{ij} \in V^{ij}(\Omega),$$

gives the estimate of the largest eigenvalue, $\lambda_{\max}(\hat{A}^{-1}A) \lesssim 1$, thus completes the proof of Theorem 7.1. □

Remark 7.2 The key step in the proof of Theorem 7.1 is the derivation of (7.29), by using the weighted discrete Helmholtz decomposition newly developed in Theorem 3.2. If the standard Helmholtz decomposition is used, no conclusion can be made about how the constant in the upper bound of (7.29) depends on the possible large jumps of the coefficient $\beta(\mathbf{x})$.

8 Numerical experiments

In this section we present some numerical experiments to verify the convergence of the newly proposed preconditioner in Sect. 7.2. For the purpose we construct a test example with an exact solution. We consider the most difficult saddle-point case of system (1.1)–(1.2), namely $\gamma_0 = 0$, and the cubic domain $\Omega = (0, 1) \times (0, 1) \times (0, 1)$. In order to test the case of non-homogeneous media, we take

$$D = \left[\frac{1}{4}, \frac{1}{2}\right]^3 \bigcup \left[\frac{1}{2}, \frac{3}{4}\right]^3,$$

and choose the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ to be

$$\alpha(\mathbf{x}) = 2\,\beta(\mathbf{x}) = \begin{cases} \alpha_0\,, & \mathbf{x} \in D \\ 1\,, & \mathbf{x} \in \Omega \backslash D. \end{cases}$$

We will test $\alpha_0 = 1$ and $\alpha_0 = 10^5$. Clearly there are no jumps in the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ for $\alpha_0 = 1$, but there is a large jump in both coefficients and two *cross-points* appear in Ω for $\alpha_0 = 10^5$. The source terms $\mathbf{f}$ and g are taken such that the exact solution $\mathbf{u} = (u_1, u_2, u_3)^T$ of the system (1.1)–(1.2) is given by

$$\begin{aligned} u_1 &= xyz(x-1)(y-1)(z-1)\,, \\ u_2 &= \sin(\pi x)\sin(\pi y)\sin(\pi z)\,, \\ u_3 &= (1-e^x)(1-e^{x-1})(1-e^y)(1-e^{y-1})(1-e^z)(1-e^{z-1}). \end{aligned}$$

To generate the subdomain decomposition of the whole domain Ω, we first partition the three edges of Ω on the x-, y- and z-axis respectively into equally distributed n subintervals, then we can naturally generate n^3 equal smaller cubes of size $d = 1/n$. This yields the desired subdomain decomposition in our experiments: $\Omega_1, \ldots, \Omega_N$, with $N = n^3$.

To generate a fine triangulation $\mathcal{T}_h$ of size h over the entire domain Ω, we divide each subdomain Ω_k into m^3 equal smaller cubes of size $h = 1/(mn)$, in the same manner as we generated the subdomains above. Then $\mathcal{T}_h$ is obtained by triangulating each small cube into 6 tetrahedra. For the easy reference we shall denote the triangulation $\mathcal{T}_h$ by $m^3(n^3)$ below.

Our numerical experiment is to solve the saddle-point system (7.5), which is equivalent to the edge and nodal element saddle-point system (7.2), by the Nonlinear Inexact Preconditioned Uzawa Algorithm mentioned in Sect. 7.1, with the preconditioners $\hat{A}$ given in Sect. 7.2 and $\hat{C}$ being the standard multigrid preconditioner for the discrete Laplacian (thus satisfying the condition (7.12) with $G(d/h) = 1$; see Remark 7.1). For any $\phi \in V_h(\Omega)$, let $\Psi(\phi)$ be an approximation of the solution ξ to the system $A\xi = \phi$ obtained by the PCG method with preconditioner $\hat{A}$. Then the inexact Uzawa-type algorithm for solving (7.5) can be described below (see [19]).

Nonlinear Inexact Preconditioned Uzawa Algorithm.

Step 1. Compute $f_i = \mathbf{f}_h - (A\mathbf{u}_h^i + Bp_h^i)$ and $\Psi(f_i)$, update $\mathbf{u}_h^{i+1} = \mathbf{u}_h^i + \Psi(f_i)$;
Step 2. Compute $g_i = B^t\mathbf{u}_h^{i+1} - g_h$, $d_i = \hat{C}^{-1}g_i$ and

$$\tau_i = \frac{1}{2}\frac{(g_i, d_i)}{(\Psi(Bd_i), Bd_i)} \quad \text{for} \;\; g_i \neq 0\,; \quad \tau_i = 1 \;\; \text{for} \;\; g_i = 0.$$

Then update $\quad p_h^{i+1} = p_h^i + \tau_i d_i.$

Note that the above algorithm involves two inner iterations, namely computing two approximations $\Psi(f_i)$ and $\Psi(Bd_i)$ by the PCG method with preconditioner $\hat{A}$. However, the approximations $\Psi(f_i)$ and $\Psi(Bd_i)$ are not required to be accurate and suffice when the energy-norm errors are reduced respectively by a factor $1/2$ and $1/3$ (cf. [19]). The inexact Uzawa iteration will terminate when the global relative residual (cf. [19]) is less than 10^{-6}, and the corresponding number of iterations will be reported;

Table 1 Number of iterations of the inexact Uzawa algorithm

$m\backslash n$	$\alpha_0 = 1$						$\alpha_0 = 10^5$					
	4			8			4			8		
	Iter	n_f	n_d	*Iter*	n_f	n_d	*Iter*	n_f	n_d	*Iter*	n_f	n_d
4	17	4.7	3.4	18	5.3	3.0	22	4.8	3.2	23	5.2	3.5
8	19	6.8	5.0	18	6.4	4.3	25	7.1	4.7	24	6.8	5.0
16	19	9.8	6.7	19	8.5	5.8	25	9.9	6.4	25	9.5	6.8

see *iter* in Table 1. To show the efficiency of the preconditioner $\hat{A}$, we will also report the averaging number of iterations for the approximations $\Psi(f_i)$ and $\Psi(Bd_i)$; see n_f and n_d in Table 1.

We can see from Table 1 that the numerical results confirm our theoretical predictions in Theorem 7.1: the preconditioner $\hat{A}$ is nearly optimal in the sense that the condition number depends weakly on the ratio $m = d/h$ (see the growth of n_f and n_d with respect to m in Table 1 when n is fixed) and is almost independent of $n = 1/d$ and the jumps of the coefficients in $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ (see the growth of n_f and n_d with respect to n or α_0 in Table 1 when m is fixed). In particular, the convergence rates for the case with jumps in the coefficients deteriorate only slightly compared to the case without jumps, namely, the condition number for the case with jumps is slightly more sensitive to the ratio $mn = 1/h$ than the case without jumps.

Finally we would like to mention the very satisfactory convergence of the resulting global inexact Uzawa algorithm, the maximum number of iterations is less than 25 iterations as we see from Table 1 for the very large discrete saddle-point system (7.2), with a total number of degrees of freedom being 14,532,992 (when $n = 8$ and $m = 16$). In addition, we can see clearly from Table 1 that the resulting global inexact Uzawa algorithm converges nearly optimally in terms of mesh size h (see the growth of *Iter* with respect to m in Table 1 when n is fixed) and subdomain size d (see the growth of *Iter* with respect to n when m is fixed), and the convergence rate is affected slightly by the jumps in the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$.

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