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Journal of Functional Analysis 263 (2012) 364–382

JOURNAL OF Functional Analysis

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Universal extension for Sobolev spaces of differential forms and applications

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Received 16 July 2009; accepted 29 April 2012

Available online 9 May 2012

Communicated by I. Rodnianski

Abstract

This article is devoted to the construction of a family of universal extension operators for the Sobolev spaces $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$ of differential forms of degree l ($0 \leqslant l \leqslant d$) in a Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$, $d \geqslant 2$) for any $k \in \mathbb{N}_0$. It generalizes the construction of the first universal extension operator for standard Sobolev spaces $H^k(\Omega)$, $k \in \mathbb{N}_0$, on Lipschitz domains, introduced by Stein [E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, NJ, 1970, Theorem 5, p. 181]. We adapt Stein's idea in the form of integral averaging over the pullback of a parametrized reflection mapping. The new theory covers extension operators for $\boldsymbol{H}^k(\mathbf{curl}; \Omega)$ and $\boldsymbol{H}^k(\mathrm{div}; \Omega)$ in $\mathbb{R}^3$ as special cases for $l = 1, 2$, respectively. Of considerable mathematical interest in its own right, the new theoretical results have many important applications: we elaborate existence proofs for generalized regular decompositions.

Keywords: Universal (Stein) extension; Sobolev spaces of differential forms; Lipschitz domains; Integral averaging; Parametrized reflection mapping; Generalized regular decomposition

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1 Supported by Hong Kong RGC grants (Projects 405110 and 404611).

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2012.04.016

1. Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}_0$, $d \geqslant 2$), Stein [27, Theorem 5, p. 181] constructed a celebrated *extension mapping*

$$\mathscr{E} : C^\infty(\overline{\Omega}) \mapsto C^\infty\big(\mathbb{R}^d\big), \quad \mathscr{E}u(\mathbf{x}) = u(\mathbf{x})\ \forall \mathbf{x} \in \overline{\Omega},$$

which fulfills that for any $m \in \mathbb{N}_0$, $1 \leqslant p < \infty$,

$$\exists C = C(m, p, \Omega) > 0: \quad \|\mathscr{E}u\|_{W^{m,p}(\mathbb{R}^d)} \leqslant C\|u\|_{W^{m,p}(\Omega)}\ \forall u \in C^\infty(\overline{\Omega}). \tag{1.1}$$

Thus, it can be naturally extended to a continuous extension operator for *any* classical Sobolev space $W^{m,p}(\Omega)$, $m \in \mathbb{N}_0$, $1 \leqslant p < \infty$. Thanks to its “one formula fits all (Sobolev spaces)” property, the operator $\mathscr{E}$ is called a *universal* extension operator. This makes it exceptional, because other designs of extension operators for Sobolev spaces by, for instance, the successive reflection method [15,30,26], or the singular integral method [7], rely on different formulas for different orders m and may hinge on smoothness of the boundary. It goes without saying that universality renders $\mathscr{E}$ a valuable tool in the theory of Sobolev spaces and their applications.

Beyond the classical articles, there are a few existing studies on extensions of general function spaces in Lipschitz domains. Let us mention, e.g., [29], where a novel extension operator was proposed to cover the most general cases, like Hardy, Sobolev, Besov, Triebel–Lizorkin spaces, etc. Extension results can find wide applications to such as interpolation spaces and regularity estimates in PDEs, see, e.g., [19,21].

The main purpose of this paper is to construct a new family of universal extension operators for Sobolev spaces $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$ of differential forms, for $l, k \in \mathbb{N}_0$, and $0 \leqslant l \leqslant d$ in Lipschitz domains $\Omega \subset \mathbb{R}^d$, see Section 2 for the precise definition. To keep our presentation succinct, we study only Hilbert spaces, that is, the case $p = 2$. We would like to point out that Sobolev spaces of differential forms are fundamental to the theoretical analysis of, e.g., electromagnetic phenomena governed by Maxwell's equation [3,16,20,22], the Navier–Stokes equation [14], and interpolation theory [19].

The paper is arranged as follows. We first present some notations and basic auxiliary tools in Section 2. In Section 3, we briefly recall Stein's approach, i.e., an integral averaging method based on local parametrized reflection mappings and then present our construction. Guided by the commuting relationship of the pullback and the exterior derivative of differential forms, the gist of our construction is to apply Stein's integral averaging to the pullback operators induced by the reflection mappings. This offers a natural generalization of Stein's formula to differential forms, see formula (3.9). With some technical effort, Stein's original analysis can be adapted, which is also done in Section 3 of this article, see Lemma 3.4 and Theorems 3.5, 3.6. From the perspective of vector fields, we demonstrate the explicit construction of those extension operators in terms of Euclidean vector proxies in $\mathbb{R}^3$ in Section 4. We point out that universal extension operators for the Sobolev spaces $\boldsymbol{H}^k(\mathbf{curl}; \Omega)$ and $\boldsymbol{H}^k(\mathrm{div}; \Omega)$ of vector fields in $\mathbb{R}^3$ are covered by our universal extension theorem as special cases for $l = 1, 2$, respectively. These new theoretical results are not only of mathematical interest in their own right, but also have important applications. We elaborate existence proofs for generalized regular decompositions in Section 5.

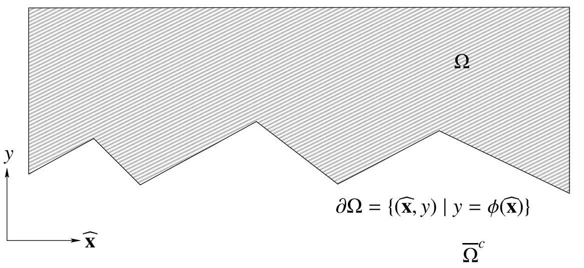

Fig. 1. Sketch of a Lipschitz epigraph.

2. Notation and preliminaries

Throughout the paper, $\mathbb{R}^d$ stands for the classical Euclidean space ($d \in \mathbb{N}$, $d \geqslant 2$), equipped with the canonical orthonormal bases e_j's, $1 \leqslant j \leqslant d$, and norm $|\mathbf{x}| := \sqrt{x_1^2 + \cdots + x_d^2}$, if $\mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$. The canonical orthonormal basis of $\mathbb{R}^d$ corresponds to a dual basis of $(\mathbb{R}^d)^*$, i.e., $\boldsymbol{d}x_1, \boldsymbol{d}x_2, \ldots, \boldsymbol{d}x_d$ with $\boldsymbol{d}x_i(e_j) = 1$ if $i = j$ and zero otherwise.

Recall that a function $f : D \mapsto \mathbb{R}$, $D \subset \mathbb{R}^{d-1}$ is called *Lipschitz* if there exists a finite constant $C > 0$ such that

$$\left|f(\mathbf{x}) - f(\mathbf{y})\right| \leqslant C|\mathbf{x} - \mathbf{y}| \quad \forall \mathbf{x}, \mathbf{y} \in D.$$

A *Lipschitz epigraph* $\Omega \subset \mathbb{R}^d$ is defined as a domain lying above the graph of a Lipschitz function $\phi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$, i.e., $\Omega = \{(\hat{\mathbf{x}}, x_d) \mid \phi(\hat{\mathbf{x}}) < x_d\}$ with $\hat{\mathbf{x}} = (x_1, \ldots, x_{d-1})$. See [27] and Fig. 1 for illustration.

A bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ is a bounded domain whose boundary $\partial\Omega$ can be covered by a finite number of open balls B_i, $1 \leqslant i \leqslant m$, so that, possibly after a proper rigid motion, $\partial\Omega \cap B_i$ is part of the graph of a Lipschitz function, above which $\Omega \cap B_i$ lies, for all i's,

Next, we introduce differential forms and associated Sobolev spaces. We will adopt some standard notations, and refer to [3,5,8,9,12,13,16,18,23,25] for more details. For $l \in \mathbb{N}_0$ and $0 \leqslant l \leqslant d$, we denote by Λ^l the vector space of real-valued (or complex-valued), alternating, l-multilinear maps on $\mathbb{R}^d$. In particular, Λ^0 and Λ^1 can be identified with $\mathbb{R}$ and $\mathbb{R}^d$, respectively. Given $\omega \in \Lambda^l$ and $\eta \in \Lambda^k$, the *exterior product* $\omega \wedge \eta \in \Lambda^{l+k}$ is defined by[2]

$$(\omega \wedge \eta)(v_1, \ldots, v_{l+k}) = \sum_{\sigma} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(l)})\eta(v_{\sigma(l+1)}, \ldots, v_{\sigma(l+k)}),$$

for any $v_1, \ldots, v_{l+k} \in \mathbb{R}^d$ where $\operatorname{sgn}(\sigma)$ indicates the signature of σ and the sum is taken over all the permutations σ of $\{1, \ldots, l+k\}$ such that $\sigma(1) < \cdots < \sigma(l)$ and $\sigma(l+1) < \cdots < \sigma(l+k)$.

[2] We adopt the convention that Roman letters denote scalar functions, and their associated spaces etc., while bold letters represent vector-valued functions, and their associated spaces etc. In particular, bold Greek letters $\boldsymbol{\omega}$, $\boldsymbol{\eta}$, $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$, etc. are reserved for differential forms, except that $\boldsymbol{\xi}$ stands for the independent variable in the frequency domain $\mathbb{R}^d$.

Given a vector $\mathbf{a} = (a_1, a_2, \ldots, a_d)^T$ and a basis l-form ($l \geqslant 1$) $\boldsymbol{\omega} = \boldsymbol{d}x_{j_1} \wedge \boldsymbol{d}x_{j_2} \wedge \cdots \wedge \boldsymbol{d}x_{j_l}$ with $j_1 < j_2 < \cdots < j_l$, the *interior product* $\mathbf{a} \lrcorner \boldsymbol{\omega} \in \Lambda^{l-1}$ and is defined by

$$\mathbf{a} \lrcorner \boldsymbol{\omega} = \sum_{k=1}^{l} (-1)^{k-1} a_{j_k} \boldsymbol{d}x_{j_1} \wedge \cdots \wedge \check{\boldsymbol{d}x}_{j_k} \wedge \cdots \wedge \boldsymbol{d}x_{j_l} \in \Lambda^{l-1},$$

where $\check{\cdot}$ indicates that $\cdot$ is dropped.

For simplicity, we will frequently use the increasing l-permutation $I = (i_1, \ldots, i_l)$, with $1 \leqslant i_1 < \cdots < i_l \leqslant d$, and denote $\boldsymbol{d}\mathbf{x}_I = \boldsymbol{d}x_{i_1} \wedge \cdots \wedge \boldsymbol{d}x_{i_l}$. Σ_I always means the summation over all the increasing l-permutations I. Therefore Λ^l can be viewed as a vector space of dimension $\binom{d}{l}$ with bases $\{\boldsymbol{d}x_I\}$ for all increasing l-permutations I.

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, spaces of differential forms are equivalent to those in the componentwise sense. We use standard function spaces $C^m(\overline{\Omega})$, $C^\infty(\overline{\Omega})$, $C_0^\infty(\Omega)$, $L^2(\Omega)$ and $H^s(\Omega)$, $s \in \mathbb{R}_0^+$ (see [1] for more details).

A differential form ω of degree l, $l \in \mathbb{N}_0$, and class C^m, $m \in \mathbb{N}_0$, in Ω is an l-form valued mapping

$$\boldsymbol{\omega} = \sum\nolimits_I \boldsymbol{\omega}_I \boldsymbol{d}\mathbf{x}_I : \mathbf{x} \in \Omega \subset \mathbb{R}^d \mapsto \boldsymbol{\omega}(\mathbf{x}) \in \Lambda^l,$$

where all the components $\boldsymbol{\omega}_I(\mathbf{x}) \in C^m(\overline{\Omega})$. Hence we write $\boldsymbol{\omega} \in \mathcal{DF}^{l,m}(\overline{\Omega})$. In an analogous way, we can define $\mathcal{DF}^{l,\infty}(\overline{\Omega})$ if all $\boldsymbol{\omega}_I(\mathbf{x}) \in C^\infty(\overline{\Omega})$, and $\mathcal{DF}_0^{l,\infty}(\Omega)$ if all $\boldsymbol{\omega}_I(\mathbf{x}) \in C_0^\infty(\Omega)$. Note that the exterior and interior products can be extended as pointwise operations to differential forms on domains in $\mathbb{R}^d$.

Likewise, $\boldsymbol{H}^s(\Omega; \Lambda^l)$ ($s \in \mathbb{R}_0^+$) denotes the space consisting of all differential forms with each component in $H^s(\Omega)$, which can be viewed as the Hilbert space obtained by means of the completion of $\mathcal{DF}^{l,\infty}(\overline{\Omega})$ with respect to the norm

$$\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\Omega;\Lambda^l)} := \sum_I \|\boldsymbol{\omega}_I\|^2_{H^s(\Omega)}.$$

In particular we use $\boldsymbol{L}^2(\Omega; \Lambda^l)$ instead of $\boldsymbol{H}^0(\Omega; \Lambda^l)$.

If $\mathscr{T} : \widehat{\Omega} \mapsto \Omega$, is a diffeomorphism between two manifolds in $\mathbb{R}^d$, then the pullback $\mathscr{T}^* : \mathcal{DF}^{l,\infty}(\overline{\Omega}) \mapsto \mathcal{DF}^{l,\infty}(\overline{\Omega})$ is given by

$$\big(\big(\mathscr{T}^*\boldsymbol{\omega}\big)(\hat{\mathbf{x}})\big)(\mathbf{v}_1, \ldots, \mathbf{v}_l) = \big(\boldsymbol{\omega}\big(\mathscr{T}(\hat{\mathbf{x}})\big)\big)\big(D\mathscr{T}(\hat{\mathbf{x}})\mathbf{v}_1, \ldots, D\mathscr{T}(\hat{\mathbf{x}})\mathbf{v}_l\big),$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_l \in \mathbb{R}^d$ and the linear map $D\mathscr{T}(\hat{\mathbf{x}}) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is the derivative (Jacobian) of $\mathscr{T}$ at $\hat{\mathbf{x}}$.

For a differential l-form $\boldsymbol{\omega} = \sum_I \boldsymbol{\omega}_I \boldsymbol{d}\mathbf{x}_I \in \mathcal{DF}^{l,\infty}(\overline{\Omega})$, its *exterior derivative* $\boldsymbol{d\omega}$ is defined by

$$\boldsymbol{d\omega} := \sum_{i=1}^{d} \sum_I \frac{\partial \omega_I}{\partial x_i} \boldsymbol{d}x_i \wedge \boldsymbol{d}\mathbf{x}_I \in \mathcal{DF}^{l+1,\infty}(\overline{\Omega}), \tag{2.1}$$

and if $l \geqslant d$, we let $\boldsymbol{d\omega} = 0$.

We recall the fact that the pullback commutes with the exterior derivative, i.e.,

$$\mathscr{T}^*(\boldsymbol{d}\boldsymbol{\omega}) = \boldsymbol{d}\big(\mathscr{T}^*\boldsymbol{\omega}\big), \quad \forall \boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega}) \tag{2.2}$$

and with the wedge product

$$\mathscr{T}^*(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) = \mathscr{T}^*\boldsymbol{\omega} \wedge \mathscr{T}^*\boldsymbol{\eta}, \quad \forall \boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega}),\ \boldsymbol{\eta} \in \mathcal{DF}^{k,\infty}(\overline{\Omega}). \tag{2.3}$$

The crucial Hilbert spaces of differential forms are

$$\boldsymbol{H}^s\big(\boldsymbol{d}, \Omega, \Lambda^l\big) := \big\{\boldsymbol{\omega} \in \boldsymbol{H}^s\big(\Omega; \Lambda^l\big) \bigm| \boldsymbol{d}\boldsymbol{\omega} \in \boldsymbol{H}^s\big(\Omega; \Lambda^{l+1}\big)\big\}, \quad s \in \mathbb{R}_0^+,$$

with the natural graph norms

$$\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)} := \|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\Omega,\Lambda^l)} + \|\boldsymbol{d}\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\Omega,\Lambda^{l+1})}.$$

Specifically, we simply put $\boldsymbol{H}(\boldsymbol{d}, \Omega, \Lambda^l)$ when $s = 0$.

Moreover, we define some important subspaces of $\boldsymbol{H}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ and $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$, $k \in \mathbb{N}$, respectively:

$$\boldsymbol{H}\big(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l\big) := \big\{\boldsymbol{\omega} \in \boldsymbol{H}\big(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l\big) \bigm| \boldsymbol{d}\boldsymbol{\omega} = 0\big\},$$
$$\boldsymbol{H}_0^k\big(\boldsymbol{d}, \Omega, \Lambda^l\big) := \text{the closure of } \mathcal{DF}_0^{l,\infty}(\Omega) \text{ in the space } \boldsymbol{H}^k\big(\boldsymbol{d}, \Omega, \Lambda^l\big).$$

In the sequel, we denote by c and C generic positive constants which may depend on the domain Ω, space dimension d, the degree of differential forms l and the order of differentiability k, but independent of the differential forms involved.

3. Universal extension of differential forms

In this section, we present in detail our construction of the universal extension operators for Sobolev spaces of differential forms. After briefly recalling essential ingredients of Stein's approach for constructing the universal extension operator for standard Sobolev spaces $H^k(\Omega)$ ($k \in \mathbb{N}_0$) (cf. [27, Chap. VI]), we first show the extension for the case of a Lipschitz epigraph with most key ingredients, and then generalize to bounded Lipschitz domains by the partition of unity.

3.1. Some technical lemmas

For a closed domain $\overline{\Omega}$, let $\delta(\mathbf{x}) := \operatorname{dist}(\mathbf{x}; \overline{\Omega})$ denote the distance of $\mathbf{x} \in \mathbb{R}^d$ from $\overline{\Omega}$. The function $\delta(\mathbf{x})$ vanishes in $\overline{\Omega}$, and, in general, will only be Lipschitz continuous, as $|\delta(\mathbf{x}) - \delta(\mathbf{y})| \leqslant |\mathbf{x} - \mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \overline{\Omega}^c$, the complement of $\overline{\Omega}$. The next lemma introduces a regularized distance with enhanced smoothness as a replacement for $\delta(\mathbf{x})$.

Lemma 3.1 *(Regularized distance). (See [27, Theorem 2, p. 171].) For a closed domain $\overline{\Omega} \in \mathbb{R}^d$, there exists a regularized distance function $\Delta(\mathbf{x}) = \Delta(\mathbf{x}, \overline{\Omega})$ such that for $\mathbf{x} \in \overline{\Omega}^c$*

i). $c\delta(\mathbf{x}) \leqslant \Delta(\mathbf{x}) \leqslant C\delta(\mathbf{x})$;
ii). $\Delta(\mathbf{x})$ *is* C^∞*-smooth in* $\overline{\Omega}^c$ *and* $|\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha}\Delta(\mathbf{x})| \leqslant C_\alpha(\delta(\mathbf{x}))^{1-|\alpha|}$,

where c *and* C *are positive constants independent of* $\overline{\Omega}$ *and* C_α *depends on the multi-index* α.[3]

The following two technical lemmas are key tools to construct universal extension operators. The first lemma introduces a suitable weighting function, in terms of which the weighted averaging integral for the construction of extension operators will be defined.

Lemma 3.2 *(Weighting function). (See [27, Lemma 1, p. 182].) The weighting function*[4]

$$\psi(\lambda) := \frac{e}{\pi\lambda}\operatorname{Im}\left(\exp\left(\frac{1}{2}\sqrt{2}(-1+i)(\lambda-1)^{1/4}\right)\right) \tag{3.1}$$

is defined in $[1,\infty)$*, and satisfies the decay property*

$$\psi(\lambda) = O\left(\lambda^{-n}\right) \quad \textit{as } \lambda \to \infty,\ \forall n \in \mathbb{N}, \tag{3.2}$$

and all its higher moments vanish

$$\int_1^\infty \lambda^k \psi(\lambda)\,\mathrm{d}\lambda = \begin{cases} 1, & \textit{for } k = 0, \\ 0, & \textit{for } k \in \mathbb{N}. \end{cases} \tag{3.3}$$

Now we consider the special case that Ω is a Lipschitz epigraph with its boundary defined by a Lipschitz function $\phi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$, see Fig. 1. We split position vectors according to $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \mathbb{R}^d$, where $\hat{\mathbf{x}} \in \mathbb{R}^{d-1}$ and $y \in \mathbb{R}$.

Lemma 3.3 *(Existence of smoothed distance function). (See [27, Lemma 2, p. 182].) For a Lipschitz epigraph* Ω*, let* $\Delta(\mathbf{x})$ *be the regularized distance given in Lemma* 3.1*. Then there exists a constant* $C_\delta = C_\delta(\phi) > 0$ *such that for* $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \overline{\Omega}^c$,

$$C_\delta \Delta(\mathbf{x}) \geqslant \phi(\hat{\mathbf{x}}) - y. \tag{3.4}$$

We define a *scaled smoothed distance* $\delta^*(\mathbf{x}) := 2C_\delta\Delta(\mathbf{x})$ with smoothness inherited from $\Delta(\mathbf{x})$. From (3.4) it is immediate to see that

$$\delta^*(\mathbf{x}) \geqslant 2\left(\phi(\hat{\mathbf{x}}) - y\right). \tag{3.5}$$

[3] $\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha}$ stands for $\frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$ with $\alpha = (\alpha_1, \ldots, \alpha_d)$ being a multi-index, $\alpha_i \geqslant 0$ for $1 \leqslant i \leqslant d$, and $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$.

[4] Im in (3.1) means taking the imaginary part.

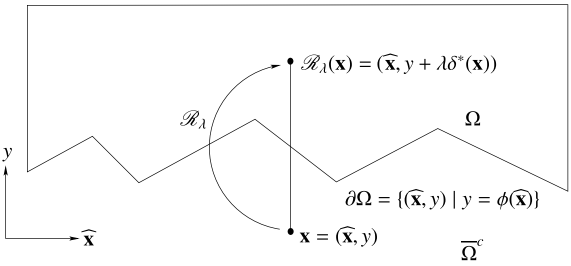

Fig. 2. Parametrized reflection mapping.

3.2. Extension formula for epigraphs

The classical Stein extension formula [27] for compactly supported[5] smooth functions f on a Lipschitz epigraph $\overline{\Omega}$ reads

$$\mathscr{E}(f)(\mathbf{x}) = \int_1^\infty f\big(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})\big)\psi(\lambda)\,\mathrm{d}\lambda. \tag{3.6}$$

To generalize this formula, let us first define a parametrized reflection mapping (see Fig. 2) for $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \overline{\Omega}^c \in \mathbb{R}^d$,

$$\mathscr{R}_\lambda(\mathbf{x}) = \big(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})\big) = \mathbf{x} + \lambda\delta^*(\mathbf{x})e_d. \tag{3.7}$$

Note that for points $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \overline{\Omega}$ we have, using the fact that $\delta^*(\mathbf{x}) = 0$,

$$\mathscr{R}_\lambda(\mathbf{x}) = (\hat{\mathbf{x}}, y + 0) = \mathbf{x}.$$

In other words, $\mathscr{R}_\lambda$ reduces to the identity operator in $\overline{\Omega}$. However, for $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \overline{\Omega}^c$ with $y < \phi(\hat{\mathbf{x}})$, due to (3.5) and the fact that $\lambda \geqslant 1$, we see that

$$y + \lambda\delta^*(\mathbf{x}) \geqslant y + 2\big(\phi(\hat{\mathbf{x}}) - y\big) \geqslant \phi(\hat{\mathbf{x}}) + \big(\phi(\hat{\mathbf{x}}) - y\big) > \phi(\hat{\mathbf{x}}).$$

Thus, the parametrized reflection mapping $\mathscr{R}_\lambda$ always maps $\mathbf{x} \in \overline{\Omega}^c$ into Ω for any $\lambda \in [1, \infty)$.

It is straightforward to calculate the Jacobian of the parametrized reflection mapping

$$D\mathscr{R}_\lambda(\mathbf{x}) = \begin{pmatrix} Id_{d-1} & 0 \\ \lambda\,\mathbf{grad}_{\hat{\mathbf{x}}}\,\delta^*(\mathbf{x})^T & 1 + \lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_d} \end{pmatrix}, \tag{3.8}$$

where $\mathbf{grad}_{\hat{\mathbf{x}}}\,\delta^*(\mathbf{x}) = (\partial\delta^*(\mathbf{x})/\partial x_1, \ldots, \partial\delta^*(\mathbf{x})/\partial x_{d-1})^T$ and 0 represents a column vector with $(d-1)$ zeros.

The function f in (3.6) can be regarded as a vector proxy of a compactly supported 0-form ω on $\overline{\Omega}$. From this perspective, $\mathbf{x} \mapsto f(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x}))$ turns out to be the vector proxy of the

[5] It is understood in the sequel as functions or differential forms compactly supported in $\mathbb{R}^d$ with restriction on Ω.

pullback $\mathscr{R}_\lambda^* f$. This immediately suggests the following generalization of (3.6) to a universal extension operator for smooth compactly supported l-forms on $\overline{\Omega}$:

$$\boxed{(\mathscr{E}_l\boldsymbol{\omega})(\mathbf{x}) := \begin{cases} \boldsymbol{\omega}(\mathbf{x}), & \mathbf{x}\in\overline{\Omega}; \\ \int_1^\infty (\mathscr{R}_\lambda^*\boldsymbol{\omega})(\mathbf{x})\psi(\lambda)\,\mathrm{d}\lambda, & \mathbf{x}\in\overline{\Omega}^c. \end{cases}} \tag{3.9}$$

For the remainder of this section we fix an increasing l-permutation $I=(i_1,\dots,i_l)$ with $1\leqslant i_1<\dots<i_l\leqslant d$. For a compactly supported differential l-form $\boldsymbol{\omega}\in\mathcal{DF}^{l,\infty}(\overline{\Omega})$ we have

$$(\mathscr{E}_l\boldsymbol{\omega})_I(\mathbf{x}) := (\mathscr{E}_l\boldsymbol{\omega})(\mathbf{x})(e_{i_1},\dots,e_{i_l}) = \int_1^\infty \big(\mathscr{R}_\lambda^*\boldsymbol{\omega}\big)(\mathbf{x})(e_{i_1},\dots,e_{i_l})\psi(\lambda)\,\mathrm{d}\lambda$$

$$= \int_1^\infty \big(\boldsymbol{\omega}\big(\mathscr{R}_\lambda(\mathbf{x})\big)\big)\big(D\mathscr{R}_\lambda(\mathbf{x})e_{i_1},\dots,D\mathscr{R}_\lambda(\mathbf{x})e_{i_l}\big)\psi(\lambda)\,\mathrm{d}\lambda.$$

From (3.8) we infer

$$\big(D\mathscr{R}_\lambda(\mathbf{x})\big)e_{i_k} = e_{i_k} + \lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_{i_k}}e_d \quad \text{for } 1\leqslant k\leqslant l, \tag{3.10}$$

which yields

$$(\mathscr{E}_l\boldsymbol{\omega})_I(\mathbf{x}) = \mathfrak{K}(\mathbf{x}) + \sum_{k=1}^{l}(-1)^{l-k}\mathfrak{J}_{i_k}(\mathbf{x}), \tag{3.11}$$

where we have used the abbreviations

$$\mathfrak{K}(\mathbf{x}) := \int_1^\infty \big(\boldsymbol{\omega}_I\big(\mathscr{R}_\lambda(\mathbf{x})\big)\big)\psi(\lambda)\,\mathrm{d}\lambda, \tag{3.12}$$

$$\mathfrak{J}_i(\mathbf{x}) := \frac{\partial\delta^*(\mathbf{x})}{\partial x_i}\int_1^\infty \big(\boldsymbol{\omega}_{\check{I}_i\cup\{d\}}\big(\mathscr{R}_\lambda(\mathbf{x})\big)\big)\lambda\psi(\lambda)\,\mathrm{d}\lambda, \quad i=1,2,\dots,d, \tag{3.13}$$

and by $\check{I}_{i_k}\cup\{d\}$ we designate the increasing l-permutation $1\leqslant i_1<\dots<\check{i}_k<\dots<i_l<d$ with i_k dropped and d included. For $i_l=d$, we have a simpler representation, viz.,

$$(\mathscr{E}_l\boldsymbol{\omega})_I(\mathbf{x}) = \mathfrak{K}(\mathbf{x}) + \mathfrak{J}_d(\mathbf{x}). \tag{3.14}$$

For $\boldsymbol{d}\boldsymbol{\omega}$, using the commuting diagram property of exterior derivative and the parametrized reflection mapping $\mathscr{R}_\lambda$ used in $\mathscr{E}_l$, we derive

$$\boldsymbol{d}(\mathscr{E}_l\boldsymbol{\omega})(\mathbf{x}) = \int_1^\infty \boldsymbol{d}\big(\mathscr{R}_\lambda^*\boldsymbol{\omega}\big)(\mathbf{x})\psi(\lambda)\,\mathrm{d}\lambda = \int_1^\infty \mathscr{R}_\lambda^*(\boldsymbol{d}\boldsymbol{\omega})(\mathbf{x})\psi(\lambda)\,\mathrm{d}\lambda, \tag{3.15}$$

which implies

$$\boldsymbol{d} \circ \mathscr{E}_l = \mathscr{E}_{l+1} \circ \boldsymbol{d}. \tag{3.16}$$

Before we proceed, we have to verify that $\mathscr{E}_l \boldsymbol{\omega}$ provides well-defined differential l-forms.

Lemma 3.4. *For a Lipschitz epigraph Ω, the extension formula in* (3.9) *is well defined in the sense that for compactly supported $\boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega})$,*

$$\mathscr{E}_l \boldsymbol{\omega} = \boldsymbol{\omega} \quad \text{in } \overline{\Omega} \quad \text{and} \quad \mathscr{E}_l \boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}\big(\mathbb{R}^d\big).$$

Proof. The bounded support and smoothness of $\boldsymbol{\omega}$ guarantee that $\mathscr{E}_l \boldsymbol{\omega}$ is well defined everywhere in $\mathbb{R}^d$. In particular, $\mathscr{E}_l \boldsymbol{\omega} = \boldsymbol{\omega}$ in $\overline{\Omega}$ due to the fact that the reflection mapping $\mathscr{R}_\lambda$ reduces to the identity operator.

The smoothness of δ^* and $\boldsymbol{\omega}$ along with the compact support of $\boldsymbol{\omega}$ ensures that $\mathscr{E}_l \boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\Omega \cup \Omega^c)$. It suffices to prove, for every I, continuity of all derivatives of $(\mathscr{E}_l \boldsymbol{\omega})_I \in C^\infty(\mathbb{R}^d)$ across $\partial\Omega$.

Since $\mathfrak{K}$ from (3.12) is the standard Stein extension (3.6) of $\boldsymbol{\omega}_I$, this term enjoys the required smoothness. For the $\mathfrak{J}_i$ terms from (3.13) we show that they vanish of infinite order on the boundary $\partial\Omega$. These $\mathfrak{J}_i$ terms are of the form

$$\mathfrak{J}(\mathbf{x}) = \frac{\partial \delta^*(\mathbf{x})}{\partial x_i} \int_1^\infty f\big(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})\big)\lambda\psi(\lambda)\,\mathrm{d}\lambda \quad \text{where } \mathbf{x} \in \mathbb{R}^d,\ f \in C_0^\infty(\overline{\Omega}). \tag{3.17}$$

Thanks to $C_0^\infty(\overline{\Omega})$ and the decay properties of ψ we conclude $\mathfrak{J} \in C^\infty(\Omega \cup \Omega^c)$. Moreover, by Lemma 3.2 $\mathfrak{J}(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega$. Hence we have to show $\frac{\partial^\alpha \mathfrak{J}}{\partial \mathbf{x}^\alpha}(\mathbf{x}) \to 0$ as $\delta(\mathbf{x}) \to 0$ (for $\mathbf{x} \in \Omega^c$) for any multi-index α. By the product rule, Leibniz rule and the chain rule of differentiation, we can write a generic partial derivative

$$\begin{aligned}
\frac{\partial^\alpha \mathfrak{J}(\mathbf{x})}{\partial \mathbf{x}^\alpha} &= \int_1^\infty \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha}\left(\frac{\partial \delta^*}{\partial x_i}(\mathbf{x}) f\big(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})\big)\lambda\psi(\lambda)\right)\mathrm{d}\lambda \\
&= \int_1^\infty \sum_{r=1}^{M(\alpha)} \left(P_r(\lambda)\frac{\partial^{\gamma_r}\delta^*}{\partial \mathbf{x}^{\gamma_r}}(\mathbf{x}) G\big(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})\big)\right)\lambda\psi(\lambda)\,\mathrm{d}\lambda,
\end{aligned} \tag{3.18}$$

where $P_r(\lambda)$ is a polynomial in λ, $G(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})) := \frac{\partial^{\beta_r}}{\partial \mathbf{x}^{\beta_r}} f(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x}))$, and $M(\alpha)$ is a fixed integer depending on α. Taylor expansion of G w.r.t. λ around $\lambda = 1$ yields

$$\begin{aligned}
G\big(\hat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})\big) &= \sum_{m=0}^{|\gamma_r|_1 - 1} \frac{1}{m!}\big[\delta^*(\mathbf{x})\big]^m (\lambda - 1)^m \frac{\partial^m}{\partial x_d^m} G\big(\hat{\mathbf{x}}, y + \delta^*(\mathbf{x})\big) \\
&\quad + \int_1^\lambda \frac{1}{(|\gamma_r|_1 - 1)!}\big[\delta^*(\mathbf{x})\big]^{|\gamma_r|_1}(\lambda - \tau)^{|\gamma_r|_1 - 1}\frac{\partial^{|\gamma_r|_1}}{\partial x_d^{|\gamma_r|_1}} G\big(\hat{\mathbf{x}}, y + \tau\delta^*(\mathbf{x})\big)\,\mathrm{d}\tau.
\end{aligned} \tag{3.19}$$

The first term on the right hand side of (3.19) is a polynomial in λ and thus its substitution into (3.18) vanishes on the boundary due to Lemma 3.2. Substituting the factor $[\delta^*(\mathbf{x})]^{|\gamma_r|_1}$ in the second term of (3.19) into (3.18) leads to the cancellation of the singularities of higher order derivatives of the smoothed distance function δ^*, see Lemmas 3.1 and 3.3:

$$\left|\frac{\partial^{\gamma_r}\delta^*}{\partial\mathbf{x}^{\gamma_r}}(\mathbf{x})\cdot\left[\delta^*(\mathbf{x})\right]^{|\gamma_r|_1}\right| \leqslant C\delta^*(\mathbf{x}).$$

This key fact makes all the remainder terms vanish uniformly on the boundary in the limit $\operatorname{dist}(\mathbf{x};\partial\Omega)\to 0$. $\square$

3.3. Continuity of extension operators

Theorem 3.5. *Let Ω be a Lipschitz epigraph in $\mathbb{R}^d$, $k\in\mathbb{N}_0$ and $0\leqslant l\leqslant d$. Then the extension operator* (3.9) *satisfies*

$$\|\mathscr{E}_l\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \leqslant C\|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)} \quad \forall \text{ compactly supported } \boldsymbol{\omega}\in\mathcal{DF}^{l,\infty}(\overline{\Omega}),$$

with a constant $C=C(\Omega,d,k,l)>0$. Thus $\mathscr{E}_l$ can be extended to a continuous extension operator

$$\mathscr{E}_l:\boldsymbol{H}^k\big(\boldsymbol{d},\Omega,\Lambda^l\big)\mapsto\boldsymbol{H}^k\big(\boldsymbol{d},\mathbb{R}^d,\Lambda^l\big).$$

Proof. The second assertion relies on a density argument since compactly supported differential forms in $\mathcal{DF}^{l,\infty}(\overline{\Omega})\cap\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)$ form a dense subset of $\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)$.

It remains to show the continuity of the extension operator. We only show the case of $k=2$. Let us exemplify the estimate of the following second order derivative of a typical part $\mathfrak{K}$ of $(\mathscr{E}_l\boldsymbol{\omega})_I$ in detail:

$$\begin{aligned}\frac{\partial^2\mathfrak{K}}{\partial x_j^2} &= \int_1^\infty\left(\frac{\partial^2((\mathscr{E}_l\boldsymbol{\omega})_I)(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_j^2}\right)\psi(\lambda)\,\mathrm{d}\lambda + \int_1^\infty\left(\frac{\partial^2((\mathscr{E}_l\boldsymbol{\omega})_I)(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_j\partial x_d}\right)\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_j}\psi(\lambda)\,\mathrm{d}\lambda\\ &\quad+\int_1^\infty\left(\frac{\partial^2((\mathscr{E}_l\boldsymbol{\omega})_I)(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d^2}\right)\left(\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_j}\right)^2\psi(\lambda)\,\mathrm{d}\lambda\\ &\quad+\int_1^\infty\left(\frac{\partial((\mathscr{E}_l\boldsymbol{\omega})_I)(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d}\right)\lambda\frac{\partial^2\delta^*(\mathbf{x})}{\partial x_j^2}\psi(\lambda)\,\mathrm{d}\lambda.\end{aligned}\tag{3.20}$$

Consider a boundary point $(\hat{\mathbf{x}}^0,y^0)\in\partial\Omega$, and assume without loss of generality that $0=y^0=\phi(\hat{\mathbf{x}}^0)$. Using the upper bounds of $\psi(\lambda)\leqslant C_1/\lambda^2, C_2/\lambda^3, C_3/\lambda^4$, respectively, the terms on the right hand side of (3.20) can be bounded as follows:

$$\left|\frac{\partial^2 \mathfrak{K}(\hat{\mathbf{x}}^0, y)}{\partial x_j^2}\right| \leqslant C \int_1^\infty \left(\left|\left(\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, y)}{\partial x_j^2}\right)\right| + \left|\left(\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, y)}{\partial x_j x_d}\right)\right| + \left|\left(\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, y)}{\partial x_d^2}\right)\right|\right) \frac{1}{\lambda^2}\,\mathrm{d}\lambda$$
$$+ \left|\int_1^\infty \left(\frac{\partial \boldsymbol{\omega}_I(\mathscr{R}_\lambda(\hat{\mathbf{x}}^0, y))}{\partial x_d}\right)\lambda \frac{\partial^2 \delta^*(\hat{\mathbf{x}}^0, y)}{\partial x_j^2}\psi(\lambda)\,\mathrm{d}\lambda\right|. \tag{3.21}$$

Only the last term has to be dealt with separately. Using the Taylor expansion with integral residual, we have

$$\frac{\partial \boldsymbol{\omega}_I}{\partial x_d}\big(\mathscr{R}_\lambda\big(\hat{\mathbf{x}}^0, y\big)\big) = \frac{\partial \boldsymbol{\omega}_I}{\partial x_d}\big(\hat{\mathbf{x}}^0, y + \delta^*\big(\hat{\mathbf{x}}^0, y\big)\big) + \int_{y+\delta^*(\hat{\mathbf{x}}^0, y)}^{y+\lambda\delta^*(\hat{\mathbf{x}}^0, y)} \frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_d^2}\big(\hat{\mathbf{x}}^0, s\big)\,\mathrm{d}s.$$

Substituting this in (3.21), we know that the integral term involving $\partial \boldsymbol{\omega}_I/\partial x_d(\hat{\mathbf{x}}^0, y + \delta^*(\hat{\mathbf{x}}^0, y))$ vanishes due to Lemma 3.2. Hence, it suffices to show the following bound (note that $|\partial^2\delta^*/\partial x_j^2| \leqslant C|\delta(\hat{\mathbf{x}}^0, y)|^{-1} \leqslant C|y|^{-1}$):

$$\begin{aligned}
&|y|^{-1}\int_1^\infty \left\{\int_{y+\delta^*(\hat{\mathbf{x}}^0, y)}^{y+\lambda\delta^*(\hat{\mathbf{x}}^0, y)} \left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right| \mathrm{d}s\right\}\frac{1}{\lambda^3}\,\mathrm{d}\lambda \\
&= |y|^{-1}\int_{y+\delta^*(\hat{\mathbf{x}}^0, y)}^{\infty} \left\{\int_{(s-y)/\delta^*(\hat{\mathbf{x}}^0, y)}^{\infty} \left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right| \frac{1}{\lambda^3}\,\mathrm{d}\lambda\right\}\mathrm{d}s \\
&\leqslant |y|^{-1}\big(\delta^*\big(\hat{\mathbf{x}}^0, y\big)\big)^2 \int_{y+\delta^*(\hat{\mathbf{x}}^0, y)}^{\infty} \left\{\left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right|\right\}\frac{1}{(s-y)^2}\,\mathrm{d}s \\
&\leqslant C|y|\int_{|y|}^{\infty} \left\{\left|\frac{\partial^2 \boldsymbol{\omega}_I(\hat{\mathbf{x}}^0, s)}{\partial x_d^2}\right|\right\}\frac{1}{s^2}\,\mathrm{d}s,
\end{aligned}$$

where we have interchanged the order of integration for the first equality, and used that $\delta^*(\hat{\mathbf{x}}^0, y) \leqslant C|y|$, $\delta^*(\hat{\mathbf{x}}^0, y) \geqslant 2|y|$ and $s - y \geqslant s$ when $y < 0$ for the second inequality. Now we can appeal to the following Hardy inequality [27, p. 272]

$$\left(\int_0^\infty \left(\int_x^\infty f(y)\,\mathrm{d}y\right)^p x^{r-1}\,\mathrm{d}x\right)^{1/p} \leqslant \frac{p}{r}\left(\int_0^\infty \big(yf(y)\big)^p y^{r-1}\,\mathrm{d}y\right)^{1/p}, \quad \forall f \geqslant 0,\ p \geqslant 1,\ r > 0$$

for (3.21) and integrate over all $\hat{\mathbf{x}} \in \mathbb{R}^{d-1}$ to obtain

$$\left\|\frac{\partial^2 (\mathscr{E}_l\boldsymbol{\omega})_I}{\partial x_j^2}\right\|^2_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^l)} \leqslant C\left(\left\|\frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_j^2}\right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)} + \left\|\frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_j x_d}\right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)} + \left\|\frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_d^2}\right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)}\right).$$

Analogously by the commuting diagram property, we have

$$\left\| \frac{\partial^2 \boldsymbol{d}(\mathscr{E}\boldsymbol{\omega})_I}{\partial x_j^2} \right\|^2_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^{l+1})} \leqslant C\left(\left\| \frac{\partial^2 \boldsymbol{d}\boldsymbol{\omega}_I}{\partial x_j^2} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^{l+1})} + \left\| \frac{\partial^2 \boldsymbol{d}\boldsymbol{\omega}_I}{\partial x_j \partial x_d} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^{l+1})} + \left\| \frac{\partial^2 \boldsymbol{d}\boldsymbol{\omega}_I}{\partial x_d^2} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^{l+1})} \right).$$

This completes the proof of the case $k = 2$.

For $k = 0$ or for general $k > 0$, differentiating (3.9) gives various order partial derivatives of the components of $\boldsymbol{\omega}$. Whenever the total differential order of $\boldsymbol{\omega}$ is less than k, we always use the Taylor expansion around the point $(\hat{\mathbf{x}}^0, y + \delta^*(\hat{\mathbf{x}}^0, y))$ and carry it up to order k with integral remainders and proceed the arguments as above. We note that the constant C involved only depends on the domain Ω, the dimension d, the order of differentiability k and the degree of differential forms l. Thus the proof is done. □

The general situation of a compact Lipschitz boundary can be tackled by a partition of unity subordinate to a finite cover of $\partial\Omega$ in the usual way. This yields our main result:

Theorem 3.6. *Let Ω be a domain with a bounded Lipschitz domain, $k \in \mathbb{N}_0$ and $0 \leqslant l \leqslant d$. Then there exists a universal extension operator*

$$\mathscr{E}_l : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$$

satisfying

1. *$\mathscr{E}_l\boldsymbol{\omega} = \boldsymbol{\omega}$ a.e. in Ω, and*
2. *the extension operator is continuous*

$$\|\mathscr{E}_l\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \leqslant C\|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)} \quad \forall \boldsymbol{\omega} \in \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l),$$

with the constant $C = C(\Omega, d, k, l)$, but independent of the differential forms involved.

Remark 3.1. Theorem 3.6 also holds for a domain Ω whose complement $\overline{\Omega}^c$ is a bounded Lipschitz domain. It is further pointed out that the commuting diagram property (3.16) no longer holds for $\mathscr{E}_l$ for general bounded Lipschitz domains due to the use of a partition of unity.

Remark 3.2. Costabel and McIntosh have recently introduced some so-called smoothed Poincaré liftings in [11]. Those offer an alternative way to define universal extension operators based on standard extension for the Sobolev spaces $H^s(\Omega)$.

4. Vector field perspective

In three-dimensional Euclidean space, we may represent the differential forms in terms of their so-called vector proxies, as shown in Table 1.

The concept of Euclidean vector proxies establishes a one-to-one correspondence between Sobolev spaces of scalar/vector functions and Sobolev spaces of differential forms, see Table 2.

Table 1
Relationship between differential forms and vectorfields (“vector proxies”) in three-dimensional Euclidean space ($\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$). The operation $\cdot$ is the canonical inner product in Euclidean space.

Differential form	Related function u/vectorfield $\mathbf{u}$
$\mathbf{x} \mapsto \boldsymbol{\omega}(\mathbf{x})$	$u(\mathbf{x}) := \boldsymbol{\omega}(\mathbf{x})$
$\mathbf{x} \mapsto \{\mathbf{v} \mapsto \boldsymbol{\omega}(\mathbf{x})(\mathbf{v})\}$	$\mathbf{u}(\mathbf{x}) \cdot \mathbf{v} := \boldsymbol{\omega}(\mathbf{x})(\mathbf{v})$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\mathbf{u}(\mathbf{x}) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$u(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

Table 2
Correspondence between Sobolev spaces of functions/fields and Sobolev spaces of differential forms in $\mathbb{R}^3$.

l	Sobolev spaces of functions	Sobolev spaces of differential forms
0	$H^{k+1}(\Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^0)$
1	$\boldsymbol{H}^k(\mathbf{curl}; \Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^1)$
2	$\boldsymbol{H}^k(\mathrm{div}; \Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^2)$
3	$H^k(\Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^3)$

Now we give special incarnations of the extension operators $\mathscr{E}_l$, $0 \leqslant l \leqslant 3$, for Lipschitz epigraphs $\Omega \subset \mathbb{R}^3$ from (3.9) in terms of vector proxies in $\mathbb{R}^3$. Of course, for $l = 0$ we recover Stein’s formula (3.6).

In the case $l = 1$, that is, for a covector field $\mathbf{u} \in \boldsymbol{H}^k(\mathbf{curl}; \Omega)$ ($k \in \mathbb{N}_0$), we have

$$\begin{aligned} \mathscr{E}_1 \mathbf{u}(\mathbf{x}) &= \int_1^\infty \big(D\mathscr{R}_\lambda(\mathbf{x})\big)^T \mathbf{u}(\hat{\mathbf{x}}, \bullet)\psi(\lambda)\,\mathrm{d}\lambda \\ &= \int_1^\infty \big(\mathbf{u}(\hat{\mathbf{x}}, \bullet) + \lambda u_3(\hat{\mathbf{x}}, \bullet)\,\mathbf{grad}\,\delta^*(\mathbf{x})\big)\psi(\lambda)\,\mathrm{d}\lambda \end{aligned} \tag{4.1}$$

for $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \overline{\Omega}^c$, where $\bullet$ stands for $y + \lambda\delta^*(\mathbf{x})$ and u_3 is the third component of $\mathbf{u}$.

For $l = 2$, that is, for a bivector field $\mathbf{u} \in \boldsymbol{H}^k(\mathrm{div}; \Omega)$ ($k \in \mathbb{N}_0$), we have

$$\begin{aligned} \mathscr{E}_2 \mathbf{u}(\mathbf{x}) &= \int_1^\infty \big(D\mathscr{R}_\lambda(\mathbf{x})\big)^{-1} \det\big(D\mathscr{R}_\lambda(\mathbf{x})\big)\mathbf{u}(\hat{\mathbf{x}}, \bullet)\psi(\lambda)\,\mathrm{d}\lambda \\ &= \int_1^\infty \left(\left(1 + \lambda\frac{\partial\delta^*}{\partial x_3}(\mathbf{x})\right)\mathbf{u}(\hat{\mathbf{x}}, \bullet) - \begin{pmatrix} 0 \\ 0 \\ \lambda\,\mathbf{grad}\,\delta^*(\mathbf{x}) \cdot \mathbf{u}(\hat{\mathbf{x}}, \bullet) \end{pmatrix}\right)\psi(\lambda)\,\mathrm{d}\lambda \end{aligned} \tag{4.2}$$

for $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \overline{\Omega}^c$, where the occurrence of $D\mathscr{R}_\lambda(\mathbf{x})^{-1}$ is merely formal. In fact, we need the adjunct Jacobian matrix of the parametrized reflection mapping

$$\det\big(D\mathscr{R}_\lambda(\mathbf{x})\big)\big(D\mathscr{R}_\lambda(\mathbf{x})\big)^{-1} = \begin{pmatrix} 1+\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_3} & 0 & 0 \\ 0 & 1+\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_3} & 0 \\ -\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_1} & -\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_2} & 1 \end{pmatrix}. \tag{4.3}$$

Lastly, for any density function $u \in H^k(\Omega)$ ($k \in \mathbb{N}_0$) we have

$$\begin{aligned} \mathscr{E}_3 u(\mathbf{x}) &= \int_1^\infty \det\big(D\mathscr{R}_\lambda(\mathbf{x})\big) u(\hat{\mathbf{x}}, \bullet)\psi(\lambda)\,\mathrm{d}\lambda \\ &= \int_1^\infty \left(1+\lambda\frac{\partial\delta^*}{\partial x_3}(\mathbf{x})\right) u(\hat{\mathbf{x}}, \bullet)\psi(\lambda)\,\mathrm{d}\lambda \end{aligned} \tag{4.4}$$

for $\mathbf{x} = (\hat{\mathbf{x}}, y) \in \overline{\Omega}^c$.

To the best knowledge of the authors, the three formulae (4.1), (4.2) and (4.4) seem to be new to the mathematical community. Applying Theorem 3.6 for the Euclidean space $\mathbb{R}^3$ leads immediately to the following "vector analytic" specializations.

Corollary 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $k \in \mathbb{N}_0$. Then there exists a family of universal extension operators $\mathscr{E}_l$ ($l = 0, 1, 2, 3$) such that*

$$\mathscr{E}_0 : H^{k+1}(\Omega) \mapsto H^{k+1}(\Omega)\mathbb{R}^3 \quad \textit{satisfying} \begin{cases} \mathscr{E}_0 u = u, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_0 u\|_{H^{k+1}(\mathbb{R}^3)} \leqslant C\|u\|_{H^{k+1}(\Omega)}; \end{cases}$$

$$\mathscr{E}_1 : \boldsymbol{H}^k(\mathbf{curl};\Omega) \mapsto \boldsymbol{H}^k\big(\mathbf{curl};\mathbb{R}^3\big) \quad \textit{satisfying} \begin{cases} \mathscr{E}_1 \mathbf{u} = \mathbf{u}, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_1 \mathbf{u}\|_{\boldsymbol{H}^k(\mathbf{curl};\mathbb{R}^3)} \leqslant C\|\mathbf{u}\|_{\boldsymbol{H}^k(\mathbf{curl};\Omega)}; \end{cases}$$

$$\mathscr{E}_2 : \boldsymbol{H}^k(\mathrm{div};\Omega) \mapsto \boldsymbol{H}^k\big(\mathrm{div};\mathbb{R}^3\big) \quad \textit{satisfying} \begin{cases} \mathscr{E}_2 \mathbf{u} = \mathbf{u}, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_2 \mathbf{u}\|_{\boldsymbol{H}^k(\mathrm{div};\mathbb{R}^3)} \leqslant C\|\mathbf{u}\|_{\boldsymbol{H}^k(\mathrm{div};\Omega)}; \end{cases}$$

$$\mathscr{E}_3 : H^k(\Omega) \mapsto H^k\big(\mathbb{R}^3\big) \quad \textit{satisfying} \begin{cases} \mathscr{E}_3 u = u, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_3 u\|_{H^k(\mathbb{R}^3)} \leqslant C\|u\|_{H^k(\Omega)} \end{cases}$$

with all the constants $C = C(k, \Omega)$ independent of the functions/fields involved.

5. Application: Regular decompositions

Regular decomposition results for $\boldsymbol{H}(\mathrm{div};\Omega)$ and $\boldsymbol{H}(\mathbf{curl};\Omega)$ and related spaces assert that those can be split into the kernel of the underlying differential operator and a complement space of H^1-regular functions. Regular decompositions, pioneered in [4], have become a powerful tool in mathematical analysis [10,6] and numerical analysis, see [16, Sect. 2.4] and the references given there.

In this section, we apply the universal extension result to establish regular decompositions of Sobolev spaces of differential forms. As a consequence, a well-known lifting lemma can be generalized to Sobolev spaces of differential forms. Throughout this section we only consider $d \geqslant 3$ and $\Omega \subset \mathbb{R}^d$ is always assumed to be a bounded Lipschitz domain.

As a tool for the subsequent analysis, we introduce the Fourier transform $\mathscr{F}$ and its inverse transform $\mathscr{F}^{-1}$. For a differential l-form $\boldsymbol{\omega} = \sum_I \boldsymbol{\omega}_I \boldsymbol{d}\mathbf{x}_I \in \boldsymbol{L}^2(\mathbb{R}^d; \Lambda^l)$, we define its Fourier transform componentwise by (cf. [28])

$$\hat{\boldsymbol{\omega}}(\boldsymbol{\xi}) := \mathscr{F}(\boldsymbol{\omega})(\boldsymbol{\xi}) = \sum_I \hat{\boldsymbol{\omega}}_I(\boldsymbol{\xi}) \boldsymbol{d}\boldsymbol{\xi}_I$$

where the component $\hat{\boldsymbol{\omega}}_I(\boldsymbol{\xi})$ is given by

$$\mathscr{F}(\boldsymbol{\omega}_I)(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\imath \boldsymbol{\xi} \cdot \mathbf{x}) \boldsymbol{\omega}_I(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Here $\imath$ is the imaginary unit, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^T$ is the vectorial angular frequency in $\mathbb{R}^d$ and $\boldsymbol{d}\boldsymbol{\xi}_I = \boldsymbol{d}\xi_{i_1} \wedge \cdots \wedge \boldsymbol{d}\xi_{i_l}$, with I being an increasing l-permutation.

It is easy to see that the Fourier transform converts the exterior derivative into an exterior product. More precisely for any $\boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}, \Omega, \Lambda^l)$, we have

$$\mathscr{F}(\boldsymbol{d}\boldsymbol{\omega}) = \imath \hat{\boldsymbol{\xi}} \wedge \mathscr{F}(\boldsymbol{\omega}), \tag{5.1}$$

where $\hat{\boldsymbol{\xi}}$ is the differential 1-form in the frequency domain, namely $\hat{\boldsymbol{\xi}} = \xi_1 \boldsymbol{d}\xi_1 + \xi_2 \boldsymbol{d}\xi_2 + \cdots + \xi_d \boldsymbol{d}\xi_d$.

The regular decomposition we are going to discuss relies on the existence of regular potentials in $\mathbb{R}^d$.

Lemma 5.1 *(Existence of regular potentials in $\mathbb{R}^d$). For $1 \leqslant l \leqslant d$, $l \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ there is a continuous lifting mapping*

$$\mathscr{L} : \boldsymbol{H}\big(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l\big) \cap \boldsymbol{H}^k\big(\mathbb{R}^d, \Lambda^l\big) \mapsto \boldsymbol{H}^{k+1}_{\mathrm{loc}}\big(\mathbb{R}^d, \Lambda^{l-1}\big)$$

such that for all $\boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l) \cap \boldsymbol{H}^k(\mathbb{R}^d, \Lambda^l)$,

$$\boldsymbol{d}\mathscr{L}\boldsymbol{\omega} = \boldsymbol{\omega}. \tag{5.2}$$

Proof. We follow the idea in the proof of [2, Lemma 3.5, p. 837]. It boils down to straightforward calculations with Fourier transforms of differential forms.

Let $\boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l) \cap \boldsymbol{H}^k(\mathbb{R}^d, \Lambda^l)$, i.e., $\boldsymbol{d}\boldsymbol{\omega} = 0$. We try to seek an $\boldsymbol{\eta} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$ such that for any compact $D \subset \mathbb{R}^d$,

$$\boldsymbol{d}\boldsymbol{\eta} = \boldsymbol{\omega} \quad \text{and} \quad \|\boldsymbol{\eta}\|_{\boldsymbol{H}^{k+1}(D, \Lambda^{l-1})} \leqslant C \|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(D, \Lambda^l)}. \tag{5.3}$$

Taking the Fourier transform on both sides of the equations $\boldsymbol{d}\boldsymbol{\eta} = \boldsymbol{\omega}$ and $\boldsymbol{d}\boldsymbol{\omega} = 0$, and using (5.1) we derive

$$\imath \hat{\boldsymbol{\xi}} \wedge \hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\omega}}, \qquad \imath \hat{\boldsymbol{\xi}} \wedge \hat{\boldsymbol{\omega}} = 0. \tag{5.4}$$

This linear system (5.4) has a solution that can be explicitly expressed as (cf. [17] for details)

$$\hat{\boldsymbol{\eta}}(\boldsymbol{\xi}) := \frac{-\imath\boldsymbol{\xi} \lrcorner \hat{\boldsymbol{\omega}}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}. \tag{5.5}$$

It remains to show that $\boldsymbol{\eta} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$. We will use the cut-off technique as in the proof of [2, Lemma 3.5]. First we observe that for any increasing l-permutation I and any j with $1 \leqslant j \leqslant d$, we get from (5.5)

$$\left|\xi_j \hat{\boldsymbol{\eta}}_I(\boldsymbol{\xi})\right| \leqslant \sum_J \left|\hat{\boldsymbol{\omega}}_J(\boldsymbol{\xi})\right|. \tag{5.6}$$

Appealing to the Fourier representation of Sobolev norms on $\mathbb{R}^d$, we can conclude $\frac{\partial \eta_I}{\partial x_j} \in H^k(\mathbb{R}^d)$ for all combinations of I and j.

Next we can choose a cut-off function $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi(\boldsymbol{\xi}) = 1$ for $|\boldsymbol{\xi}| \leqslant 1$, and $\psi(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}| \geqslant 2$. Then split $\hat{\boldsymbol{\omega}}$ according to

$$\hat{\boldsymbol{\eta}}(\boldsymbol{\xi}) = \psi(\boldsymbol{\xi})\hat{\boldsymbol{\eta}}(\boldsymbol{\xi}) + \big(1 - \psi(\boldsymbol{\xi})\big)\hat{\boldsymbol{\eta}}(\boldsymbol{\xi}). \tag{5.7}$$

Note that each component of the differential form $\psi(\boldsymbol{\xi})\hat{\boldsymbol{\eta}}(\boldsymbol{\xi})$ has a compact support and belongs to $L^1(\mathbb{R}^d)$ ($d \geqslant 3$!), so that its inverse Fourier transform is analytic. Hence, the restriction of $\mathscr{F}^{-1}(\psi(\cdot)\hat{\boldsymbol{\eta}}(\cdot))$ to any compact $D \subset \mathbb{R}^d$ belongs to $H^m(D)$ for any $m \in \mathbb{N}_0$. It goes without saying that the inverse Fourier transform of the second term $(1 - \psi(\boldsymbol{\xi}))\hat{\boldsymbol{\eta}}(\boldsymbol{\xi})$ yields a form in $\boldsymbol{H}^k(\mathbb{R}^d, \Lambda^{l-1})$. Summing up, we have shown that $\mathscr{F}^{-1}(\hat{\boldsymbol{\eta}}(\cdot)) \in \boldsymbol{H}^k_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$. This completes our proof. □

The following theorem is a fairly straightforward generalization of the regular decomposition lemma [16, Lemma 2.4]:

Theorem 5.2 *(Lifted regular decompositions). For every $k \in \mathbb{N}_0$, $1 \leqslant l \leqslant d$, there exist continuous maps $\mathsf{R} : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}(\Omega, \Lambda^l)$ and $\mathsf{N} : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}(\Omega, \Lambda^{l-1})$ such that*

$$\mathsf{R} + \boldsymbol{d} \circ \mathsf{N} = Id \quad \text{on } \boldsymbol{H}^k\big(\boldsymbol{d}, \Omega, \Lambda^l\big). \tag{5.8}$$

In addition, there are continuous maps $\mathsf{R}_0 : \boldsymbol{H}^k_0(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}_0(\Omega, \Lambda^l)$ and $\mathsf{N}_0 : \boldsymbol{H}^k_0(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}_0(\Omega, \Lambda^{l-1})$ such that

$$\mathsf{R}_0 + \boldsymbol{d} \circ \mathsf{N}_0 = Id \quad \text{on } \boldsymbol{H}^k_0\big(\boldsymbol{d}, \Omega, \Lambda^l\big). \tag{5.9}$$

Proof. We first prove (5.8): pick $\boldsymbol{\omega} \in \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$ and extend it to $\tilde{\boldsymbol{\omega}} \in \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ using the universal extension of Theorem 3.6. Then set

$$\mathsf{R}\boldsymbol{\omega} := (\mathscr{L}\boldsymbol{d}\tilde{\boldsymbol{\omega}})|_\Omega, \qquad \mathsf{N}\boldsymbol{\omega} := \mathscr{L}(\tilde{\boldsymbol{\omega}} - \mathscr{L}\boldsymbol{d}\tilde{\boldsymbol{\omega}})|_\Omega. \tag{5.10}$$

It is easy to check that $\boldsymbol{d}(\tilde{\boldsymbol{\omega}} - \mathsf{R}\tilde{\boldsymbol{\omega}}) = 0$ in Ω in view of (5.2). The continuity properties of these operators and (5.8) are straightforward from Lemma 5.1.

Next we show (5.9) following the proof of Lemma 2.4 in [16]: for any $\boldsymbol{\mu} \in \boldsymbol{H}_0^k(\boldsymbol{d}, \Omega, \Lambda^l)$, we extend it by zero to $\tilde{\boldsymbol{\mu}} \in \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ and define $\tilde{\boldsymbol{\omega}} = \boldsymbol{d}\tilde{\boldsymbol{\mu}} \in \boldsymbol{H}^k(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^{l+1})$. There exist from Lemma 5.1 $\boldsymbol{\eta} \in \boldsymbol{H}_{\mathrm{loc}}^{k+1}(\mathbb{R}^d, \Lambda^l)$ and $\tilde{\boldsymbol{\omega}} = \boldsymbol{d}\boldsymbol{\eta}$, which implies that $\boldsymbol{d}(\tilde{\boldsymbol{\mu}} - \boldsymbol{\eta}) = 0$. Applying Lemma 5.1 again yields $\boldsymbol{\rho} \in \boldsymbol{H}_{\mathrm{loc}}^{k+1}(\mathbb{R}^d, \Lambda^l)$ satisfying $\tilde{\boldsymbol{\mu}} - \boldsymbol{\eta} = \boldsymbol{d}\boldsymbol{\rho}$. The fact that $\tilde{\boldsymbol{\mu}} = 0$ in $\mathbb{R}^d \setminus \Omega$ leads to $\boldsymbol{\rho} \in \boldsymbol{H}_{\mathrm{loc}}^{k+1}(\boldsymbol{d}, \mathbb{R}^d \setminus \Omega, \Lambda^l)$. By extending $\boldsymbol{\rho}|_{\mathbb{R}^d \setminus \Omega}$ into the *interior* of Ω by Theorem 3.6, denoted by $\tilde{\boldsymbol{\rho}}$, we can define

$$\mathsf{R}_0\boldsymbol{\mu} := \boldsymbol{\eta} + \boldsymbol{d}\tilde{\boldsymbol{\rho}}_{\cdot|\Omega} \in \boldsymbol{H}^{k+1}(\Omega, \Lambda^l), \qquad \mathsf{N}_0\boldsymbol{\mu} := \boldsymbol{\rho} - \tilde{\boldsymbol{\rho}} \in \boldsymbol{H}^{k+1}(\Omega, \Lambda^{l-1}).$$

The identity (5.9) is a consequence of the construction. Continuity of the extension translates into the asserted continuity properties of the operators. Finally, we note that $\boldsymbol{\rho} - \tilde{\boldsymbol{\rho}} = 0$ on $\mathbb{R}^d \setminus \Omega$, which implies the homogeneous boundary condition for $\mathsf{N}_0\boldsymbol{\mu}$. Similarly, we have $\boldsymbol{\eta} + \boldsymbol{d}\tilde{\boldsymbol{\rho}} = 0$ in $\mathbb{R}^d \setminus \Omega$, and $\boldsymbol{\eta} + \boldsymbol{d}\tilde{\boldsymbol{\rho}} \in \boldsymbol{H}_{\mathrm{loc}}^{k+1}(\mathbb{R}^d, \Lambda^l)$, hence $\mathsf{R}_0\boldsymbol{\mu} \in \boldsymbol{H}_0^{k+1}(\Omega, \Lambda^l)$. □

Let us recall the classical version of a lifting lemma which is important in the analysis of the Navier–Stokes equations, see Corollary 2.4 and Lemma 2.2 in [14].

Lemma 5.3 *(Classical lifting lemma). Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then for any $p \in L^2(\Omega)$ and $p \in L^2(\Omega)$ with $\int_\Omega p\,\mathrm{d}\mathbf{x} = 0$, there exist $\mathbf{v} \in \boldsymbol{H}^1(\Omega)$ and $\mathbf{v} \in \boldsymbol{H}_0^1(\Omega)$, respectively, satisfying*

$$\operatorname{div}\mathbf{v} = p \quad \text{and} \quad \|\mathbf{v}\|_{H^1(\Omega)} \leqslant C\|p\|_{L^2(\Omega)}, \tag{5.11}$$

where C is a positive constant independent of p.

It is remarked that the original proof due to Nečas establishes an equivalent assertion by showing that the range space of **grad**, the adjoint operator of div, is closed in $\boldsymbol{H}_0^1(\Omega)$, which is the dual space of $\boldsymbol{H}^{-1}(\Omega)$ (cf. [24]). Nečas' proof is rather lengthy and quite complicated for the technical treatment of Lipschitz boundary. Lemma 5.3 is of crucial importance for the treatment of the constraints of compressibility and incompressibility in mechanics and fluid mechanics. With the regular decomposition lemma we have established earlier, a generalized version of Lemma 5.3 can be deduced easily.

Corollary 5.4 *(Generalized lifting lemma). Let $k \in \mathbb{N}_0$ and $1 \leqslant l \leqslant d$, and consider a bounded Lipschitz domain $\Omega \in \mathbb{R}^d$ of full topological generality. Then for all $\boldsymbol{\omega} \in \boldsymbol{d}\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^{l-1})$ and $\boldsymbol{d}\boldsymbol{H}_0^s(\boldsymbol{d}, \Omega, \Lambda^{l-1})$ $(1 \leqslant l < d)$ with $\int_\Omega \boldsymbol{\omega} = 0$ for $l = d$, there are an $\boldsymbol{\eta} \in \boldsymbol{H}^{s+1}(\Omega, \Lambda^{l-1})$ and $\boldsymbol{H}_0^{s+1}(\Omega, \Lambda^{l-1})$ respectively, and a positive constant C independent of $\boldsymbol{\eta}$ such that*

$$\boldsymbol{d}\boldsymbol{\eta} = \boldsymbol{\omega}, \tag{5.12}$$

$$\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{k+1}(\Omega, \Lambda^{l-1})} \leqslant C\|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\Omega, \Lambda^l)}. \tag{5.13}$$

Proof. By Theorem 5.2, we can define $\boldsymbol{\eta} = \mathsf{R}\boldsymbol{\omega}$ and $\boldsymbol{\eta} = \mathsf{R}_0\boldsymbol{\omega}$ to show (5.12) and (5.13), respectively. □

It is natural to derive from Corollary 5.4 a similar result for the **curl** operator to Lemma 5.3.

Corollary 5.5. *Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then for any* $\mathbf{v} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl};\Omega)$ *and* $\mathbf{v} \in \mathbf{curl}\, \boldsymbol{H}_0(\mathbf{curl};\Omega)$*, there exist* $\mathbf{u} \in \boldsymbol{H}^1(\Omega)$ *and* $\mathbf{u} \in \boldsymbol{H}_0^1(\Omega)$ *respectively, satisfying*

$$\mathbf{curl}\,\mathbf{u} = \mathbf{v} \quad \textit{and} \quad \|\mathbf{u}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\|\mathbf{v}\|_{\boldsymbol{L}^2(\Omega)}, \tag{5.14}$$

where C is a positive constant independent of $\mathbf{v}$.

Remark 5.1. Theorem 5.2 also holds for $d = 2$, but with a different proof invoking analyticity of Fourier transforms (cf. [14, Sect. I.3.1]).

Remark 5.2. We emphasize that Theorem 5.2 can also be deduced from [11, Theorem 4.6] by appealing to the so-called Bogovskii operators.

Acknowledgments

The authors are grateful to the anonymous referee for many constructive comments that have led to a much improved presentation of the paper.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, London, 1975.
[2] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (1988) 823–864.
[3] D.N. Arnold, R.S. Falk, R. Winther, Finite element exterior calculus, homological techniques, and applications, Acta Numer. (2006) 1–155.
[4] M. Birman, M. Solomyak, Construction in a piecewise smooth domain of a function of the class H^2 from the value of the conormal derivative, J. Math. Sov. 49 (1990) 1128–1136.
[5] A. Bossavit, Discretization of Electromagnetic Problems: The "Generalized Finite Differences" Approach, in: Handb. Numer. Anal., vol. 13, Elsevier, Amsterdam, 2005, pp. 105–197.
[6] A. Buffa, M. Costabel, D. Sheen, On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains, J. Math. Anal. Appl. 276 (2002) 845–867.
[7] A.P. Calderón, Lebesgue spaces of differentiable functions and distributions, Proc. Sympos. Pure Math. 5 (1961) 33–49.
[8] E. Cartan, Sur certaines expressions différentielles et le problèe de pfaff, Ann. Sci. École Norm. Sup. 16 (1899) 239–332.
[9] H. Cartan, Differential Forms, Hermann/H. Mifflin Co., Paris, Boston, 1970.
[10] M. Costabel, A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains, Math. Methods Appl. Sci. 12 (1990) 365–368.
[11] M. Costabel, A. McIntosh, On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains, Math. Z. 265 (2) (2010) 297–320.
[12] S. Ding, Weighted imbedding theorems in the space of differential forms, J. Math. Anal. Appl. 262 (2001) 435–445.
[13] J. Dodziuk, Sobolev spaces of differential forms and de Rahm–Hodge isomorphism, J. Differential Geom. 16 (1981) 73–83.
[14] V. Girault, P. Raviart, Finite Element Methods for Navier–Stokes Equations, Springer, Berlin, 1986.
[15] M. Hestenes, Extension of the range of a differentiable function, Duke Math. J. 8 (1941) 183–192.
[16] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numer. 11 (2002) 237–339.
[17] R. Hiptmair, J. Li, J. Zou, Universal extension for Sobolev spaces of differential forms and applications, Tech. Report, 2009-22, SAM ETH, Zürich, ftp://ftp.scsc.ethz.ch/pub/sam-reports/reports/reports2009/2009-22.pdf.
[18] T. Iwaniec, Nonlinear Differential Forms, Lect. Notes International Summer School in Jyväskylä, Department of Mathematics, University of Jyväskylä, Finland, 1998.

[19] N. Kalton, S. Mayboroda, M. Mitrea, Interpolation of Hardy–Sobolev–Besov–Triebel–Lizorkin spaces and applications to problems in partial differential equations, in: Interpolation Theory and Applications, Contemp. Math. 445 (2007) 121–177.
[20] M. Mitrea, Sharp hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds, Duke Math. J. 125 (2004) 467–547.
[21] M. Mitrea, S. Monniaux, The regularity of the stokes operator and the Fujita–Kato approach to the Navier–Stokes initial value problem in Lipschitz domains, J. Funct. Anal. 254 (2008) 1522–1574.
[22] P. Monk, Finite Element Methods for Maxwell's Equations, Clarendon Press, Oxford, 2003.
[23] C.B.J. Morrey, Multiple Integrals in the Calculus of Variations, Springer, Berlin, 1966.
[24] J. Nečas, Équations aux dérivées partielles, Presses de l'Université de Montréal, Montréal, 1966.
[25] C. Scott, L^p theory of differential forms on manifolds, Trans. Amer. Math. Soc. 347 (1995) 2075–2096.
[26] S. Seeley, Extensions of C^∞ functions defined on a half space, Proc. Amer. Math. Soc. 15 (1961) 625–626.
[27] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.
[28] E.M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton University Press, Princeton, 1971.
[29] V.S. Rychkov, On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains, J. London Math. Soc. 60 (1999) 237–257.
[30] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc. 36 (1934) 63–89.