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Convergence rates of Tikhonov regularizations for parameter identification in a parabolic-elliptic system

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Abstract

We shall study the convergence rates of the Tikhonov regularizations for the identification of the diffusivity $q(\mathbf{x})$ in a parabolic–elliptic system. The H^1 regularization and a mixed L^p–H^1 regularization are considered. For the H^1 regularization, we present a simple and easily interpretable source condition, under which the regularized solutions will be shown to converge at the standard rate in terms of the noise level of the data. The convergence is analyzed in three different approaches, which result in the same convergence rate but require quite different conditions on the measurement time and the identifying parameters. For the mixed L^p–H^1 regularization, we will achieve some desired convergence rate by using the Bregman distance and some new source condition and nonlinearity condition.

1. Introduction

In this work, we are interested in the following coupled parabolic–elliptic system:

$$\begin{cases} -\nabla \cdot (q_0(\mathbf{x})\nabla u) = 0 & \text{in} \quad \Omega \setminus \bar{\omega}, \\ \dfrac{\partial u}{\partial t} - \nabla \cdot (q(\mathbf{x})\nabla u) = 0 & \text{in} \quad \omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded and connected domain in $\mathbf{R}^d$ $(d \leqslant 3)$ occupied by two materials, one in a small domain ω that sits completely in the interior of Ω, and the other in $\Omega \setminus \bar{\omega}$. The two materials are of different physical properties: one with the physical diffusivity $q(\mathbf{x})$ in ω and the other with the diffusivity $q_0(\mathbf{x})$. Function $u(\mathbf{x}, t)$ represents the profile of some physical quantity at time t and location $\mathbf{x}$. The coupled parabolic–elliptic system (1.1) arises in many engineering and industrial applications; see [14–18] and references therein. One important application is from electromagnetic metal forming [16–18], where the evolution of the deformation field of a mechanical structure of some conducting material is coupled with an electromagnetic field that generates a Lorentz force, thus driving the metal forming process. Inside the conducting material region ω, diffusion of the electromagnetic field takes place, leading to a model equation of parabolic type, while the equilibrium state is instantaneously assumed outside, resulting in a model equation of elliptic type; see (1.1).

We shall complement the system (1.1) with the following initial and boundary conditions:

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \text{in} \quad \omega; \quad u(\mathbf{x}, t) = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \tag{1.2}$$

and the following physical interface condition on the interface $\partial\omega$:

$$u^- = u^+, \quad q_0 \frac{\partial u^-}{\partial n} = q \frac{\partial u^+}{\partial n} \quad \text{on} \quad \partial\omega \times (0, T), \tag{1.3}$$

where u^- and u^+ denote the restriction of u from $\Omega \setminus \bar{\omega}$ and ω onto $\partial\omega$, respectively.

When the physical property $q(\mathbf{x})$ of the medium inside ω is known, one can solve the system (1.1)–(1.3) to find the profile of the physical quantity $u(q)$ in Ω. This is called the direct problem, and there is a unique solution $u(q)$ to the system (1.1). But in this work we are interested in the following inverse problem: the material property $q(\mathbf{x})$ of the medium occupied by ω is not available, so we need to recover the distribution $q(\mathbf{x})$ of the material property in ω by using some extra measurement data. This identification problem is severely unstable, i.e. small perturbations in the measurement data can cause tremendous effects on the parameter $q(\mathbf{x})$. Therefore some type of regularization has to be introduced in the numerical identification process [3]. One most stable and effective approach is to transform it into stabilized output least-squares problems with Tikhonov regularization. Our major interest in this work is to study the convergence and convergence rate of the regularized approximation to the true physical diffusivity $q(\mathbf{x})$ in terms of the noise level in the observation data.

Intensive studies on the stability and convergence of the Tiknohov regularization can be found in the literature, see, e.g., [19, 4, 12, 2, 7, 10, 11, 13]. Next, we shall briefly review some existing theories and then point out the new contribution of this work.

Consider a nonlinear ill-posed equation

$$u(q) = z, \tag{1.4}$$

where $u : K \subset Q \to U$ is a nonlinear mapping between Hilbert spaces Q and U, and K is some admissible set of the parameter q. Let q^* be some *a priori* estimate of the true parameter q; then we consider the output least-squares formulation with Tikhonov regularization

$$\min_{q \in K} \|u(q) - z^\delta\|_U^2 + \beta \|q - q^*\|_Q^2, \tag{1.5}$$

where z^δ is the measurement of the exact data z with an error of level δ, i.e. $\|z - z^\delta\|_U \leqslant \delta$.

Let q^+ be an exact solution of (1.4). If u is Fréchet differentiable and its Fréchet derivative u' is Lipschitz continuous with Lipschitz constant L, and there exists a $w \in U$ such that the so-called source condition

$$q^+ - q^* = u'(q^+)^* w \tag{1.6}$$

holds and such a w is small enough, i.e.

$$L\|w\|_U < 1, \tag{1.7}$$

then the regularized minimizers q_β^δ of (1.5) converge to q^+ with the rate $\sqrt{\delta}$ (cf [3, 4]):

$$\|q^+ - q_\beta^\delta\|_Q = O(\sqrt{\delta}). \tag{1.8}$$

This convergence can be naturally extended to the seminorm-regularization case [12], namely replacing $q - q^*$ in (1.5) by $D(q - q^*)$ for some linear operator D.

It is widely known (cf [5]) that it is not easy to apply the above classical theory for general inverse problems and to verify the source condition and smallness condition, especially when the space dimension is higher than 1. In addition, there is no clear comprehension of the derivative $u'(q)$ and the adjoint $u'(q)^*$ required in the source condition (1.6), so it is too complicated to lead to some reasonable geometric or physical interpretation. More importantly, the smallness condition (1.7) for the source function w appears to be quite restrictive.

Some new techniques were introduced in [5] to avoid the use of those restrictive requirements in the general convergence theory for a parabolic system. With these new techniques, the Fréchet differentiability of $u(q)$ and the uniform Lipschitz continuity of the Fréchet derivative $u'(q)$ can be much weakened or removed, and a simpler source condition was proposed to get rid of the smallness condition for the source function. Moreover, the source condition in [5] uses the parameter-to-solution map $u(q)$ itself, instead of its derivative $u'(q)$ and the adjoint $u'(q)^*$, so that it can be interpreted more easily and physically. All these still ensure the usual convergence rates $O(\sqrt{\delta})$ under weaker and more realistic conditions. However, this new theory does not apply directly to elliptic inverse problems, due to the difficulty of its construction of source functions, which are required to meet homogeneous boundary conditions. By demanding the identifying parameter take a specified boundary value, the homogeneous boundary conditions for the source functions in [5] can be relaxed and the convergence theory can be established; see some recent interesting and important developments in [8, 9] for identifying diffusivity and radiativity in elliptic systems. But the identifying parameter may not always be available on the entire boundary.

Next we consider the following Tikhonov regularization with a general penalty term in Banach spaces:

$$\min_{q\in K} \|u(q) - z^\delta\|_U^2 + \beta\mathfrak{R}(q), \tag{1.9}$$

where $\mathfrak{R}$: $Q \to [0, \infty]$ is a convex and lower semi-continuous functional. For further discussions, we introduce two fundamental concepts. Let $\partial\mathfrak{R}(q)$ be the subdifferential of $\mathfrak{R}(q)$ at q, i.e.

$$\partial\mathfrak{R}(q) = \{\xi \in Q^* : \mathfrak{R}(\tilde{q}) \geqslant \mathfrak{R}(q) + \langle \xi, \tilde{q} - q\rangle, \forall \tilde{q} \in Q\}.$$

Here and in the following, we use the notation $\langle\cdot, \cdot\rangle$ to denote both inner products and duality pairing involved if no confusion is caused. Using the subdifferential, the convergence result (1.8) was established in [2] under the following source condition: there exists a source function $\omega \in U$ such that

$$u'(q^+)^* w = \xi \in \partial\mathfrak{R}(q^+),$$

and the nonlinearity condition of the form

$$\langle u(q) - u(q^+) - u'(q^+)(q - q^+), \omega\rangle \leqslant C\|u(q) - u(q^+)\|_U\|\omega\|_U,$$

instead of the smallness condition (1.7). By using the Bregman distance

$$D_\xi(q, \tilde{q}) = \mathfrak{R}(q) - \mathfrak{R}(\tilde{q}) - \langle \xi, q - \tilde{q}\rangle, \forall \xi \in \partial\mathfrak{R}(q), \tag{1.10}$$

a new source condition was formulated in [7] as follows: there exist numbers $\beta_1 \in [0, 1)$, $\beta_2 \geqslant 0$, and $\xi \in \partial\mathfrak{R}(q^+)$ such that

$$\langle \xi, q - q^+\rangle_{(Q^*,Q)} \leqslant \beta_1 D_\xi(q, q^+) + \beta_2\|u(q) - u(q^+)\|_U \quad \forall q \in K.$$

Under this source condition, convergence and convergence rate were achieved and the results were applied to a phase retrieval problem and an inverse option pricing problem in [7]. Recently, another weaker nonlinearity condition (see (3.9)) than the smallness requirement (1.7) was proposed in [10], under which the classical convergence results were derived for a general class of nonlinear parameter identification problems, and verified for some concrete elliptic inverse problems.

In this work, we shall analyze the convergence rate of the Tikhonov regularized solutions for the identification of the diffusivity coefficient in (1.1). To the best of our knowledge, existing convergence theories cannot be applied for the coupled parabolic–elliptic system (1.1). We will establish the convergence for several different regularization techniques such as the H^1 regularization and the mixed L^p–H^1 regularization, under a simple and easily interpretable source condition, and without smallness requirement on the source function. We shall carry out the convergence analysis using three different approaches, which result in the same convergence rate, but under some quite different conditions on the length σ of the observation time $[T-\sigma, T]$: the first one requires $\sigma \leqslant C\beta$ for a generic constant C and the regularization parameter β; the second one requires $\sigma \leqslant C$ for a generic constant C that depends on the forward solution u; the last one imposes no restriction on σ.

Throughout the paper, we shall use the symbols $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{0,\omega}$ to denote the L^2-norm in domains Ω and ω, respectively.

2. Convergence for the H^1 regularization

In this section, we shall formulate the inverse problem of recovering the magnetic diffusivity $q(\mathbf{x})$ in (1.1) in the interior magnetic diffusion region ω, using the observation data ∇z^δ of ∇u over a time range $[T-\sigma, T]$. Assume that q^+ is the exact coefficient and the data noise level is δ, namely

$$\int_{T-\sigma}^{T} \|\nabla u(q^+) - \nabla z^\delta\|_{0,\Omega}^2 \,\mathrm{d}t \leqslant \delta^2. \tag{2.1}$$

As the medium is known in $\Omega \setminus \omega$, we may assume that the medium is known up to the boundary of the magnetic diffusion region ω, namely $q(\mathbf{x}) = q_0$ on $\partial\omega$. So we may consider the following constrained sets of parameters for the desired unknown parameter:

$$K_0 = \{q \in H^1(\omega);\, 0 < \underline{q} \leqslant q \leqslant \bar{q} \ \text{ a.e. in } \omega,\, q = q_0 \ \text{ on } \partial\omega\}, \tag{2.2}$$

$$K = \{q \in H^1(\omega);\, 0 < \underline{q} \leqslant q \leqslant \bar{q} \ \text{ a.e. in } \omega\}. \tag{2.3}$$

We shall first handle the simpler case with the constraint set K_0 (see theorems 2.2–2.3), and then deal with the more general case with the constraint set K (see theorem 2.4).

Now we can formulate our interested parameter identification process into the following regularized output least-squares minimization:

$$\min_{q\in K_0} J(q) = \min_{q\in K_0} \int_{T-\sigma}^{T} \|\nabla u(q) - \nabla z^\delta\|_{0,\Omega}^2 \,\mathrm{d}t + \frac{\beta}{2}\|\nabla q - \nabla q^*\|_{0,\omega}^2, \tag{2.4}$$

where $\beta > 0$ is the regularization parameter and q^* is an *a priori* estimate of the true parameter q^+. We shall write the minimizer of (2.4) as q_β^δ. And for convenience, we shall often use

$$G(q) = \int_{T-\sigma}^{T} \|\nabla u(q) - \nabla z^\delta\|_{0,\Omega}^2 \,\mathrm{d}t. \tag{2.5}$$

We end this section by giving the important equation satisfied by the Fréchet derivative of $u(q)$. For every $q \in K_0$ and each direction $h \in H_0^1(\omega)$, the Fréchet derivative $\eta \equiv u'(q)h$ of u at q in direction h satisfies

$$\begin{cases} -\nabla \cdot (q_0 \nabla \eta) = 0 & \text{in} \quad \Omega \setminus \bar{\omega}, \\ \eta_t - \nabla \cdot (q(\mathbf{x}) \nabla \eta) = \nabla \cdot (h \nabla u(q)) & \text{in} \quad \omega, \\ \eta(\mathbf{x}, t) = 0 & \text{on} \quad \partial\Omega \times (0, T), \\ \eta^+ - \eta^- = 0 & \text{on} \quad \partial\omega \times (0, T), \\ q_0 \dfrac{\partial \eta^-}{\partial n} - q \dfrac{\partial \eta^+}{\partial n} = 0 & \text{on} \quad \partial\omega \times (0, T), \\ \eta(\mathbf{x}, 0) = 0 & \text{in} \quad \omega, \end{cases} \tag{2.6}$$

where η^- and η^+ denote the restriction of η from $\Omega \setminus \bar{\omega}$ and ω onto $\partial\omega$ respectively. We shall often need the variational form of (2.6): for any $v \in H_0^1(\Omega)$, η satisfies

$$\int_\omega \eta_t v \,\mathrm{d}\mathbf{x} + \int_\omega q \nabla \eta \cdot \nabla v \,\mathrm{d}\mathbf{x} + \int_{\Omega \setminus \bar{\omega}} q_0 \nabla \eta \cdot \nabla v \,\mathrm{d}\mathbf{x} = -\int_\omega h \nabla u(q) \cdot \nabla v \,\mathrm{d}\mathbf{x}. \tag{2.7}$$

2.1. Source conditions

In this section, we shall introduce some source condition required for the subsequent convergence analysis. First, we define an adjoint operator ∇^* of ∇ by (cf [5])

$$\langle \nabla^* \xi, \varphi \rangle_{L^2(\omega)} = \langle \xi, \nabla \varphi \rangle_{L^2(\omega)^d}, \qquad \forall \xi \in L^2(\omega)^d, \forall \varphi \in H^1(\omega) \tag{2.8}$$

and the scalar product

$$\langle u, v \rangle_{L^2(T-\sigma,T;H^1(\omega))} = \int_{T-\sigma}^T \int_\omega (uv + \nabla u \cdot \nabla v) \,\mathrm{d}\mathbf{x} \,\mathrm{d}t, \quad \forall u, v \in L^2(T-\sigma, T; H^1(\omega)).$$

Let $H_0^1(T-\sigma, T; L^2(\omega)) = \{v \in H^1(T-\sigma, T; L^2(\omega));\ v(\cdot, T-\sigma) = v(\cdot, T) = 0 \text{ in } \omega\}$. Then we can formulate the source condition that is crucial to the establishment of our convergence results: there exists $v \in L^2(T-\sigma, T; H^1(\omega)) \cap H_0^1(T-\sigma, T; L^2(\omega))$ such that

$$-\int_{T-\sigma}^T \nabla u(q^+) \cdot \nabla v \,\mathrm{d}t = \nabla^* \nabla (q^+ - q^*). \tag{2.9}$$

This condition was first proposed in [5]. Note that (2.9) uses the forward operator $u(q)$ directly, instead of the non-physical quantities $u'(q)$ and $u'(q)^*$ as adopted in most existing source conditions. Also unlike the existing ones, condition (2.9) agrees with the important fact that the true parameter q^+ is unidentifiable in those subregions of ω where $\nabla u(q^+)$ vanishes, except that an accurate *a priori* estimate q^* of q^+ is available.

Now we cite a useful result from [9] that may help us verify the source condition (2.9).

Lemma 2.1. *Assume that $\mathcal{O}$ is an open bounded domain with a C^1-boundary, and $u \in W^{2,\infty}(\mathcal{O})$ such that $|\nabla u| \geqslant C_0 > 0$ a.e. in $\mathcal{O}$. Then for any $q \in H^1(\mathcal{O})$, there exists $v \in H^1(\mathcal{O})$ satisfying the equation $\nabla u \cdot \nabla v = q$.*

Theorem 2.1. *Consider an open bounded domain ω with a C^1-boundary, and a given function $u \in L^2(T-\sigma, T; W^{2,\infty}(\omega))$ satisfying $\left|\int_{T-\sigma}^T (T-t)(T-\sigma-t)\varphi(t) \nabla u \,\mathrm{d}t\right| \geqslant C_0$ a.e. in ω for some $C_0 > 0$ and any $\varphi \in H^1(T-\sigma, T)$. Then for any $q \in H^1(\omega)$, there exists $v \in L^2(T-\sigma, T; H^1(\omega)) \cap H_0^1(T-\sigma, T; L^2(\omega))$ such that*

$$\int_{T-\sigma}^T \nabla u \cdot \nabla v \,\mathrm{d}t = q. \tag{2.10}$$

Proof. Let $P = \int_{T-\sigma}^{T} (T-t)(T-\sigma-t)\varphi(t)\nabla u\,\mathrm{d}t$ and $v = (T-t)(T-\sigma-t)\varphi(t)R$, with $R \in H^1(\omega)$ to be decided. By the assumption we have $|P| \geqslant C_0 > 0$, and $v \in L^2(T-\sigma, T; H^1(\omega)) \cap H_0^1(T-\sigma, T; L^2(\omega))$. It is easy to see that

$$\nabla v = (T-t)(T-\sigma-t)\varphi(t)\nabla R,$$

and that (2.10) is equivalent to

$$\int_{T-\sigma}^{T} \nabla u \cdot \nabla v\,\mathrm{d}t = P \cdot \nabla R = q. \tag{2.11}$$

By the assumption, we can verify that $P \in W^{1,\infty}(\omega)$, which ensures the existence of a solution $R \in H^1(\omega)$ to equation (2.11) by lemma 2.1, or equivalently a desired solution v to equation (2.10). □

We shall need the following important consequence of the source condition (2.9).

Lemma 2.2. *If there exists a solution* $v \in L^2(T-\sigma, T; H^1(\omega)) \cap H_0^1(T-\sigma, T; L^2(\omega))$ *to the source condition* (2.9), *then we can find a function* $\tilde{v} \in L^2\big(T-\sigma, T; H_0^1(\omega)\big)$ *such that*

$$\big\langle \nabla(q^+ - q^*), \nabla\big(q^+ - q_\beta^\delta\big)\big\rangle_{L^2(\omega)^2} = \big\langle \tilde{v}, u'(q^+)\big(q^+ - q_\beta^\delta\big)\big\rangle_{L^2(T-\sigma,T;H_0^1(\omega))}. \tag{2.12}$$

Proof. It follows from (2.9) that

$$-\int_{T-\sigma}^{T}\int_\omega \big(q^+ - q_\beta^\delta\big)\,\nabla u(q^+)\cdot\nabla v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \big\langle \nabla(q^+ - q^*), \nabla\big(q^+ - q_\beta^\delta\big)\big\rangle_{L^2(\omega)^2}. \tag{2.13}$$

Using (2.6) we know that $\eta = u'(q^+)(q^+ - q_\beta^\delta)$ solves the equation:

$$\eta_t - \nabla\cdot(q^+\nabla\eta) = \nabla\cdot\big(\big(q^+ - q_\beta^\delta\big)\,\nabla u(q^+)\big) \quad \text{in } \omega, \tag{2.14}$$

and $\eta(\mathbf{x}, t) = 0$ on $\partial\omega\times(0, T)$, $\eta(\mathbf{x}, 0) = 0$ in ω, and $\eta \in L^2\big(0, T; H_0^1(\omega)\cap H^2(\omega)\big)$. Multiplying both sides of equation (2.14) by v and integrating by parts, we obtain

$$\int_\omega \big(u'(q^+)\big(q^+ - q_\beta^\delta\big)\big)_t v\,\mathrm{d}\mathbf{x} + \int_\omega q^+\nabla u'(q^+)\big(q^+ - q_\beta^\delta\big)\cdot\nabla v\,\mathrm{d}\mathbf{x}$$
$$- \int_{\partial\omega} q^+ \frac{\partial u'(q^+)\big(q^+ - q_\beta^\delta\big)}{\partial n} v\,\mathrm{d}s = -\int_\omega \big(q^+ - q_\beta^\delta\big)\nabla u(q^+)\cdot\nabla v\,\mathrm{d}\mathbf{x}.$$

Using this relation and (2.13), we derive

$$\begin{aligned}\big\langle \nabla(q^+ - q^*), \nabla\big(q^+ - q_\beta^\delta\big)\big\rangle_{L^2(\omega)^2} = &-\int_{T-\sigma}^{T}\int_\omega u'(q^+)\big(q^+ - q_\beta^\delta\big)v_t\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &+\int_{T-\sigma}^{T}\int_\omega q^+\nabla u'(q^+)\big(q^+ - q_\beta^\delta\big)\cdot\nabla v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &-\int_{T-\sigma}^{T}\int_{\partial\omega} q_0\frac{\partial u'(q^+)\big(q^+ - q_\beta^\delta\big)}{\partial n} v\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{2.15}$$

Now we define a linear functional $F : L^2\big(T-\sigma, T; H_0^1(\omega)\cap H^2(\omega)\big) \to R^1$ by

$$F(\xi) = -\int_{T-\sigma}^{T}\int_\omega \xi v_t\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{T-\sigma}^{T}\int_\omega q^+\nabla\xi\cdot\nabla v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_{T-\sigma}^{T}\int_{\partial\omega} q_0\frac{\partial\xi}{\partial n}v\,\mathrm{d}s\,\mathrm{d}t \tag{2.16}$$

for a given $v \in L^2(T-\sigma, T; H^1(\omega))\cap H_0^1(T-\sigma, T; L^2(\omega))$. It is easy to see that F is bounded in $L^2\big(T-\sigma, T; H_0^1(\omega)\cap H^2(\omega)\big)$. By the Riesz representation theorem, there exists an element $\mu \in L^2\big(T-\sigma, T; H^2(\omega)\cap H_0^1(\omega)\big)$ such that

$$F(\xi) = \langle \mu, \xi\rangle_{L^2(T-\sigma,T;H^2(\omega))} \quad \forall\xi \in L^2\big(T-\sigma, T; H_0^1(\omega)\cap H^2(\omega)\big). \tag{2.17}$$

On the other hand, for any $\phi \in L^2(T-\sigma, T; H_0^1(\omega))$, define $\varphi : L^2(T-\sigma, T; H_0^1(\omega) \cap H^2(\omega)) \to R^1$ by

$$\varphi(v) = \langle \phi, v\rangle_{L^2(T-\sigma,T;H^1(\omega))}. \tag{2.18}$$

Clearly, φ is linear and bounded in $L^2(T-\sigma, T; H_0^1(\omega) \cap H^2(\omega))$; then by the Riesz representation theorem, there exists an element $\Phi \in L^2(T-\sigma, T; H^2(\omega) \cap H_0^1(\omega))$ such that

$$\varphi(v) = \langle \Phi, v\rangle_{L^2(T-\sigma,T;H^2(\omega))}, \tag{2.19}$$

and $\|\Phi\|_{L^2(T-\sigma,T;H^2(\omega))} \leqslant \|\phi\|_{L^2(T-\sigma,T;H^1(\omega))}$. So the above process defines a linear mapping

$$G : L^2(T-\sigma, T; H_0^1(\omega)) \to L^2(T-\sigma, T; H_0^1(\omega) \cap H^2(\omega))$$

by $G(\phi) = \Phi$, which satisfies

$$\|G(\phi)\|_{L^2(T-\sigma,T;H^2(\omega))} \leqslant \|\phi\|_{L^2(T-\sigma,T;H^1(\omega))}.$$

Using the density of $L^2(T-\sigma, T; H_0^1(\omega)\cap H^2(\omega))$ in $L^2(T-\sigma, T; H_0^1(\omega))$ and the definition of G, we can easily see the existence of the inverse G^{-1}. By the invertibility of G, we can find an element $\tilde{v} \in L^2(T-\sigma, T; H_0^1(\omega))$ corresponding to $\mu \in L^2(T-\sigma, T; H^2(\omega)\cap H_0^1(\omega))$ in (2.17) such that $G(\tilde{v}) = \mu$. It follow from (2.18) and (2.19) that

$$\langle \tilde{v}, v\rangle_{L^2(T-\sigma,T;H^1(\omega))} = \langle \mu, v\rangle_{L^2(T-\sigma,T;H^2(\omega))}. \tag{2.20}$$

Then taking $v = u'(q^+)(q^+ - q_\beta^\delta)$ above gives

$$\begin{aligned} F(u'(q^+)\,(q^+ - q_\beta^\delta)\,) &= \langle \mu, u'(q^+)(q^+ - q_\beta^\delta)\rangle_{L^2(T-\sigma,T;H^2(\omega))} \\ &= \langle \tilde{v}, u'(q^+)(q^+ - q_\beta^\delta)\rangle_{L^2(T-\sigma,T;H^1(\omega))}, \end{aligned} \tag{2.21}$$

which along with (2.15) and (2.16) implies the desired relation (2.12) for some $\tilde{v} \in L^2(T-\sigma, T; H_0^1(\omega))$. □

2.2. Convergence rates

We are now going to estimate the convergence rate of the regularized solution to the system (2.4) in terms of the noise level δ and the regularization parameter β. For the purpose, we first estimate $u(q^+ + h) - u(q^+) - u'(q^+)h$ for a general direction $h \in H_0^1(\omega)$, under the following regularity for the unique solution $u(q)$ of system (1.1) at $q = q^+$:

$$u(q^+) \in L^\infty(0, T; L^2(\omega)) \cap L^2(0, T; H_0^1(\Omega) \cap H^2(\omega)). \tag{2.22}$$

Lemma 2.3. *For the Fréchet derivative $u'(q^+)h$ which satisfies (2.6), we have the following estimate:*

$$\begin{aligned} &\|u(q^+ + h) - u(q^+) - u'(q^+)h\|_{0,\omega}^2 + \int_0^T \|\nabla(u(q^+ + h) - u(q^+) - u'(q^+)h)\|_{0,\Omega}^2\,\mathrm{d}t \\ &\quad\leqslant C\|\nabla h\|_{L^2(\omega)}^2 \int_0^T \|u(q^+)\|_{H^2(\omega)}^2\,\mathrm{d}t. \end{aligned} \tag{2.23}$$

Proof. Letting $\eta = u'(q^+)h$, then choosing $v = \eta$ in (2.7) and integrating over $(0, t)$, we have

$$\|\eta\|_{0,\omega}^2 + \int_0^t \|\nabla\eta\|_{0,\Omega}^2\,\mathrm{d}t \leqslant C\int_0^T\int_\omega |h\nabla u(q^+)|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant C\|h\|_{L^4(\omega)}^2 \int_0^T \|\nabla u(q^+)\|_{L^4(\omega)}^2\,\mathrm{d}t.$$

Then using the upper bound of h, the Poincaré inequality and Sobolev embedding theorem, we obtain

$$\|\eta\|_{0,\omega}^2 + \int_0^t \|\nabla\eta\|_{0,\Omega}^2 \,\mathrm{d}t \leqslant C\|h\|_{L^2(\omega)}^2 \int_0^T \|\nabla u(q^+)\|_{L^4(\omega)}^2 \,\mathrm{d}t$$
$$\leqslant C\|\nabla h\|_{L^2(\omega)}^2 \int_0^T \|u(q^+)\|_{H^2(\omega)}^2 \,\mathrm{d}t. \qquad (2.24)$$

Now subtracting the variational equation of (1.1) associated with $u(q^+)$ from the one associated with $u(q^+ + h)$, we derive for any $v \in H_0^1(\Omega)$ that

$$\int_0^t \int_\omega (u(q^+ + h) - u(q^+))_t v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^t \int_\omega (q^+ + h)\nabla(u(q^+ + h) - u(q^+)) \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_0^t \int_{\Omega\setminus\bar{\omega}} q_0 \nabla(u(q^+ + h) - u(q^+)) \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$= -\int_0^t \int_\omega h\nabla u(q^+) \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$= \int_0^t \int_\omega \eta_t v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^t \int_\omega q^+ \nabla\eta \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^t \int_{\Omega\setminus\bar{\omega}} q_0 \nabla\eta \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

This implies

$$\int_0^t \int_\omega (u(q^+ + h) - u(q^+) - \eta)_t v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_0^t \int_\omega (q^+ + h)\nabla(u(q^+ + h) - u(q^+) - \eta) \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_0^t \int_{\Omega\setminus\bar{\omega}} q_0 \nabla(u(q^+ + h) - u(q^+) - \eta) \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$= -\int_0^t \int_\omega h\nabla\eta \cdot \nabla v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \qquad (2.25)$$

Taking $v = u(q^+ + h) - u(q^+) - \eta$ in (2.25) and using (2.24) and the upper bound of h, we obtain

$$\|u(q^+ + h) - u(q^+) - \eta\|_{0,\omega}^2 + \int_0^T \|\nabla(u(q^+ + h) - u(q^+) - \eta)\|_{0,\Omega}^2 \,\mathrm{d}t$$
$$\leqslant C \int_0^T \int_\Omega |h\nabla\eta|_{0,\omega}^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant C \int_0^T \|\nabla\eta\|_{L^2(\omega)}^2 \,\mathrm{d}t$$
$$\leqslant C\|\nabla h\|_{L^2(\omega)}^2 \int_0^T \|u(q^+)\|_{H^2(\omega)}^2 \,\mathrm{d}t.$$

□

Now we are ready to estimate the convergence rate of the regularized solution to the system (2.4).

Theorem 2.2. *Under the source condition (2.9), there exists a constant $\sigma_0 > 0$ such that for $\sigma \leqslant \sigma_0\beta$,*

$$\left\|\nabla\left(q_\beta^\delta - q^+\right)\right\|_{0,\omega}^2 = O\left(\frac{\delta^2}{\beta} + \beta\right),$$
$$\int_{T-\sigma}^T \int_\Omega \left|\nabla u\left(q_\beta^\delta\right) - \nabla u(q^+)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t = O(\delta^2 + \beta^2).$$

Proof. As q_β^δ is a minimizer of (2.4), then we have $J\big(q_\beta^\delta\big) \leqslant J(q^+)$, which, along with (2.12) and (2.5), implies the existence of an element $\tilde{v} \in L^2\big(T-\sigma, T; H_0^1(\omega)\big)$ such that

$$G\big(q_\beta^\delta\big) + \frac{\beta}{2}\big\|\nabla\big(q^+ - q_\beta^\delta\big)\big\|_{0,\omega}^2 \leqslant G(q^+) + \beta\langle\nabla(q^+ - q^*), \nabla(q^+ - q_\beta^\delta)\rangle_{L^2(\omega)^2}$$
$$= G(q^+) + \beta\big\langle\tilde{v}, u'(q^+)\big(q^+ - q_\beta^\delta\big)\big\rangle_{L^2(T-\sigma,T;H_0^1(\omega))}. \tag{2.26}$$

For any $\beta > 0$, we may choose $\psi_\beta \in H_0^1\big(T-\sigma, T; H_0^2(\omega)\big)$ by the density result such that

$$\int_{T-\sigma}^{T} \|\psi_\beta - \tilde{v}\|_{1,\omega}^2 \,\mathrm{d}t \leqslant \beta^2. \tag{2.27}$$

Using ψ_β, we define ϕ_β which solves the following elliptic interface problem:

$$\begin{cases} -\nabla\cdot(q_0\nabla\phi_\beta) = 0 & \text{in} \quad \Omega\setminus\bar{\omega}, \\ -\nabla\cdot(q^+\nabla\phi_\beta) = \psi_\beta - \triangle\psi_\beta & \text{in} \quad \omega, \\ \phi_\beta^+ - \phi_\beta^- = 0, & \text{on} \quad \partial\omega\times(0,T), \\ q_0\dfrac{\partial\phi_\beta^-}{\partial n} - q^+\dfrac{\partial\phi_\beta^+}{\partial n} = 0 & \text{on} \quad \partial\omega\times(0,T), \\ \phi_\beta = 0 & \text{on} \quad \partial\Omega\times(0,T). \end{cases} \tag{2.28}$$

Clearly, ϕ_β has the following variational form:

$$\int_{\Omega\setminus\bar{\omega}} q_0\nabla\phi_\beta\cdot\nabla v \,\mathrm{d}\mathbf{x} + \int_\omega q^+\nabla\phi_\beta\cdot\nabla v\,\mathrm{d}\mathbf{x} = \int_\omega \psi_\beta v + \nabla\psi_\beta\cdot\nabla v\,\mathrm{d}\mathbf{x} = \langle\psi_\beta, v\rangle_{H^1(\omega)} \tag{2.29}$$

for any $v \in H_0^1(\Omega)$. Taking $v = \phi_\beta$ above and using the Poincaré inequality, we obtain

$$\int_{T-\sigma}^{T} \|\phi_\beta\|_{1,\Omega}^2 \,\mathrm{d}t \leqslant C\int_{T-\sigma}^{T} \|\nabla\phi_\beta\|_{0,\Omega}^2 \,\mathrm{d}t \leqslant C\int_{T-\sigma}^{T} \|\psi_\beta\|_{1,\omega}^2 \,\mathrm{d}t. \tag{2.30}$$

By taking the time derivative for each equation in (2.28), the same derivation as for (2.30) gives

$$\int_{T-\sigma}^{T} \|(\phi_\beta)_t\|_{1,\Omega}^2 \,\mathrm{d}t \leqslant C\int_{T-\sigma}^{T} \|\nabla(\phi_\beta)_t\|_{0,\Omega}^2 \,\mathrm{d}t \leqslant C\int_{T-\sigma}^{T} \|(\psi_\beta)_t\|_{1,\omega}^2 \,\mathrm{d}t. \tag{2.31}$$

To continue our estimate using (2.26), we take $v = u'(q^+)\big(q^+ - q_\beta^\delta\big)$ in (2.29) and use (2.7) to obtain

$$\begin{aligned}\big\langle\psi_\beta, u'(q^+)\big(q^+ - q_\beta^\delta\big)\big\rangle_{H^1(\omega)} &= \int_\omega q^+\nabla\phi_\beta\cdot\nabla u'(q^+)\big(q^+ - q_\beta^\delta\big)\,\mathrm{d}\mathbf{x} + \int_{\Omega\setminus\bar{\omega}} q_0\nabla\phi_\beta\cdot\nabla u'(q^+)\big(q^+ - q_\beta^\delta\big)\,\mathrm{d}\mathbf{x} \\ &= -\int_\omega \big(u'(q^+)\big(q^+ - q_\beta^\delta\big)\big)_t\phi_\beta\,\mathrm{d}\mathbf{x} - \int_\omega \big(q^+ - q_\beta^\delta\big)\nabla u(q^+)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x},\end{aligned}$$

and then we derive from (2.26) that

$$\begin{aligned}G\big(q_\beta^\delta\big) + \frac{\beta}{2}\big\|q^+ - q_\beta^\delta\big\|_{0,\omega}^2 &\leqslant G(q^+) + \beta\big\langle\tilde{v}, u'(q^+)\big(q^+ - q_\beta^\delta\big)\big\rangle_{L^2(T-\sigma,T;H_0^1(\omega))} \\ &= G(q^+) + \beta\langle\tilde{v} - \psi_\beta, u'(q^+)(q^+ - q_\beta^\delta)\rangle_{L^2(T-\sigma,T;H_0^1(\omega))} \\ &\quad - \beta\int_{T-\sigma}^{T}\int_\omega \big(u'(q^+)\big(q^+ - q_\beta^\delta\big)\big)_t\phi_\beta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad - \beta\int_{T-\sigma}^{T}\int_\omega \big(q^+ - q_\beta^\delta\big)\nabla u(q^+)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.\end{aligned}$$

But it is easy to see by using the variational equations of (1.1) associated with $u(q^+)$ and $u(q_\beta^\delta)$ that

$$\int_\omega \big(u(q^+) - u(q_\beta^\delta)\big)_t \phi_\beta \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\omega \big(q^+\nabla u(q^+) - q_\beta^\delta \nabla u(q_\beta^\delta)\big)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x} + \int_{\Omega\setminus\bar\omega} q_0\big(\nabla u(q^+) - \nabla u(q_\beta^\delta)\big)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x} = 0,$$

which enables us to write

$$\begin{aligned} G(q_\beta^\delta) + \frac{\beta}{2}\big\|\nabla\big(q^+ - q_\beta^\delta\big)\big\|^2 &\leqslant \delta^2 + \beta\langle\tilde v - \psi_\beta, u'(q^+)(q^+ - q_\beta^\delta)\rangle_{L^2(T-\sigma,T;H_0^1(\omega))} \\ &\quad + \beta\int_{T-\sigma}^T\int_\omega \big(u(q^+) - u(q_\beta^\delta) - u'(q^+)(q^+ - q_\beta^\delta)\big)_t\phi_\beta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad + \beta\int_{T-\sigma}^T\int_\omega q_\beta^\delta\big(\nabla u(q^+) - \nabla u(q_\beta^\delta)\big)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad + \beta\int_{T-\sigma}^T\int_{\Omega\setminus\bar\omega} q_0\big(\nabla u(q^+) - \nabla u(q_\beta^\delta)\big)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\equiv \delta^2 + I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.32}$$

Next, we estimate all the terms I_i for $i = 1, 2, 3, 4$. First for I_1, we can deduce readily from (2.24) and (2.27) that

$$\begin{aligned} |I_1| &\leqslant \beta\left(\int_{T-\sigma}^T \|\psi_\beta - \tilde v\|_{1,\omega}^2\,\mathrm{d}t\right)^{\frac12}\left(\int_{T-\sigma}^T \|u'(q^+)(q^+ - q_\beta^\delta)\|_{1,\omega}^2\,\mathrm{d}t\right)^{\frac12} \\ &\leqslant C\beta^2\left(\int_0^T \|u(q^+)\|_{1,\omega}^2\,\mathrm{d}t\right)^{\frac12}. \end{aligned}$$

For I_3 and I_4, we use the Cauchy–Schwarz and triangle inequalities

$$\begin{aligned} |I_3| &\leqslant \beta\left|\int_{T-\sigma}^T\int_\omega q_\beta^\delta\big(\nabla u(q^+) - \nabla z^\delta + \nabla z^\delta - \nabla u(q_\beta^\delta)\big)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \\ &\leqslant \frac14\delta^2 + \frac14 G(q_\beta^\delta) + C\beta^2\int_{T-\sigma}^T \|\nabla\phi_\beta\|_{0,\omega}^2\,\mathrm{d}t, \\ |I_4| &\leqslant \beta\left|\int_{T-\sigma}^T\int_{\Omega\setminus\bar\omega} q_0\big(\nabla u(q^+) - \nabla z^\delta + \nabla z^\delta - \nabla u(q_\beta^\delta)\big)\cdot\nabla\phi_\beta\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \\ &\leqslant \frac14\delta^2 + \frac14 G(q_\beta^\delta) + C\beta^2\int_{T-\sigma}^T \|\nabla\phi_\beta\|_{0,\Omega}^2\,\mathrm{d}t. \end{aligned}$$

Finally, for I_2, we integrate by parts with respect to t over $(T-\sigma, T)$, and then use the Cauchy–Schwarz inequality to derive

$$\begin{aligned} |I_2| &= \left|-\beta\int_{T-\sigma}^T\int_\omega \big(u(q^+) - u(q_\beta^\delta) - u'(q^+)(q^+ - q_\beta^\delta)\big)(\phi_\beta)_t\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \\ &\leqslant \int_{T-\sigma}^T \big\|u(q^+) - u(q_\beta^\delta) - u'(q^+)(q^+ - q_\beta^\delta)\big\|_{0,\omega}^2\,\mathrm{d}t + \frac{\beta^2}{4}\int_{T-\sigma}^T \|(\phi_\beta)_t\|_{0,\omega}^2\,\mathrm{d}t. \end{aligned}$$

But using $h = q^+ - q_\beta^\delta$ in (2.23), we have

$$\big\|u(q^+) - u(q_\beta^\delta) - u'(q^+)(q^+ - q_\beta^\delta)\big\|_{0,\omega}^2 \leqslant C\big\|\nabla\big(q^+ - q_\beta^\delta\big)\big\|_{0,\omega}^2\int_0^T \|u(q^+)\|_{H^2(\omega)}^2\,\mathrm{d}t,$$

which implies

$$\int_{T-\sigma}^T \big\|u(q^+) - u(q_\beta^\delta) - u'(q^+)(q^+ - q_\beta^\delta)\big\|_{0,\omega}^2\,\mathrm{d}t \leqslant C\sigma\big\|\nabla\big(q^+ - q_\beta^\delta\big)\big\|_{0,\omega}^2.$$

Now we can readily see from (2.32) and the above estimates for I_i for $i = 1, 2, 3, 4$ when σ is appropriately small in terms of β that

$$G\big(q_\beta^\delta\big) + \beta\big\|\nabla\big(q^+ - q_\beta^\delta\big)\big\|_{0,\omega}^2 \leqslant C(\beta^2 + \delta^2),$$

which verifies the convergences in theorem 2.2. □

The convergence rate in theorem 2.2 was achieved under the condition that $\sigma \leqslant \sigma_0\beta$ for some constant σ_0. This holds obviously when $\sigma = 0$, namely the observation data are available only at the terminal time $t = T$. Otherwise this may be a restrictive condition. Next we shall relax this condition by imposing a stronger regularity assumption on the forward solution $u(q^+)$ than (2.22):

$$u(q^+) \in L^\infty(0,T;L^2(\omega)) \cap L^2(0,T;H_0^1(\Omega)\cap W^{1,\infty}(\omega)) \text{ and } \triangle u(q^+) \in L^2(0,T;L^\infty(\omega)). \tag{2.33}$$

Under this assumption, we will obtain an improved estimate for $u(q^+ + h) - u(q^+) - u(q^+)'h$ over (2.23), leading to the same convergence rate as in theorem 2.2 but with a weaker requirement on σ.

Lemma 2.4. *For the Fréchet derivative $u'(q^+)h$ which satisfies (2.6), we have the following estimate:*

$$\begin{aligned}\|u(q^+ + h) - u(q^+) - \eta\|_{0,\omega}^2 &+ \int_0^T \|\nabla(u(q^+ + h) - u(q^+) - \eta)\|_{0,\Omega}^2\,\mathrm{d}t\\ &\leqslant C\|\nabla h\|_{L^2(\omega)}^4 \int_0^T \big(\|\nabla u(q^+)\|_{L^\infty(\omega)}^2 + \|\triangle u(q^+)\|_{L^\infty(\omega)}^2\big)\,\mathrm{d}t.\end{aligned} \tag{2.34}$$

Proof. Using (2.6), we know $\eta = u'(q^+)h$ satisfies the parabolic equation in ω:

$$\eta_t - \nabla\cdot(q^+(\mathbf{x})\nabla\eta) = \nabla\cdot(h\nabla u(q^+)) \quad \text{in } \omega \tag{2.35}$$

and the boundary and initial conditions $\eta(\mathbf{x},t) = 0$ on $\partial\omega\times(0,T)$ and $\eta(\mathbf{x},0) = 0$ in ω. Then we derive from the *a priori* estimates [6] that

$$\begin{aligned}\int_0^T\int_\omega \|\eta_t\|_{0,\omega}^2\,\mathrm{d}t + \int_0^t \|\eta\|_{2,\omega}^2\,\mathrm{d}t &\leqslant C\int_0^T\int_\omega |\nabla\cdot(h\nabla u(q^+))|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\ &\leqslant C\|\nabla h\|_{L^2(\omega)}^2 \int_0^T \big(\|\nabla u(q^+)\|_{L^\infty(\omega)}^2 + \|\triangle u(q^+)\|_{L^\infty(\omega)}^2\big)\,\mathrm{d}t.\end{aligned} \tag{2.36}$$

Taking $v = u(q^+ + h) - u(q^+) - \eta$ in (2.25) and using (2.36), we obtain

$$\begin{aligned}\|u(q^+ + h) - u(q^+) - \eta\|_{0,\omega}^2 &+ \int_0^T \|\nabla(u(q^+ + h) - u(q^+) - \eta)\|_{0,\Omega}^2\,\mathrm{d}t\\ &\leqslant C\int_0^T\int_\Omega |h\nabla\eta|_{0,\omega}^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant C\|\nabla h\|_{L^2(\omega)}^2 \int_0^T \|\eta\|_{H^2(\omega)}^2\,\mathrm{d}t\\ &\leqslant C\|\nabla h\|_{L^2(\omega)}^4 \int_0^T \big(\|\nabla u(q^+)\|_{L^\infty(\omega)}^2 + \|\triangle u(q^+)\|_{L^\infty(\omega)}^2\big)\,\mathrm{d}t.\end{aligned}$$

□

Theorem 2.3. *Under the source condition (2.9), the following estimates hold for all $\sigma \leqslant c_0$ for some constant $c_0 > 0$:*

$$\left\|\nabla\left(q_\beta^\delta - q^+\right)\right\|_{0,\omega}^2 = O\left(\frac{\delta^2}{\beta} + \beta\right),$$

$$\int_{T-\sigma}^T \int_\Omega \left|\nabla u\left(q_\beta^\delta\right) - \nabla u(q^+)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t = O(\delta^2 + \beta^2).$$

Proof. The proof is nearly the same as the one for theorem 2.2, except that we can achieve a different estimate for I_2 using lemma 2.4. To do so, we first obtain by the Cauchy–Schwarz inequality that

$$|I_2| \leqslant C\beta\left(\int_{T-\sigma}^T \left\|u(q^+) - u\big(q_\beta^\delta\big) - u'(q^+)\big(q^+ - q_\beta^\delta\big)\right\|_{0,\omega}^2 \mathrm{d}t\right)^{\frac{1}{2}}.$$

But taking $h = q^+ - q_\beta^\delta$ in (2.34), we further derive

$$|I_2| \leqslant C\beta\sigma^{\frac{1}{2}}\left(\int_0^T \left(\|\nabla u(q^+)\|_{L^\infty(\omega)}^2 + \|\Delta u(q^+)\|_{L^\infty(\omega)}^2\right)\mathrm{d}t\right)^{\frac{1}{2}} \left\|\nabla\left(q^+ - q_\beta^\delta\right)\right\|_{0,\omega}^2.$$

Now the desired estimates follow by following the proof of theorem 2.2 and choosing σ such that

$$C\sigma^{\frac{1}{2}}\left(\int_0^T \left(\|\nabla u(q^+)\|_{L^\infty(\omega)}^2 + \|\Delta u(q^+)\|_{L^\infty(\omega)}^2\right)\mathrm{d}t\right)^{\frac{1}{2}} \leqslant \frac{1}{4}.$$ □

For the estimates of convergence rates in theorems 2.2 and 2.3, we have imposed the restrictive condition that $q = q_0$ on $\partial\omega$ in the constrained set, namely the set K_0 in (2.2), and the length of the observation time σ should be either smaller than $\sigma_0\beta$ or *a priori* upper bound. In the remainder of this section, we present a new analysis which enables us to have the same convergence rates as in theorems 2.2 and 2.3, but without the restriction that $q = q_0$ on $\partial\omega$ in the constrained set, namely the set K in (2.3), nor any restriction on σ.

Theorem 2.4. *Under the source condition (2.9), the minimizer q_β^δ of the system (2.4) with K_0 replaced by K has the following approximation properties:*

$$\left\|\nabla\left(q_\beta^\delta - q^+\right)\right\|_{0,\omega}^2 = O\left(\frac{\delta^2}{\beta} + \beta\right),$$

$$\int_{T-\sigma}^T \int_\Omega \left|\nabla u\big(q_\beta^\delta\big) - \nabla u(q^+)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t = O(\delta^2 + \beta^2).$$

Proof. As q_β^δ is a minimizer of (2.4), then we have $J\left(q_\beta^\delta\right) \leqslant J(q^+)$, i.e.

$$G\big(q_\beta^\delta\big) + \frac{\beta}{2}\left\|\nabla\big(q_\beta^\delta - q^*\big)\right\|_{0,\omega}^2 \leqslant G(q^+) + \frac{\beta}{2}\|\nabla(q^+ - q^*)\|_{0,\omega}^2.$$

By using the source condition, we have an element $v \in L^2(T-\sigma, T; H^1(\omega)) \cap H_0^1(T-\sigma, T; L^2(\omega))$ such that (2.9) is satisfied. Combining (2.9) with the above inequality gives

$$\begin{aligned} G\big(q_\beta^\delta\big) + \frac{\beta}{2}\left\|\nabla\big(q^+ - q_\beta^\delta\big)\right\|_{0,\omega}^2 &\leqslant G(q^+) + \beta\big\langle\nabla(q^+ - q^*), \nabla\big(q^+ - q_\beta^\delta\big)\big\rangle_{L^2(\omega)^2} \\ &= G(q^+) - \beta\int_{T-\sigma}^T \int_\omega \big(q^+ - q_\beta^\delta\big)\nabla u(q^+)\cdot\nabla v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \end{aligned} \tag{2.37}$$

Now, we construct a new element $\tilde{v}$ which extends v from ω to Ω such that $\tilde{v} = v$ in ω and $\tilde{v} \in L^2\big(T-\sigma, T; H_0^1(\Omega)\big) \cap H_0^1(T-\sigma, T; L^2(\Omega))$. To do so, we first extend $v \in L^2(T-\sigma, T; H^1(\omega)) \cap H_0^1(T-\sigma, T; L^2(\omega))$ to $\hat{v} \in L^2(T-\sigma, T; H^1(\mathbf{R}^d)) \cap H_0^1(T-\sigma, T; L^2(\mathbf{R}^d))$ by the standard extension (cf [1]). Then we introduce a cut-off function $\zeta \in C_0^\infty(\Omega)$ such that $\zeta(\mathbf{x}) = 1$ in ω, and we can verify that $\tilde{v} = \zeta\,\hat{v} \in L^2(T-\sigma, T; H_0^1(\Omega)) \cap H_0^1(T-\sigma, T; L^2(\Omega))$.

Using this extension $\tilde{v}$ as a test function, we derive from the variational equation associated with (1.1) that

$$\begin{aligned}\int_\omega u\big(q_\beta^\delta\big)_t \tilde{v}\,\mathrm{d}\mathbf{x} + \int_\omega q_\beta^\delta \nabla u\big(q_\beta^\delta\big)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x} + \int_{\Omega\backslash\bar{\omega}} q_0 \nabla u\big(q_\beta^\delta\big)\big)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x} \\ = \int_\omega u(q^+)_t\tilde{v}\,\mathrm{d}\mathbf{x} + \int_\omega q^+\nabla u(q^+)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x} + \int_{\Omega\backslash\bar{\omega}} q_0\nabla u(q^+)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x},\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\int_\omega \big(u(q^+) - u\big(q_\beta^\delta\big)\big)_t \tilde{v}\,\mathrm{d}\mathbf{x} + \int_\omega \big(q^+\nabla u(q^+) - q_\beta^\delta\nabla u\big(q_\beta^\delta\big)\big)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x} \\ + \int_{\Omega\backslash\bar{\omega}} q_0\big(\nabla u(q^+) - \nabla u\big(q_\beta^\delta\big)\big)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x} = 0.\end{aligned}$$

Integrating both sides of the above equation over $t \in (T-\sigma, T)$, we have

$$\begin{aligned}\beta\int_{T-\sigma}^T\int_\omega \big(u(q^+) - u\big(q_\beta^\delta\big)\big)_t v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \beta\int_{T-\sigma}^T\int_\omega \big(q^+\nabla u(q^+) - q_\beta^\delta\nabla u\big(q_\beta^\delta\big)\big)\cdot\nabla v\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ + \beta\int_{T-\sigma}^T\int_{\Omega\backslash\bar{\omega}} q_0\big(\nabla u(q^+) - \nabla u\big(q_\beta^\delta\big)\big)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0.\end{aligned} \tag{2.38}$$

It follows easily from (2.38) and (2.37) that

$$\begin{aligned}G\big(q_\beta^\delta\big) + \frac{\beta}{2}\big\|\nabla\big(q^+ - q_\beta^\delta\big)\big\|^2 &\leqslant \delta^2 + \beta\int_{T-\sigma}^T\int_\omega \big(u(q^+) - u\big(q_\beta^\delta\big)\big)_t\tilde{v}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad + \beta\int_{T-\sigma}^T\int_\omega q_\beta^\delta\big(\nabla u(q^+) - \nabla u\big(q_\beta^\delta\big)\big)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad + \beta\int_{T-\sigma}^T\int_{\Omega\backslash\bar{\omega}} q_0\big(\nabla u(q^+) - \nabla u\big(q_\beta^\delta\big)\big)\cdot\nabla\tilde{v}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\equiv \delta^2 + I_1 + I_2 + I_3.\end{aligned} \tag{2.39}$$

Next we estimate the three terms I_1, I_2 and I_3 above. For I_1, we deduce by integration by parts with respect to t, using the fact that $\tilde{v} \in H_0^1(T-\sigma, T; L^2(\Omega))$ and the Poincaré inequality

$$\begin{aligned}|I_1| &= \left|-\beta\int_{T-\sigma}^T\int_\omega \big(u(q^+) - u\big(q_\beta^\delta\big)\big)\tilde{v}_t\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| \leqslant \beta\int_{T-\sigma}^T \big\|u(q^+) - u\big(q_\beta^\delta\big)\big\|_{0,\omega}\|\tilde{v}_t\|_{0,\omega}\,\mathrm{d}t \\ &\leqslant C\beta\int_{T-\sigma}^T \big\|\nabla u(q^+) - \nabla u\big(q_\beta^\delta\big)\big\|_{0,\Omega}\|\tilde{v}_t\|_{0,\Omega}\,\mathrm{d}t.\end{aligned}$$

Furthermore, using the Young inequality and the triangle inequality we obtain

$$\begin{aligned}|I_1| &\leqslant \frac{1}{8}\int_{T-\sigma}^T \big\|\nabla u(q^+) - \nabla z^\delta + \nabla z^\delta - \nabla u\,\big(q_\beta^\delta\big)\,\big\|_{0,\Omega}^2\,\mathrm{d}t + 2C\beta^2\int_{T-\sigma}^T \|\tilde{v}_t\|_{0,\Omega}^2\,\mathrm{d}t \\ &\leqslant \frac{1}{4}\delta^2 + \frac{1}{4}G\big(q_\beta^\delta\big) + 2C\beta^2\int_{T-\sigma}^T \|\tilde{v}_t\|_{0,\Omega}^2\,\mathrm{d}t.\end{aligned}$$

For I_2, using the fact that $q_\beta^\delta \in K$, the Cauchy–Schwarz and triangle inequalities we can derive

$$\begin{aligned}|I_2| &\leqslant \bar{q}\beta \int_{T-\sigma}^{T} \int_{\omega} \left|\nabla u(q^+) - \nabla z^\delta + \nabla z^\delta - \nabla u\left(q_\beta^\delta\right)\right| |\nabla \tilde{v}| \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ &\leqslant \frac{1}{4} G(q^+) + \frac{1}{4} G\left(q_\beta^\delta\right) + 2\bar{q}^2 \beta^2 \int_{T-\sigma}^{T} \|\nabla \tilde{v}\|_{0,\Omega}^2 \, \mathrm{d}t \\ &\leqslant \frac{1}{4}\delta^2 + \frac{1}{4} G\left(q_\beta^\delta\right) + 2\bar{q}^2 \beta^2 \int_{T-\sigma}^{T} \|\nabla \tilde{v}\|_{0,\Omega}^2 \, \mathrm{d}t.\end{aligned}$$

Finally for I_3, we can similarly deduce

$$\begin{aligned}|I_3| &\leqslant \bar{\alpha}\beta \int_{T-\sigma}^{T} \int_{\Omega\setminus\bar{\omega}} \left|\left(\nabla u(q^+) - \nabla z^\delta + \nabla z^\delta - \nabla u\left(q_\beta^\delta\right)\right) \cdot \nabla \tilde{v}\right| \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ &\leqslant \frac{1}{4}\delta^2 + \frac{1}{4} G\left(q_\beta^\delta\right) + 2\bar{\alpha}^2 \beta^2 \int_{T-\sigma}^{T} \|\nabla \tilde{v}\|_{0,\Omega}^2 \, \mathrm{d}t.\end{aligned}$$

By the above estimates for I_1, I_2 and I_3, we immediately obtain from (2.39) that

$$\tfrac{1}{4} G\left(q_\beta^\delta\right) + \tfrac{1}{2}\beta \left\|\nabla\left(q^+ - q_\beta^\delta\right)\right\|_{0,\omega}^2 = O(\beta^2 + \delta^2),$$

which completes the proof of theorem 2.4. □

Remark 2.1. One can easily see that $\left\|\nabla\left(q^+ - q_\beta^\delta\right)\right\|_{0,\omega} = O(\sqrt{\delta})$ and $\int_{T-\sigma}^{T} \int_\Omega \left|\nabla u\left(q_\beta^\delta\right) - \nabla u(q^+)\right|^2 \mathrm{d}\mathbf{x} \, \mathrm{d}t = O(\delta^2)$ by choosing $\beta \sim \delta$ in theorems 2.2–2.4.

3. Regularization in Banach spaces

In this section, we shall consider some regularizations in Banach spaces and derive the convergence rate of the corresponding regularized solutions. We will focus on the following formulation of identifying the diffusivity parameter $q(\mathbf{x})$ in the system (1.1), using the mixed L^p–H^1 regularization:

$$\min_{q\in K} J(q) = G(q) + \gamma \mathfrak{R}(q) + \frac{\beta}{2} \|\nabla q - \nabla q^*\|_{0,\omega}^2, \tag{3.1}$$

where $G(q)$ is given in (2.5) and $\mathfrak{R}(q) = \frac{1}{p}\|q\|_{L^p(\omega)}^p$, with $p = 1$ or $p \geqslant 2$. The case with $1 < p < 2$ is not much used in applications, and it also happens that our subsequent analysis does not extend to this case. The case with $\gamma = 0$ was considered in section 2, which results usually in a reconstructed parameter $q(\mathbf{x})$ that is globally smooth and overly diffusive. But this may not be physically interesting in some applications. The L^p penalty may promote some special feature of the identifying parameter. For instance, the L^1 regularization preserves the sparsity or some localized oscillating profiles of the parameter. However, if the L^1 penalty is used alone, i.e. $\beta = 0$ in (3.1), the solution tends to be unstable and spiky, and may miss some physically relevant clustering feature of the parameter. In general the solution to the system (3.1) will be unstable when only the L^p penalty is used. In view of these facts, we propose a combined L^p–H^1 regularization in (3.1), and the effect of either the L^p regularization or the H^1 regularization may be realized by reinforcing the magnitude of one parameter over the other between γ and β.

We shall first consider the case $p = 1$ and write the solution to the system (3.1) as $q_{\beta,\gamma}^\delta$, for which we have the following useful estimate.

Lemma 3.1. *There exists a constant $C > 0$ such that the following inequality holds:*

$$\left|\gamma\left\langle 1, q_{\beta,\gamma}^\delta - q^+\right\rangle_{L^2(\omega)}\right| \leqslant \tfrac{1}{4} G\left(q_{\beta,\gamma}^\delta\right) + C(\delta^2 + \gamma^2). \tag{3.2}$$

Proof. By theorem 2.1 there exists some $\pi \in L^2(T-\sigma, T; H^1(\omega)) \cap H_0^1(T-\sigma, T; L^2(\omega))$ such that $\int_{T-\sigma}^T \nabla u \cdot \nabla \pi \,\mathrm{d}t = 1$. Using this we can write

$$\langle 1, q^+ - q_{\beta,\gamma}^\delta \rangle_{L^2(\omega)} = \int_{T-\sigma}^T \int_\omega (q^+ - q_{\beta,\gamma}^\delta) \nabla u \cdot \nabla \pi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{3.3}$$

Next, we construct a new element $\tilde{\pi}$ which extends π from ω to Ω such that $\tilde{\pi} = \pi$ in ω and $\tilde{\pi} \in L^2\big(T-\sigma, T; H_0^1(\Omega)\big) \cap H_0^1\big(T-\sigma, T; L^2(\Omega)\big)$. Then we can estimate the right-hand side of (3.3) in a similar manner to the estimate of the last integral term in (2.37) to obtain (see (2.39)):

$$\begin{aligned}\gamma \big|\langle 1, q^+ - q_{\beta,\gamma}^\delta \rangle_{L^2(\omega)}\big| \leqslant{}& \gamma \int_{T-\sigma}^T \int_\omega \big(u(q^+) - u(q_{\beta,\gamma}^\delta)\big)_t \tilde{\pi} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &+ \gamma \int_{T-\sigma}^T \int_\omega q_{\beta,\gamma}^\delta \big(\nabla u(q^+) - \nabla u(q_{\beta,\gamma}^\delta)\big) \cdot \nabla \tilde{\pi} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &+ \gamma \int_{T-\sigma}^T \int_{\Omega\backslash\bar{\omega}} q_0 \big(\nabla u(q^+) - \nabla u(q_{\beta,\gamma}^\delta)\big) \cdot \nabla \tilde{\pi} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t.\end{aligned}$$

Now the desired result (3.2) can be derived by following a similar estimate to the one for (2.39) in the proof of theorem 2.4. □

3.1. Estimate of rates of convergence

We first study the convergence of the regularized solution to system (3.1) with L^1–H^1 regularization.

Theorem 3.1. *Let $\Re(q) = \|q\|_{L^1(\omega)}$; then the following convergence holds for the solution $q_{\beta,\gamma}^\delta$ to the system (3.1) under the source condition (2.9):*

$$\begin{aligned}&\|\nabla q_{\beta,\gamma}^\delta - \nabla q^+\|_{0,\omega}^2 = O\left(\beta + \frac{\delta^2}{\beta} + \frac{\gamma^2}{\beta}\right), \\ &\int_{T-\sigma}^T \int_\Omega \big|\nabla u(q_{\beta,\gamma}^\delta) - \nabla u(q^+)\big|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = O(\delta^2 + \gamma^2 + \beta^2), \\ &\left|\int_\omega (q_{\beta,\gamma}^\delta - q^+)\,\mathrm{d}\mathbf{x}\right| = O\left(\gamma + \frac{\delta^2}{\gamma} + \frac{\beta^2}{\gamma}\right).\end{aligned}$$

Proof. As $q_{\beta,\gamma}^\delta$ is a minimizer of (3.1), $J\big(q_{\beta,\gamma}^\delta\big) \leqslant J(q^+)$, which with the fact $q_{\beta,\gamma}^\delta, q^+ \in K$ implies that

$$G\big(q_{\beta,\gamma}^\delta\big) + \gamma \int_\omega q_{\beta,\gamma}^\delta \mathrm{d}x + \frac{\beta}{2}\|\nabla q_{\beta,\gamma}^\delta - \nabla q^*\|_{0,\omega}^2 \leqslant \delta^2 + \gamma \int_\omega q^+ \mathrm{d}x + \frac{\beta}{2}\|\nabla q^+ - \nabla q^*\|_{0,\omega}^2,$$

which can be rewritten as

$$\begin{aligned}&G\big(q_{\beta,\gamma}^\delta\big) + \frac{\beta}{2}\|\nabla q_{\beta,\gamma}^\delta - \nabla q^+\|_{0,\omega}^2 \\ &\quad\leqslant \delta^2 - \gamma \int_\omega \big(q_{\beta,\gamma}^\delta - q^+\big)\mathrm{d}x - \beta\big\langle\nabla(q^+ - q^*), \nabla\big(q^+ - q_{\beta,\gamma}^\delta\big)\big\rangle_{L^2(\omega)} \\ &\quad= \delta^2 - \gamma\langle 1, q_{\beta,\gamma}^\delta - q^+\rangle_{L^2(\omega)} - \beta\big\langle\nabla(q^+ - q^*), \nabla\big(q^+ - q_{\beta,\gamma}^\delta\big)\big\rangle_{L^2(\omega)}.\end{aligned} \tag{3.4}$$

By lemma 3.1 we have

$$\big|\gamma\langle 1, q_{\beta,\gamma}^\delta - q^+\rangle_{L^2(\omega)}\big| \leqslant \tfrac{1}{4} G\big(q_{\beta,\gamma}^\delta\big) + C(\delta^2 + \gamma^2). \tag{3.5}$$

On the other hand, we can derive in a similar manner to the proof of theorem 2.4 that

$$\left|\beta\langle\nabla(q^+ - q^*), \nabla\big(q^+ - q^\delta_{\beta,\gamma}\big)\rangle_{L^2(\omega)}\right| \leqslant \tfrac{1}{4}G\big(q^\delta_{\beta,\gamma}\big) + C(\delta^2 + \beta^2). \tag{3.6}$$

Now it follows from (3.4)–(3.6) that

$$\frac{1}{2}G\big(q^\delta_{\beta,\gamma}\big) + \frac{\beta}{2}\left\|\nabla q^\delta_{\beta,\gamma} - \nabla q^+\right\|^2_{0,\omega} \leqslant C(\delta^2 + \gamma^2 + \beta^2),$$

which leads immediately to the first result in theorem 3.1 and $G\big(q^\delta_{\beta,\gamma}\big) = O(\delta^2 + \gamma^2 + \beta^2)$. Using this latter estimate, the triangle inequality and (3.2) we come directly to the second and third results in theorem 3.1. □

Remark 3.1. One can easily see that $\left\|\nabla\big(q^+ - q^\delta_\beta\big)\right\|_{0,\omega} = O(\sqrt{\delta})$, $\left|\int_\omega \big(q^\delta_{\beta,\gamma} - q^+\big)\,\mathrm{d}\mathbf{x}\right| = O(\delta)$ and $\int_{T-\sigma}^T \int_\Omega \left|\nabla u\big(q^\delta_\beta\big) - \nabla u(q^+)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t = O(\delta^2)$, by choosing $\beta \sim \delta \sim \gamma$ in theorem 3.1.

In the remainder of this section, we shall study the convergence of the regularized solution to the system (3.1) for the L^p–H^1 regularization with $p \geqslant 2$, but the constraint set K is replaced by K_0.

We now recall the definition of the subdifferential $\partial\mathfrak{R}(q)$ of $\mathfrak{R}(q)$ and the Bregman distance in (1.10). For $p \geqslant 2$, $\mathfrak{R}(q)$ is Fréchet differentiable, and we shall write its Fréchet derivative as $\mathfrak{R}'(q)$. As $\mathfrak{R}'(q) \in \partial\mathfrak{R}(q)$, we have for any $q, q^+ \in K_0$ that

$$\begin{aligned} D_{\mathfrak{R}'(q^+)}(q, q^+) &= \mathfrak{R}(q) - \mathfrak{R}(q^+) - \langle\mathfrak{R}'(q^+), q - q^+\rangle \\ &= \int_\omega \left(\frac{1}{p}q^p - \frac{1}{p}(q^+)^p - (q^+)^{p-1}(q - q^+)\right)\mathrm{d}\mathbf{x}. \end{aligned} \tag{3.7}$$

Next we shall develop the convergence of the regularized solution to the system (3.1) under the following source and nonlinearity conditions:

Source condition. There exists θ such that the exact solution q^+ satisfies

$$\int_{T-\sigma}^T u'(q^+)^*\theta\,\mathrm{d}t = \mathfrak{R}'(q^+). \tag{3.8}$$

Nonlinearity condition. There exists some $\varepsilon > 0$ and $c_r > 0$ such that

$$\begin{aligned} &\frac{c_r}{4}\left\|\nabla u\big(q^\delta_{\beta,\gamma}\big) - \nabla u(q^+)\right\|^2_{L^2(T-\sigma,T;L^2(\Omega))} + D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) - \int_{T-\sigma}^T \big\langle\theta, E\big(q^\delta_{\beta,\gamma}, q^+\big)\big\rangle\,\mathrm{d}t \\ &\qquad \geqslant \varepsilon D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big), \end{aligned} \tag{3.9}$$

where $E\big(q^\delta_{\beta,\gamma}, q^+\big) = u\big(q^\delta_{\beta,\gamma}\big) - u(q^+) - u'(q^+)\big(q^\delta_{\beta,\gamma} - q^+\big)$ and $c_r = \frac{1-c_s}{\gamma}$ for some $0 < c_s < 1$ can be very large when γ is small.

We shall verify the above source and nonlinearity conditions mathematically in the following section. These conditions improve the conditions proposed in [10], which seem to be more restrictive and cannot be verified for our current inverse problem.

Theorem 3.2. *Let* $\mathfrak{R}(q) = \frac{1}{p}\|q\|^p_{L^p(\omega)}$ *for* $2 \leqslant p < \infty$*; then the following results hold for the solution* $q^\delta_{\beta,\gamma}$ *to the system (3.1) under conditions (3.8) and (3.9):*

$$D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) = O\left(\gamma + \frac{\delta^2}{\gamma} + \frac{\beta^2}{\gamma}\right), \qquad \left\|\nabla q^\delta_{\beta,\gamma} - \nabla q^+\right\|^2_{0,\omega} = O\left(\beta + \frac{\delta^2}{\beta} + \frac{\gamma^2}{\beta}\right),$$

$$\int_{T-\sigma}^T \int_\Omega \left|\nabla u\big(q^\delta_{\beta,\gamma}\big) - \nabla u(q^+)\right|^2 \mathrm{d}\mathbf{x}\,\mathrm{d}t = O(\delta^2 + \gamma^2 + \beta^2).$$

Proof. As $q^\delta_{\beta,\gamma}$ is a minimizer of (3.1), we have $J(q^\delta_{\beta,\gamma}) \leqslant J(q^+)$, which, along with (1.10), implies that

$$G\big(q^\delta_{\beta,\gamma}\big) + \gamma D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) + \frac{\beta}{2}\big\|\nabla q^\delta_{\beta,\gamma} - \nabla q^+\big\|^2_{0,\omega}$$
$$\leqslant \delta^2 - \gamma\big\langle \mathfrak{R}'(q^+), q^\delta_{\beta,\gamma} - q^+\big\rangle - \beta\big\langle \nabla(q^+ - q^*), \nabla\big(q^+ - q^\delta_{\beta,\gamma}\big)\big\rangle_{L^2(\omega)^2}.$$

Then it follows from (3.6), (3.8) and (3.9) that

$$\begin{aligned}
&\frac{1}{2}G\left(q^\delta_{\beta,\gamma}\right) + \gamma D_{\mathfrak{R}'(q^+)}\left(q^\delta_{\beta,\gamma}, q^+\right) + \frac{\beta}{2}\left\|\nabla q^\delta_{\beta,\gamma} - \nabla q^+\right\|^2_{0,\omega}\\
&\quad\leqslant C(\delta^2 + \beta^2) - \gamma\big\langle \mathfrak{R}'(q^+), q^\delta_{\beta,\gamma} - q^+\big\rangle\\
&\quad= C(\delta^2 + \beta^2) - \gamma\int_{T-\sigma}^{T}\big\langle\theta, u'(q^+)\big(q^\delta_{\beta,\gamma} - q^+\big)\big\rangle\,\mathrm{d}t\\
&\quad= C(\delta^2 + \beta^2) + \gamma\int_{T-\sigma}^{T}\big\langle\theta, E\big(q^\delta_{\beta,\gamma}, q^+\big)\big\rangle\,\mathrm{d}t + \gamma\int_{T-\sigma}^{T}\big\langle\theta, u(q^+) - u\big(q^\delta_{\beta,\gamma}\big)\big\rangle\,\mathrm{d}t\\
&\quad\leqslant C(\delta^2 + \beta^2) + \frac{c_r}{4}\gamma\big\|\nabla u\big(q^\delta_{\beta,\gamma}\big) - \nabla u(q^+)\big\|^2_{L^2(T-\sigma,T;L^2(\Omega))} + \gamma D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big)\\
&\qquad - \varepsilon\gamma D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) + \gamma\int_{T-\sigma}^{T}\big\langle\theta, u(q^+) - u\big(q^\delta_{\beta,\gamma}\big)\big\rangle\,\mathrm{d}t.
\end{aligned} \tag{3.10}$$

By using the Cauchy–Schwarz inequality and the relation $c_r = (1 - c_s)/\gamma$, we further derive from (3.10) that

$$\begin{aligned}
&\frac{c_s}{2}G\big(q^\delta_{\beta,\gamma}\big) + \varepsilon\gamma D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) + \frac{\beta}{2}\big\|\nabla q^\delta_{\beta,\gamma} - \nabla q^+\big\|^2_{0,\omega}\\
&\quad\leqslant C(\delta^2 + \beta^2) + C\gamma\int_{T-\sigma}^{T}\|\theta\|\,\big\|\nabla u(q^+) - \nabla u\big(q^\delta_{\beta,\gamma}\big)\big\|\,\mathrm{d}t\\
&\quad\leqslant C(\delta^2 + \beta^2) + C\frac{2\gamma^2}{c_s}\int_{T-\sigma}^{T}\|\theta\|^2\,\mathrm{d}t + \frac{c_s}{8}\int_{T-\sigma}^{T}\big\|\nabla u(q^+) - \nabla u\big(q^\delta_{\beta,\gamma}\big)\big\|^2\,\mathrm{d}t.
\end{aligned}$$

Now we can use the assumption (2.1) and the triangle inequality to obtain

$$\begin{aligned}
&\frac{c_s}{4}G\big(q^\delta_{\beta,\gamma}\big) + \varepsilon\gamma D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) + \frac{\beta}{2}\big\|\nabla q^\delta_{\beta,\gamma} - \nabla q^+\big\|^2_{0,\omega}\\
&\quad\leqslant C(\delta^2 + \beta^2) + C\frac{2\gamma^2}{c_s}\int_{T-\sigma}^{T}\|\theta\|^2\,\mathrm{d}t,
\end{aligned}$$

which yields immediately the three desired estimates in theorem 3.2. □

Remark 3.2. One can easily see that $D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) = O(\delta)$, $\big\|\nabla\big(q^+ - q^\delta_\beta\big)\big\|_{0,\omega} = O(\sqrt{\delta})$ and $\int_{T-\sigma}^{T}\int_\Omega\big|\nabla u\big(q^\delta_\beta\big) - \nabla u(q^+)\big|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = O(\delta^2)$, by choosing $\beta \sim \delta \sim \gamma$ in theorem 3.2.

3.2. Verification of source and nonlinearity conditions

In this section we demonstrate the source condition (3.8) and nonlinearity condition (3.9). Let

$$e(q, u) = \begin{cases} -\nabla\cdot(q_0\nabla u) & \text{in} \quad \Omega\setminus\bar{\omega},\\ u_t - \nabla\cdot(q(\mathbf{x})\nabla u) & \text{in} \quad \omega, \end{cases} \tag{3.11}$$

We know from (1.1) that $e(q, u) = 0$ in Ω. Taking the derivative on both sides of $e(q, u) = 0$ with respect to q in direction h, we obtain

$$e_q(q, u(q))h + e_u(q, u(q))u'(q)h = 0.$$

It is direct to verify the existence of the inverse of the operator $e_u(q, u(q))$, so we derive by taking this inverse on both sides of the above equation:

$$u'(q)h = -(e_u(q, u(q)))^{-1} e_q(q, u(q))h;$$

then the adjoint operator $u'(q)^*$ can be given by

$$u'(q)^*\theta = -e_q(q, u(q))^*((e_u(q, u(q)))^{-1})^*\theta.$$

Therefore, the source condition (3.9) can be expressed explicitly as

$$-\int_{T-\sigma}^{T} e_q(q^+, u(q^+))^*((e_u(q, u(q)))^{-1})^*\theta \, \mathrm{d}t = \mathfrak{R}'(q^+).$$

Setting $\rho = -((e_u(q, u(q)))^{-1})^*\theta$, we can then replace the source condition (3.8) by requiring the existence of ρ such that

$$\int_{T-\sigma}^{T} e_q(q^+, u(q^+))^*\rho \, \mathrm{d}t = \mathfrak{R}'(q^+). \tag{3.12}$$

By directly computing, we have

$$\begin{aligned}\langle\theta, E(q, q^+)\rangle &= \langle\theta, u(q) - u(q^+) - u'(q^+)(q - q^+)\rangle \\ &= \langle(((e_u(q, u(q)))^{-1})^*\theta, e_u(q^+, u(q^+))(u(q) - u(q^+) - u'(q^+)(q - q^+))\rangle \\ &= -\langle\rho, e_u(q^+, u(q^+))(u(q) - u(q^+) - u'(q^+)(q - q^+))\rangle,\end{aligned}$$

so the nonlinearity condition (3.9) can be rewritten as

$$\begin{aligned}&\frac{c_r}{4}\left\|\nabla u\big(q^\delta_{\beta,\gamma}\big) - \nabla u(q^+)\right\|^2_{L^2(T-\sigma,T;L^2(\Omega))} + D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big) \\ &\qquad + \int_{T-\sigma}^{T} \big\langle\rho, e_u(q^+, u(q^+))\big(u\big(q^\delta_{\beta,\gamma}\big) - u(q^+) - u'(q^+)\big(q^\delta_{\beta,\gamma} - q^+\big)\big)\big\rangle \, \mathrm{d}t \\ &\geqslant \varepsilon D_{\mathfrak{R}'(q^+)}\big(q^\delta_{\beta,\gamma}, q^+\big).\end{aligned} \tag{3.13}$$

It is easy to verify that

$$e_q(q, u(q))h = \begin{cases} 0 & \text{in} \quad \Omega \setminus \bar{\omega} \\ -\nabla\cdot(h\nabla u(q)) & \text{in} \quad \omega \end{cases}$$

and

$$e_u(q, u(q))\delta u = \begin{cases} -\nabla\cdot(q_0\nabla\delta u) & \text{in} \quad \Omega \setminus \bar{\omega}, \\ (\delta u)_t - \nabla\cdot(q\nabla\delta u) & \text{in} \quad \omega. \end{cases}$$

Hence, for any $\rho \in \mathrm{Range}(e_q(q, u(q))^*)$, we obtain

$$e_q(q, u(q))^*\rho = \begin{cases} 0 & \text{in} \quad \Omega \setminus \bar{\omega}, \\ \nabla u(q)\cdot\nabla\rho & \text{in} \quad \omega \end{cases}$$

by noting that $h = 0$ on $\partial\omega$ and using the equality

$$\langle e_q(q, u(q))h, \rho\rangle_\Omega = \langle-\nabla\cdot(h\nabla u(q)), \tilde{\rho}\rangle_\omega = \langle h, \nabla u(q)\cdot\nabla\tilde{\rho}\rangle_\omega,$$

where $\tilde{\rho} = \rho|_\omega$. Now we can see that (3.12) is equivalent to the existence of ρ such that

$$\int_{T-\sigma}^{T} \nabla u(q)\cdot\nabla\tilde{\rho} \, \mathrm{d}t = \mathfrak{R}'(q^+). \tag{3.14}$$

But such a $\tilde{\rho} \in L^2(T - \sigma, T; H^1(\omega)) \cap H^1_0(T - \sigma, T; L^2(\omega))$ is guaranteed by theorem 2.1.

Finally, we are going to verify the nonlinearity condition (3.13). By direct computings, we have

$$\begin{aligned}&\left\langle\rho, e_u(q^+, u(q^+))u\left(q^\delta_{\beta,\gamma}\right)\right\rangle\\&\qquad= \int_\omega u\left(q^\delta_{\beta,\gamma}\right)_t\rho\,\mathrm{d}\mathbf{x} + \int_\omega q^+\nabla u\left(q^\delta_{\beta,\gamma}\right)\cdot\nabla\rho\,\mathrm{d}\mathbf{x} + \int_{\Omega\setminus\bar\omega} q_0\nabla u\left(q^\delta_{\beta,\gamma}\right)\cdot\nabla\rho\,\mathrm{d}\mathbf{x},\\&\langle\rho, e_u(q^+, u(q^+))u(q^+)\rangle\\&\qquad= \int_\omega u(q^+)_t\rho\,\mathrm{d}\mathbf{x} + \int_\omega q^+\nabla u(q^+)\cdot\nabla\rho\,\mathrm{d}\mathbf{x} + \int_{\Omega\setminus\bar\omega} q_0\nabla u(q^+)\cdot\nabla\rho\,\mathrm{d}\mathbf{x}.\end{aligned}$$

But using (2.7) with $v = \rho$ and $h = q^\delta_{\beta,\gamma} - q^+$, we derive

$$\begin{aligned}&\left\langle\rho, e_u(q^+, u(q^+))u'(q^+)\left(q^\delta_{\beta,\gamma} - q^+\right)\right\rangle\\&\qquad= \int_\omega \left(u'(q^+)\left(q^\delta_{\beta,\gamma} - q^+\right)\right)_t\rho\,\mathrm{d}\mathbf{x} + \int_\omega q^+\nabla u'(q^+)\left(q^\delta_{\beta,\gamma} - q^+\right)\cdot\nabla\rho\,\mathrm{d}\mathbf{x}\\&\qquad\quad+ \int_{\Omega\setminus\bar\omega} q_0\nabla u'(q^+)\left(q^\delta_{\beta,\gamma} - q^+\right)\cdot\nabla\rho\,\mathrm{d}\mathbf{x}\\&\qquad= -\int_\omega \left(q^\delta_{\beta,\gamma} - q^+\right)\nabla u(q^+)\cdot\nabla\rho\,\mathrm{d}\mathbf{x}.\end{aligned}$$

Now it follows from the above three relations that

$$\begin{aligned}I &\equiv \int_{T-\sigma}^T \left\langle\rho, e_u(q^+, u(q^+))\left(u\left(q^\delta_{\beta,\gamma}\right) - u(q^+) - u'(q^+)\left(q^\delta_{\beta,\gamma} - q^+\right)\right)\right\rangle\mathrm{d}t\\&= \int_{T-\sigma}^T\int_\omega \left(q^+ - q^\delta_{\beta,\gamma}\right)\nabla\left(u\left(q^\delta_{\beta,\gamma}\right) - u(q^+)\right)\cdot\nabla\rho\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\\&\leqslant \frac{c_r}{8}\int_{T-\sigma}^T \left\|\nabla u\left(q^\delta_{\beta,\gamma}\right) - \nabla u(q^+)\right\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{2}{c_r}\left\|q^+ - q^\delta_{\beta,\gamma}\right\|^2_{L^2(\omega)}\int_{T-\sigma}^T \|\nabla\rho\|^2_{L^\infty(\omega)}\,\mathrm{d}t.\end{aligned}$$

Next we estimate $\left\|q^+ - q^\delta_{\beta,\gamma}\right\|^2_{L^2(\omega)}$. We have (cf [13]) the following inequality for $p \geqslant 2$:

$$D_{\mathfrak{R}'(q^+)}\left(q^\delta_{\beta,\gamma}, q^+\right) \geqslant C\left(\left\|q^+ - q^\delta_{\beta,\gamma}\right\|^p_{L^p(\omega)} + \int_\omega \left|q^+ - q^\delta_{\beta,\gamma}\right|^2|q^+|^{p-2}\,\mathrm{d}\mathbf{x}\right)$$

for some generic constant $C > 0$. Noting $q^+,\ q^\delta_{\beta,\gamma} \in K$, we deduce

$$\int_\omega \left|q^+ - q^\delta_{\beta,\gamma}\right|^2\mathrm{d}\mathbf{x} \leqslant \frac{1}{\underline{q}^{p-2}}\int_\omega \left|q^+ - q^\delta_{\beta,\gamma}\right|^2|q^+|^{p-2}\,\mathrm{d}\mathbf{x} \leqslant CD_{\mathfrak{R}'(q^+)}\left(q^\delta_{\beta,\gamma}, q^+\right).$$

This, with the relation $c_r = (1 - c_s)/\gamma$, gives

$$I \leqslant \frac{c_r}{8}\int_{T-\sigma}^T \left\|\nabla u\left(q^\delta_{\beta,\gamma}\right) - \nabla u(q^+)\right\|^2_{L^2(\Omega)}\mathrm{d}t + \frac{2C\gamma}{1 - c_s}D_{\mathfrak{R}'(q^+)}\left(q^\delta_{\beta,\gamma}, q^+\right)\int_{T-\sigma}^T \|\nabla\rho\|^2_{L^\infty(\omega)}\,\mathrm{d}t.$$

Now the nonlinearity condition (3.13) follows from the following estimate

$$\begin{aligned}&\frac{c_r}{4}\left\|\nabla u\left(q^\delta_{\beta,\gamma}\right) - \nabla u(q^+)\right\|_{L^2(T-\sigma,T;L^2(\Omega))} + (1 - \varepsilon)D_{\mathfrak{R}'(q^+)}\left(q^\delta_{\beta,\gamma}, q^+\right)\\&\qquad+ \int_{T-\sigma}^T \left\langle\rho, e_u(q^+, u(q^+))\left(u\left(q^\delta_{\beta,\gamma}\right) - u(q^+) - u'(q^+)\left(q^\delta_{\beta,\gamma} - q^+\right)\right)\right\rangle\mathrm{d}t\\&\qquad\geqslant \frac{c_r}{8}\left\|\nabla u\left(q^\delta_{\beta,\gamma}\right) - \nabla u(q^+)\right\|_{L^2(T-\sigma,T;L^2(\Omega))}\\&\qquad\quad+ \left(1 - \varepsilon - \frac{2C\gamma}{1 - c_s}\int_{T-\sigma}^T \|\nabla\rho\|^2_{L^\infty(\omega)}\,\mathrm{d}t\right)D_{\mathfrak{R}'(q^+)}\left(q^\delta_{\beta,\gamma}, q^+\right)\\&\qquad\geqslant 0\end{aligned}$$

by taking γ to be small enough.

Remark 3.3. From the derivation of this subsection, we can see that the source condition (3.8) can be eventually converted into condition (3.14), which is similar to the source condition (2.9), for the formulation (3.1) with the L^p–H^1 regularization for $p \geqslant 2$. And conditions (2.9) and (3.8) are consistent when the regularization $\mathfrak{R}(q)$ is taken to be $\|\nabla q - \nabla q^*\|^2_{0,\omega}$.

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