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Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers

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Abstract
This paper addresses the uniqueness for an inverse acoustic obstacle scattering problem. It is proved that a general sound-hard polyhedral scatterer in $\mathbb{R}^N (N \geqslant 2)$, possibly consisting of finitely many solid polyhedra and subsets of $(N-1)$-dimensional hyperplanes, is uniquely determined by N far-field measurements corresponding to N incident plane waves given by a fixed wave number and N linearly independent incident directions. A simple proof, which is quite different from that in Alessandrini and Rondi (2005 *Proc. Am. Math. Soc.* **6** 1685–91), is also provided for the unique determination of a general sound-soft polyhedral scatterer by a single incoming wave.

1. Introduction

In this paper, we are interested in an inverse acoustic scattering problem by an impenetrable obstacle D. To describe the scattering system, we shall use u^i, u^s and u to represent the incident, scattered and total field, respectively, where $u = u^i + u^s$, and $u^i(x) = \exp\{\mathrm{j}kx \cdot d\}$ with $\mathrm{j} = \sqrt{-1}$, $d \in \mathbb{S}^{N-1}$ being the incident direction and $k > 0$ being the wave number. Then, the direct scattering problem is described by the following Helmholtz equation:

$$\Delta u + k^2 u = 0 \qquad \text{in} \quad G = \mathbb{R}^N \backslash D. \tag{1}$$

The Helmholtz equation (1) is complemented by the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{(N-1)/2}\left(\frac{\partial u^s}{\partial r} - \mathrm{j}ku^s\right) = 0, \tag{2}$$

with $r = |x|$ for $x \in \mathbb{R}^N$ and either of the following boundary conditions:

$$u = 0 \qquad \text{on} \quad \partial G \text{ (the sound-soft obstacle)}, \tag{3}$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on} \quad \partial G \text{ (the sound-hard obstacle)}, \tag{4}$$

where ν is the unit normal to ∂G pointing to the interior of G.

Throughout, we assume that the obstacle D is a general compact set in $\mathbb{R}^N (N \geqslant 2)$ with an open connected complement $G = \mathbb{R}^N \backslash D$.

It is known (cf [9]) that there exists a unique solution $u = u(D; k, d) \in H^1_{\mathrm{loc}}(G)$ to (1)–(3) or (1), (2) and (4) if ∂G is Lipschitz continuous, and u is analytic on any compact set in G. The Sommerfeld radiation condition (2) characterizes the outgoing wave and enables us to have the following asymptotic behaviour for the scattered wave u^s:

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{j}k|x|}}{|x|^{(N-1)/2}} \left\{ u_\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\} \qquad \text{as} \quad |x| \to \infty, \tag{5}$$

where $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^{N-1}$ and $u_\infty(\hat{x})$ is defined on the unit sphere $\mathbb{S}^{N-1}$, known as the far-field pattern (cf [2]). We shall also write $u_\infty(\hat{x}; D, k, d)$ to specify its dependence on the obstacle D, the wave number k and the incident direction d.

Now the inverse acoustic obstacle scattering problem (IAOSP) is to determine ∂G from the far-field pattern $u_\infty(\hat{x}; D, k, d)$ which can be observed. We remark that, due to the analyticity of the solution to the Helmholtz equation, if the far-field pattern is available in a surface element of the unit sphere $\mathbb{S}^{N-1}$, then it is also known in the whole unit sphere by the unique continuation. An important theoretical issue in IAOSP is the *uniqueness*, i.e., *is the correspondence between* $u_\infty(\hat{x}; D, k, d)$ *and* D *one to one?* This uniqueness is also closely related to finding effective reconstruction algorithms in practical applications.

This paper shall consider the uniqueness issue for the IAOSP with *polyhedral scatterers*. Let us first follow [1] to exactly describe the terminology *polyhedral scatterer*. An obstacle D is said to be a *polyhedral scatterer* if it is a compact subset of $\mathbb{R}^N$ with connected complement $G = \mathbb{R}^N \backslash D$, and the boundary of G is composed of a finite union of cells. A cell, as defined in [1], is the closure of an open subset of an $(N-1)$-dimensional hyperplane. Based on this definition, we can write a two-dimensional polyhedral scatterer D as

$$D = \left(\bigcup_{i=1}^{m} S_i \right) \cup \left(\bigcup_{l=1}^{n} L_l \right),$$

where each S_i is a polygon (screen) and each L_l is a line segment (crack), and write a three-dimensional polyhedral scatterer D as

$$D = \left(\bigcup_{i=1}^{m} P_i \right) \cup \left(\bigcup_{l=1}^{n} S_l \right),$$

where each P_i is a polyhedron (real body) and each S_l is a cell (screen). We emphasize that a cell need not be an $(N-1)$-dimensional polyhedron. Clearly, such a *polyhedral scatterer* is very general and it admits the simultaneous presence of finitely many solid- and crack-type obstacles. A very important and sharp result about the uniqueness for such general sound-soft *polyhedral scatterers* was obtained recently in [1], where it was proved that a single far-field measurement of one single incident plane wave with a fixed wave number and incident direction is sufficient for the unique determination of such a scatterer D. The proof in [1] is based on the study of the structure of the nodal set $\mathcal{N}_u$ (see definition 2.3 in [1]) of u in the interior of G. A key step is to construct a so-called 'hidden path' which connects a point on ∂D to infinity, avoiding the critical points of $\mathcal{N}_u$ but intersecting $\mathcal{N}_u$ orthogonally. However, such construction heavily depends on ordering all the nodal domains, i.e., the connected components of the open set $G \backslash \mathcal{N}_u$, in a special desired manner. But to our regret, there seems to be a gap in the proof of such an ordering. More accurately speaking, the induction argument of [1] (see the proof of proposition 3.2 in [1]) does not necessarily go through all the nodal domains, but only a countable subset of them. This is one of the barriers for the extension of the method in [1] to our current sound-hard case. In fact, there are more difficulties caused by the essential difference between the Dirichlet problem and the Neumann problem.

There are few results concerning the unique determination of a sound-hard obstacle with a finite number of incident waves. The uniqueness for the simple balls with a single incident wave was given in [10]. In [4], a uniqueness result for a two-dimensional sound-hard polygon is presented by two incident plane waves under an extra 'non-trapping' condition, which was then relaxed in [6]. A more recent important advance in the uniqueness for the sound-hard polyhedral obstacle case was announced in [7]. It was demonstrated that a single sound-hard two-dimensional polygon D is uniquely determined by one single incident plane wave. The proof in [7] was based on the investigation of behaviours of the Neumann hyperplanes of the solution u (see definition 1) near ∂G. It is hard to extend the proof of [7] to higher dimensions. The main difficulty is caused by the much more complicated behaviours of the Neumann hyperplanes near ∂G in higher dimensions, and most of the arguments for the $\mathbb{R}^2$ case in [7] seem not to work for the higher dimensions.

The focus of this paper is on the uniqueness of an inverse acoustic scattering problem for a very general sound-hard case: the space can be any dimension larger than 1; the obstacle D is a general *polyhedral scatterer* as described earlier. For example, in two dimensions, D may contain finitely many polygons and line segments. Our main result will demonstrate that N far-field patterns, corresponding to N incident plane waves given by a fixed wave number and N linearly independent incident directions, uniquely determine a *polyhedral scatterer* D in $\mathbb{R}^N$. This seems to be the best known uniqueness result in the literature for sound-hard scatterers of our general setting in $\mathbb{R}^N (N \geqslant 2)$. Our proof shall rely on the reflection principle for the solutions to the Helmholtz equation, the same as in [1, 7]. But our arguments are carried out in a more elementary and simple manner, and work for both sound-hard and sound-soft cases, as well as for general dimensions and general *polyhedral scatterers*.

The rest of the paper is organized as follows. The next section is devoted to the sound-hard case. In section 3, uniqueness for the sound-soft case is treated.

2. Uniqueness for the sound-hard case

We first introduce some notation and definitions for the subsequent use. Let $u_l(x)$, $l = 1, 2, \ldots, N$, be the total fields of (1), (2) and (4) corresponding to the incident waves $\exp\{\mathrm{j}kx \cdot d_l\}$, where $\{d_l\}_{l=1}^N$, with each $d_l \in \mathbb{S}^{N-1}$, are assumed to be linearly independent. We shall write $\mathcal{U} = \{u_1, u_2, \ldots, u_N\}$ and the operations on $\mathcal{U}$ are always understood to be elementwise. For example, for any $\nu \in \mathbb{S}^{N-1}$,

$$\frac{\partial \mathcal{U}}{\partial \nu} = \left\{\frac{\partial u_1}{\partial \nu}, \frac{\partial u_2}{\partial \nu}, \ldots, \frac{\partial u_N}{\partial \nu}\right\},$$

and

$$\frac{\partial \mathcal{U}}{\partial \nu} = 0 \text{ on } S \text{ implies } \frac{\partial u_l}{\partial \nu} = 0 \text{ on } S \text{ for } l = 1, 2, \ldots, N,$$

where S can be any hypersurface in G and ν is its outward normal. Throughout, we will denote an open ball in $\mathbb{R}^N$ with centre x and radius r by $B_r(x)$, the closure of $B_r(x)$ by $\bar{B}_r(x)$ and the boundary of $B_r(x)$ by $S_r(x)$. Based on the earlier definition of a general polyhedral scatterer D of our interest, we can write the boundary of $G = \mathbb{R}^N \backslash D$ as

$$\partial G = \bigcup_{l=1}^{n} C_l \tag{6}$$

where each C_l is a cell in $\mathbb{R}^N$.

Definition 1. *$\mathcal{Z}_{\mathcal{U}}$ is called a Neumann set of $\mathcal{U}$ in G if*

$$\mathcal{Z}_{\mathcal{U}} = \left\{ x \in G;\ \frac{\partial \mathcal{U}}{\partial \nu}\bigg|_{\Pi \cap B_r(x) \cap G} = 0 \text{ for some } r > 0 \text{ and hyperplane } \Pi \text{ passing through } x \right\}.$$

We have the following useful result:

Lemma 1. *For any $x \in \mathcal{Z}_{\mathcal{U}}$, let Π be the corresponding hyperplane involved in the definition of $\mathcal{Z}_{\mathcal{U}}$ and $\widetilde{\Pi}$ be the open connected component of $\Pi\backslash D$ containing x, then*

$$\frac{\partial \mathcal{U}}{\partial \nu}\bigg|_{\widetilde{\Pi}} = 0. \tag{7}$$

Proof. By definition 1, we know that $\frac{\partial \mathcal{U}}{\partial \nu} = 0$ on $\Pi \cap B_r(x) \cap G$. Since $\mathcal{U}$ is analytic in G (cf [2]), then $\frac{\partial \mathcal{U}}{\partial \nu}$ is analytic in $N - 1$ variables on $\Pi\backslash D$, which clearly lies in G. Now observing that $\Pi \cap B_r(x) \cap G$ is an open set on $\Pi \cap D$, we have $\frac{\partial \mathcal{U}}{\partial \nu} = 0$ on $\widetilde{\Pi}$ by analytic continuation. □

We will refer to $\widetilde{\Pi}$ in the above lemma as the *Neumann hyperplane* in what follows, and obviously, it must be an open connected subset of a hyperplane and its boundary lies on ∂G.

Now, we derive some important properties of the Neumann set $\mathcal{Z}_{\mathcal{U}}$.

Lemma 2. *The Neumann set $\mathcal{Z}_{\mathcal{U}}$ and all Neumann hyperplanes are bounded. And $\mathcal{Z}_{\mathcal{U}}$ is closed in the sense that for any sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{Z}_{\mathcal{U}}$, which converges to a point $x_0 \in G$, we must have $x_0 \in \mathcal{Z}_{\mathcal{U}}$, i.e., there exists a Neumann hyperplane $\widetilde{\Pi}_0$ passing through x_0.*

Proof. We first show the boundedness of $\mathcal{Z}_{\mathcal{U}}$. Set $\mathcal{U}^s(x) = \mathcal{U} - \exp\{\mathrm{j}kx \cdot d\}$, with $d = \{d_1, d_2, \ldots, d_N\}$ be the scattered fields, then we have

$$\lim_{|x|\to\infty} |\nabla \mathcal{U}^s(x)| = 0, \tag{8}$$

i.e., $\lim_{|x|\to\infty} \left|\nabla u_l^s(x)\right| = 0$ $(l = 1, 2, \ldots, N)$. The limit (8) can be shown following the proof of lemma 9 in [4]. Now we demonstrate the boundedness of $\mathcal{Z}_{\mathcal{U}}$ by contradiction. If $\mathcal{Z}_{\mathcal{U}}$ is unbounded, then there must exist a Neumann hyperplane $\widetilde{\Pi}$ which connects to infinity. To see this, we first note that D is bounded, so one can bound D by a ball $B_R(0)$ with sufficiently large radius R. By the unboundedness of $\mathcal{Z}_{\mathcal{U}}$, we know there must exist a point $y \in \mathcal{Z}_{\mathcal{U}} \cap (\mathbb{R}^N\backslash \bar{B}_R(0))$, then the corresponding Neumann hyperplane $\widetilde{\Pi}_y$ containing y must connect to infinity. Next, using (7) and (8), we have

$$\lim_{x\in\widetilde{\Pi}:|x|\to\infty} |\partial_\nu \exp\{\mathrm{j}kx \cdot d\}| = 0,$$

where $\nu \in \mathbb{S}^{N-1}$ is the unit normal to $\widetilde{\Pi}$. Hence,

$$\lim_{x\in\widetilde{\Pi}:|x|\to\infty} |\mathrm{j}k(d \cdot \nu) \exp\{\mathrm{j}kx \cdot d\}| = 0.$$

Noting that $k \neq 0$, we have $d \cdot \nu = 0$, or equivalently,

$$\nu \cdot d_l = 0, \qquad l = 1, 2, \ldots, N.$$

But this is impossible since $\nu \in \mathbb{S}^{N-1}$ and $\{d_l\}_{l=1}^N$ are linearly independent. Therefore, $\mathcal{Z}_{\mathcal{U}}$ must be bounded. Clearly, the above proof has also demonstrated that all Neumann hyperplanes must be bounded.

Next, we shall show the closeness of $\mathcal{Z}_{\mathcal{U}}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{Z}_{\mathcal{U}}$ and $x_0 \in G$, such that $\lim_{n\to\infty} x_n = x_0$. Taking a sufficiently small hypercube $T_r(x_0)$ of edge length r and centred at x_0 such that the closure of $T_r(x_0)$ lies entirely in G. Without loss of generality, we may assume that $\{x_n\}_{n=1}^{\infty} \subset T_r(x_0)$. Let $\widetilde{\Pi}_n$ be the Neumann hyperplane through x_n such that

$$\frac{\partial \mathcal{U}}{\partial \nu_n}\bigg|_{\widetilde{\Pi}_n \cap T_r(x_0)} = 0,$$

where ν_n is the unit normal to $\widetilde{\Pi}_n$. Let us write $\nu(x_n) = \nu_n$, then by possibly extracting a subsequence, we may assume that $\nu(x_n) \to \nu_0$ as $n \to \infty$ and write $\nu(x_0) = \nu_0 \in \mathbb{S}^{N-1}$. Let Π_0 be a hyperplane through x_0 and have ν_0 as its normal, then we can show that for any $P_0 \in \Pi_0 \cap T_r(x_0)$, there exists a sequence of points $\{P_n\}_{n=1}^{\infty}$ such that $P_n \in \Pi_n$ for each n, where Π_n is the hyperplane in $\mathbb{R}^N$ containing the Neumann hyperplane $\widetilde{\Pi}_n$ and $\lim_{n\to\infty} P_n = P_0$. To see this, let L be the straight line through P_0 with direction ν_0, then any point $P \in L$ is given by

$$P = P_0 + t\nu_0 \qquad \text{for some} \quad t \in \mathbb{R}.$$

Noting that the equation for the hyperplane Π_n is given by

$$(P - x_n) \cdot \nu_n = 0 \qquad \text{for any} \quad P \in \Pi_n.$$

Since $\nu_n \to \nu_0$ as $n \to \infty$, we can assume that $\nu_n \cdot \nu_0 \neq 0$ for $n \in \mathbb{N}$. Then by straightforward calculations, we can show that L intersects with each Π_n, and the intersection point is given by

$$P_n = P_0 + t_n\nu_0 \qquad \text{with} \quad t_n = \tfrac{(x_n - P_0)\cdot\nu_n}{\nu_0\cdot\nu_n}, \quad n = 1, 2, \ldots.$$

Using the facts that

$$(P_0 - x_0) \cdot \nu_0 = 0, \qquad \lim_{n\to\infty} x_n = x_0,$$

we see

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{(x_n - P_0)\cdot\nu_n}{\nu_0\cdot\nu_n} = \lim_{n\to\infty} \frac{(x_n - x_0)\cdot\nu_n}{\nu_0\cdot\nu_n} + \lim_{n\to\infty} \frac{(x_0 - P_0)\cdot\nu_n}{\nu_0\cdot\nu_n} = 0,$$

this implies

$$\lim_{n\to\infty} P_n = P_0.$$

Since P_n converges to $P_0 \in \Pi_0 \cap T_r(x_0)$ along the ν_0-direction, we may assume that for all n, $P_n \in T_r(x_0)$, i.e., $P_n \in \widetilde{\Pi}_n \cap T_r(x_0)$. Noting that $\nabla\mathcal{U}$ is continuous in the closure of $T_r(x_0)$, we have

$$\frac{\partial \mathcal{U}}{\partial \nu_0}(P_0) = \nabla\mathcal{U}(P_0)\cdot\nu_0 = \lim_{n\to\infty} \nabla\mathcal{U}(P_n)\cdot\nu_n = 0.$$

Thus, we have $x_0 \in \mathcal{Z}_{\mathcal{U}}$. The proof is completed. □

Next, we recall a fundamental property for a connected set (see theorem 3.19.9 in [5]), which will be used in our subsequent arguments.

Lemma 3. *Let E be a metric space, $A \subset E$ be a subset and $B \subset E$ be a connected set such that $A \cap B \neq \emptyset$ and $(E\backslash A) \cap B \neq \emptyset$, then $\partial A \cap B \neq \emptyset$.*

Now, we are ready to present our main uniqueness result for a sound-hard polyhedral scatterer.

Theorem 1. *Let $d_l \in \mathbb{S}^{N-1}$, $l = 1, 2, \ldots, N$, be N linearly independent directions and $k > 0$ be fixed. A polyhedral scatterer D described in (6) is uniquely determined by the far-field patterns $\mathcal{U}_\infty = \{u_{1,\infty}, u_{2,\infty}, \ldots, u_{N,\infty}\}$.*

Proof. We shall prove the theorem by contradiction. First, we follow [1] and [7] to show that if theorem 1 does not hold, then we can assume that there exists a Neumann hyperplane $\widetilde{\Pi}_1$ in $G = \mathbb{R}^N \backslash D$. To see this, let D' be a polyhedral scatterer different from D and u' be the solution to (1), (2) and (4) when D is replaced by D'. And similarly, we write $\mathcal{U}' = \{u'_1, u'_2, \ldots, u'_N\}$ and $\mathcal{U}'_\infty = \{u'_{1,\infty}, u'_{2,\infty}, \ldots, u'_{N,\infty}\}$ for the total fields and far-field patterns corresponding to the incident waves $\exp\{\mathrm{j}kx \cdot d_l\}$, $l = 1, 2, \ldots, N$.

If the theorem is not true, then we can assume $\mathcal{U}_\infty = \mathcal{U}'_\infty$ for N given linearly independent $d_l \in \mathbb{S}^{N-1}, l = 1, 2, \ldots, N$, and fixed $k > 0$. Letting Ω be the unbounded connected component of $\mathbb{R}^N \backslash (D \cup D')$, then by theorem 2.13 [2] we infer that $\mathcal{U} = \mathcal{U}'$ over Ω.

First, we can see $\partial\Omega \not\subset D \cap D'$ from the connectedness of both G and $G' = \mathbb{R}^N \backslash D'$. Indeed, if $\partial\Omega \subset D \cap D'$, then we must have $\Omega = \mathbb{R}^N \backslash D = \mathbb{R}^N \backslash D'$. To see this, we first observe that $\Omega \subset \mathbb{R}^N \backslash D$ and $\Omega \subset \mathbb{R}^N \backslash D'$ by noting $\Omega \subset \mathbb{R}^N \backslash (D \cup D')$. On the other hand, if there exist $x \in \mathbb{R}^N \backslash D$ and $x' \in \mathbb{R}^N \backslash D'$ such that $x \not\in \Omega$ and $x' \not\in \Omega$, we obtain from lemma 3 (with $A = \Omega$ and $B = \mathbb{R}^N \backslash D$ or $B = \mathbb{R}^N \backslash D'$) that $\partial\Omega \cap (\mathbb{R}^N \backslash D) \neq \emptyset$ and $\partial\Omega \cap (\mathbb{R}^N \backslash D') \neq \emptyset$, which contradicts the assumption that $\partial\Omega \subset D \cap D'$, thus leading to $\mathbb{R}^N \backslash D \subset \Omega$ and $\mathbb{R}^N \backslash D' \subset \Omega$. Therefore, $\Omega = \mathbb{R}^N \backslash D = \mathbb{R}^N \backslash D'$, which implies $D = D' = \mathbb{R}^N \backslash \Omega$. But this contradicts the fact that D and D' are two different polyhedral scatterers.

Using the previous conclusion that $\partial\Omega \not\subset D \cap D'$, we must have $(\partial G' \backslash D) \cap \partial\Omega \neq \emptyset$ or $(\partial G \backslash D') \cap \partial\Omega \neq \emptyset$. Without loss of generality, we may assume the first case held and therefore there exists a point $\tilde{x}' \in (\partial G' \backslash D) \cap \partial\Omega$. We can also assume that $\tilde{x}'$ belongs to the interior of one of the cells composing $\partial G'$, and so there exists a hyperplane Π_1 and $r > 0$ such that $\tilde{x}' \in S_1 = \Pi_1 \cap B_r(\tilde{x}') \subset (\partial G' \backslash D) \cap \partial\Omega$. Since $\mathcal{U} = \mathcal{U}'$ in Ω, by noting $\frac{\partial \mathcal{U}'}{\partial \nu} = 0$ on $S_1 \subset \partial G'$, we have $\frac{\partial \mathcal{U}}{\partial \nu} = 0$ on S_1. Hence, $\tilde{x}'$ is contained in the Neumann set of $\mathcal{U}$ in G and S_1 is contained in a Neumann hyperplane of $\mathcal{U}$ in G, which we denote by $\widetilde{\Pi}_1$.

Next, we start from this Neumann hyperplane $\widetilde{\Pi}_1$ to build up a contradiction.

In the following, a curve $\gamma = \gamma(t)(t \geqslant 0)$ is said to be regular if it is C^1-smooth and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \neq 0$. And the notation Π_l, with l being an integer, shall always represent a hyperplane in $\mathbb{R}^N$, which contains a Neumann hyperplane $\widetilde{\Pi}_l$. Since G is an unbounded open connected set, hence the open set $G \backslash \widetilde{\Pi}_1$ must contain an open connected component, denoted as $\widetilde{G}$, which connects to the infinity. In fact, $\widetilde{G}$ is unique because $\widetilde{\Pi}_1$ is bounded by lemma 2 and G cannot be divided by $\widetilde{\Pi}_1$ into more than one unbounded open component, otherwise ∂G is unbounded. Thus, $\widetilde{\Pi}_1$ lies on $\partial\widetilde{G}$, due to the fact that every point on $\widetilde{\Pi}_1$ is in G and so can be connected to the infinity. Next, we fix an arbitrary point $x_1 \in \widetilde{\Pi}_1$. Let $\gamma = \gamma(t)(t \geqslant 0)$ be a regular curve such that $\gamma(0) = x_1$, $\gamma(t)(t > 0)$ lies entirely in $\widetilde{G}$ and $\lim_{t\to\infty} |\gamma(t)| = +\infty$. Clearly, γ lies on one side of Π_1, that is, $\gamma(t) \in \Pi_1$ iff $t = 0$, and we set $t_1 = 0$ (we refer to [8, 11] for the properties of open connected set). For convenience, we choose $\gamma(t)$ to be as 'straight' as possible in the sense that there are as few snakelike portions as possible. For example, we may let $\gamma(t)$ be given by consecutively connected line segments in $\widetilde{G}$ but with C^1-smooth junctions to connect two neighbouring line segments and let it be a straight line outside a sufficiently large ball containing D.

Next, define the distance between two sets A and B in $\mathbb{R}^N$ as usual:

$$\mathbf{d}(A, B) = \inf_{x\in A, y\in B} |x - y|.$$

Let

$$\mathbf{d}_l = \mathrm{d}(\gamma, C_l), \qquad l = 1, 2, \ldots, n, \tag{9}$$

and

$$r_0 = \tfrac{1}{2} \min_{1\leqslant l\leqslant n} \mathbf{d}_l. \tag{10}$$

Noting that γ is a closed set in $\mathbb{R}^N$ and $\{C_l\}_{l=1}^n$, which form ∂G, are compact sets, it can be readily seen that $\mathbf{d}_l > 0, l = 1, 2, \ldots, n$, are attainable. Hence, $r_0 > 0$ and for any point $x \in \gamma(t)$, we have $\bar{B}_{r_0}(x) \subset G$.

Let $\tilde{x}_2^+ = \gamma(\tilde{t}_2) \in S_{r_0}(x_1) \cap \gamma$, and $\tilde{x}_2^- \in S_{r_0}(x_1)$ be the symmetric point of $\tilde{x}_2^+$ with respect to Π_1. We remark that by lemma 3, γ must intersect $S_{r_0}(x_1)$, but the intersection need not necessarily be a unique point. For definiteness, we take $\tilde{t}_2 = \max\{t > 0; \gamma(t) \in S_{r_0}(x_1)\}$. Now, let G_1^+ be the connected component of $G\backslash\widetilde{\Pi}_1$ containing $\tilde{x}_2^+$ and G_1^- be the connected component of $G\backslash\widetilde{\Pi}_1$ containing $\tilde{x}_2^-$. It is remarked that it may happen that $G_1^+ = G_1^-$. We denote by R_1 the reflection with respect to Π_1, then let E_1^+ be the connected component of $G_1^+ \cap R_1(G_1^-)$ containing $\tilde{x}_2^+$ and E_1^- be the connected component of $G_1^- \cap R_1\big(G_1^+\big)$ containing $\tilde{x}_2^-$. Observe that $E_1^+ = R_1(E_1^-)$, and if we set $E_1 = E_1^+ \cup \widetilde{\Pi}_1 \cup E_1^-$, then E_1 contains the closed ball $\bar{B}_{r_0}(x_1)$. Moreover, E_1 is a connected open set with the boundary composed of subsets of the cells $\{C_l\}_{l=1}^n$ and $\{R_1(C_l)\}_{l=1}^n$. One can easily verify that $\mathcal{U}(x) - R_1\mathcal{U}(x)$, where $R_1\mathcal{U}(x) = \mathcal{U}(R_1(x))$, is a solution to the Helmholtz equation in E_1 with zero Dirichlet and Neumann data on $\widetilde{\Pi}_1 \cap \bar{B}_{r_0}(x_1)$, therefore $\mathcal{U}(x) = R_1\mathcal{U}(x)$ in E_1 by Holmgren's theorem (cf theorem 6.12 in [3]), i.e., $\mathcal{U}$ is even symmetric in E_1 with respect to the hyperplane Π_1. This indicates $\frac{\partial\mathcal{U}}{\partial\nu_1}\big|_{E_1\cap\Pi_1} = 0$, where ν_1 is the unit normal to Π_1. Next, we show that E_1 is bounded. Clearly, we first see ∂E_1, $\partial G_1^\pm$ and $R_1\big(\partial G_1^\pm\big)$ are bounded by our construction. If E_1 is unbounded, then E_1 would contain $\mathbb{R}^N\backslash B_r(x_1)$ for some sufficiently large $r > 0$. Then using $\frac{\partial\mathcal{U}}{\partial\nu_1}\big|_{E_1\cap\Pi_1} = 0$ and analytic continuation, $\Pi_1\backslash B_r(x_1)$ are parts of some Neumann hyperplanes. This contradicts lemma 2, and so E_1 is bounded. Now by the unboundedness of γ, there must exist a $t_2 > \tilde{t}_2$, such that $x_2 = \gamma(t_2) \in \partial E_1$. Noting ∂E_1 is composed of subsets of the cells $\{C_l\}_{l=1}^n$ and $\{R_1(C_l)\}_{l=1}^n, \mathcal{U}$ takes zero Neumann data on ∂E_1 by using the fact that $R_1\mathcal{U}(x) = \mathcal{U}(x)$ in E_1. Thus by analytic continuation, $x_2 \in \partial E_1$ implies the existence of a Neumann hyperplane passing through x_2, which we denote by $\widetilde{\Pi}_2$, and we have $x_2 = \gamma(t_2) \in \mathcal{Z}_{\mathcal{U}}$. Furthermore, we may assume that $\gamma(t_2)$ is the 'last' point on γ to intersect $\widetilde{\Pi}_2$, that is,

$$t_2 = \max\{t > 0; \gamma(t) \in \widetilde{\Pi}_2\} < \infty.$$

The following two facts shall be crucial: $\widetilde{\Pi}_2$ is different from $\widetilde{\Pi}_1$, since $\widetilde{\Pi}_1$ intersects γ only at x_1; the length of $\gamma(t)$ from t_1 to t_2 is larger than r_0, i.e.,

$$|\gamma(t_1 \leqslant t \leqslant t_2)| \geqslant |\gamma(t_1 \leqslant t \leqslant \tilde{t}_2)| \geqslant r_0.$$

Next, let $\tilde{x}_3^+ = \gamma(\tilde{t}_3) \in S_{r_0}(x_2) \cap \gamma$, and $\tilde{x}_3^- \in S_{r_0}(x_2)$ be the symmetric point of $\tilde{x}_3^+$ with respect to Π_2, then let G_2^+ be the connected component of $G\backslash\widetilde{\Pi}_2$ containing $\tilde{x}_3^+$ and G_2^- be the connected component of $G\backslash\widetilde{\Pi}_2$ containing $\tilde{x}_3^-$. Denote by R_2 the reflection with respect to Π_2, and let E_2^+ be the connected component of $G_2^+ \cap R_2(G_2^-)$ containing $\tilde{x}_3^+$ and E_2^- be the connected component of $G_2^- \cap R_2\big(G_2^+\big)$ containing $\tilde{x}_3^-$. Set $E_2 = E_2^+ \cup \widetilde{\Pi}_2 \cup E_2^-$, then we see that E_2 contains the closed ball $\bar{B}_{r_0}(x_2)$ and its boundary is composed of subsets of the cells $\{C_l\}_{l=1}^n$ and $\{R_2(C_l)\}_{l=1}^n$. By a similar argument as used earlier for deriving $x_2 = \gamma(t_2)$ and $\widetilde{\Pi}_2$, there exists a point $x_3 = \gamma(t_3)$ $(t_3 > t_2)$ and a Neumann hyperplane $\widetilde{\Pi}_3$ passing through x_3. Again, we may assume that x_3 is the 'last' point to pass through Π_3. We see that $\widetilde{\Pi}_3$ is different from $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$, since $x_1 = \gamma(t_1)$ and $x_2 = \gamma(t_2)$ are, respectively, the last point to pass through $\widetilde{\Pi}_1$ and $\widetilde{\Pi}_2$, and the length of $\gamma(t)$ from t_2 to t_3 is larger than r_0, i.e.,

$$|\gamma(t_2 \leqslant t \leqslant t_3)| \geqslant r_0.$$

Continuing with the above procedure, we can construct a strictly increasing sequence $\{t_n\}_{n=1}^\infty$ such that for any $n, x_n = \gamma(t_n) \in \mathcal{Z}_{\mathcal{U}}$ and $\widetilde{\Pi}_n$ is a Neumann hyperplane passing through x_n. Moreover, those Neumann hyperplanes are different from each other, and the length of $\gamma(t)$ from t_n to t_{n+1} is not less than r_0, i.e.,

$$|\gamma(t_n \leqslant t \leqslant t_{n+1})| \geqslant r_0. \tag{11}$$

Since $\mathcal{Z}_{\mathcal{U}}$ is bounded and $\lim_{t\to\infty} |\gamma(t)| = +\infty$, so we must have $\lim_{n\to\infty} t_n = t_0$ for some finite t_0. Otherwise, we would have $\lim_{n\to\infty} t_n = +\infty$ due to the fact that t_n is strictly increasing and this further implies $\lim_{n\to\infty} |\gamma(t_n)| = +\infty$, contradicting that $\gamma(t_n) = x_n \in \mathcal{Z}_{\mathcal{U}}$ for each n and the boundedness of $\mathcal{Z}_{\mathcal{U}}$. Finally, because $\gamma(t)$ is a C^1-smooth curve, we must have that

$$\lim_{n\to\infty} |\gamma(t_n \leqslant t \leqslant t_{n+1})| = \lim_{n\to\infty} \int_{t_n}^{t_{n+1}} |\gamma'(t)|\,\mathrm{d}t = 0, \tag{12}$$

which contradicts the inequality (11), thus completes the proof of theorem 1. □

3. Uniqueness for the sound-soft case

In this section, we extend the arguments in the previous section to the sound-soft case, but with some adaptations, which, we think, might provide some alternative thinking for the further study of the uniqueness issues for IAOSP. Such a uniqueness result was given in [1], but there seems to exist some gap in its proof, as we have pointed out in the introduction. Below we shall provide a different and relatively simpler proof.

In correspondence with the *Neumann set* and *Neumann hyperplane* for the sound-hard case, we introduce the *Dirichlet set* and *Dirichlet hyperplane* for the current sound-soft case. Let $u(x)$ be the total field to (1)–(3) associated with a single incident wave $u^i(x) = \exp\{\mathrm{j}kx \cdot d\}$ with fixed k and d.

Definition 2. *$\mathcal{D}_u$ is called a Dirichlet set of u in G, if*

$$\mathcal{D}_u = \{x \in G;\ u|_{\Pi\cap B_r(x)\cap G} = 0 \text{ for some } r > 0 \text{ and hyperplane } \Pi \text{ passing through } x\}.$$

Similar to lemma 1, we have

Lemma 4. *For any $x \in \mathcal{D}_u$, let $\widetilde{\Pi}$ be the corresponding open connected component of $\Pi\backslash D$ containing x, then the following holds:*

$$u|_{\widetilde{\Pi}} = 0. \tag{13}$$

We will refer to $\widetilde{\Pi}$ in the above lemma as the *Dirichlet hyperplane* in the following. Obviously, the same as the *Neumann hyperplane*, a Dirichlet hyperplane must be an open connected subset of a hyperplane and its boundary lies on ∂G.

The following lemma is a counterpart of lemma 2 for the sound-hard case.

Lemma 5. *The Dirichlet set $\mathcal{D}_u$ and all Dirichlet hyperplanes are bounded. And $\mathcal{D}_u$ is closed in the sense that for any sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}_u$, which converges to a point $x_0 \in G$, then we have $x_0 \in \mathcal{D}_u$, i.e., there exists a Dirichlet hyperplane Π_0 passing through x_0.*

Proof. For the boundedness of $\mathcal{D}_u$, we refer to lemma 3.1 in [1]. And the closeness of $\mathcal{D}_u$ can be proved in a similar way to the proof of lemma 2. □

Now, the uniqueness result is stated in the following theorem.

Theorem 2. *The polyhedral scatterer D described in (6) is uniquely determined by a single far-field pattern u_∞ corresponding to an incident wave* $\exp\{jkx \cdot d\}$ *with* $k > 0$ *and* $d \in \mathbb{S}^{N-1}$ *fixed.*

Proof. By contradiction, similar to the proof of theorem 1, we can assume that there exists a Dirichlet hyperplane $\widetilde{\Pi}_1$ in $G = \mathbb{R}^N \backslash D$. Then the rest of the proof can be carried out in the same way as for the sound-hard case in theorem 1. But to provide a possible alternative thinking for the uniqueness for the inverse acoustic scattering problem, we shall present a different analysis below to prove theorem 2. As done in theorem 1, with the help of the reflection principle for the Dirichlet problem and the auxiliary function $u(x) + Ru(x)$, we can find a countable set of distinct Dirichlet hyperplanes $\{\widetilde{\Pi}_n\}_{n=1}^{\infty}$ and a sequence of points $x_n = \gamma(t_n)$ with $\{t_n\}_{n=1}^{\infty}$ being a strictly increasing sequence and x_n lying on $\widetilde{\Pi}_n$. Like in the proof of theorem 1, we can choose a uniform radius r_0 for the balls $B_{r_0}(x_n)(n = 1, 2, \ldots)$. But for our purpose, we specifically choose a sequence of balls $B_{r_n}(x_n)$ lying entirely in G with distinct radii r_n. Also, we can find a finite t_0 such that $t_n \to t_0$ as $n \to \infty$ and a Dirichlet hyperplane $\widetilde{\Pi}_0$ passing through $x_0 = \gamma(t_0)$. Further, there exists a sufficiently small hypercube $T_r(x_0)$ such that $\widetilde{\Pi}_n \cap T_r(x_0) \to \widetilde{\Pi}_0 \cap T_r(x_0)$ as $n \to \infty$, in the sense that for any $P_0 \in \widetilde{\Pi}_0 \cap T_r(x_0)$, there exists in the unit normal ν_0-direction to $\widetilde{\Pi}_0$ a sequence $\{P_n\}_{n=1}^{\infty}$, with $P_n \in \Pi_n$ for each $n \in \mathbb{N}$, given by

$$P_n = P_0 + t_n \nu_0, \qquad t_n = \frac{(x_n - P_0) \cdot \nu_n}{\nu_0 \cdot \nu_n}, \quad n = 1, 2, \ldots, \tag{14}$$

and P_n converges to P_0. Now, if there are infinitely many P_n s which are different from each other, then using $P_n \to P_0$ as $n \to \infty$, we may assume that $P_n \in \widetilde{\Pi}_n \cap T_r(x_0)$ for all $n \in \mathbb{N}$. This implies

$$\frac{\partial u}{\partial \nu_0}(P_0) = \lim_{n \to \infty} \frac{u(P_n) - u(P_0)}{t_n} = 0.$$

Due to our construction, all Dirichlet hyperplanes $\widetilde{\Pi}_n$ are different from each other. Next, we claim that $\frac{\partial u}{\partial \nu_0} = 0$ a.e. on $\widetilde{\Pi}_0 \cap T_r(x_0)$. If this is not true, we would have a sufficiently small ball $B_{\tilde{r}}(Q_1) \subset T_r(x_0)$, where $Q_1 \in \widetilde{\Pi}_0$, such that for every $P_0 \in B_{\tilde{r}}(Q_1) \cap \widetilde{\Pi}_0$, only finitely many out of the sequence $\{P_n\}_{n=1}^{\infty}$ given in (14) are different from each other. Thus, there exists an $N_0 \in \mathbb{N}$ such that $t_n = 0$ for $n > N_0$, namely, $P_0 \in \Pi_n$ for $n > N_0$. Now, we choose N points $Q_l \in B_{\tilde{r}}(Q_1) \cap \widetilde{\Pi}_0 (l = 1, 2, \ldots, N)$ such that the vectors $Q_1 Q_l (2 \leqslant l \leqslant N)$ are linearly independent. Let $N_l \in \mathbb{N}$ be the integer such that $Q_l \in \Pi_n$ for $n > N_l, l = 1, 2, \ldots, N$, and $M = \max\{N_1, \ldots, N_l\}$, then for $n > M$, we have $Q_1 Q_l \subset \Pi_n$, for all $l \geqslant 2$. Since $Q_1 Q_l, l = 2, \ldots, N$, are linearly independent and all lie on both Π_n and Π_0, then Π_n must coincide with Π_0 when $n > M$. This contradicts our construction that $\{\widetilde{\Pi}_n\}_{n=1}^{\infty}$ are countable different Dirichlet hyperplanes. So, we have demonstrated that $\frac{\partial u}{\partial \nu_0} = 0$ a.e. on $\widetilde{\Pi}_0 \cap T_r(x_0)$, which implies $\frac{\partial u}{\partial \nu_0} = 0$ in $\widetilde{\Pi}_0 \cap T_r(x_0)$ by the analyticity of u. Noting that we also have $u = 0$ on $\widetilde{\Pi}_0 \cap T_r(x_0)$, then by Holmgren's theorem we must have $u = 0$ over G. But this contradicts lemma 5, so completes the proof of theorem 2. □

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