A new approach to convergence rate analysis of Tikhonov regularization for parameter identification in heat conduction

Heinz W Engl† and Jun Zou‡

† Institut für Industriemathematik, Johannes Kepler Universität, A-4040 Linz, Austria
‡ Department of Mathematics, The Chinese University of Hong Kong, Shatin, NT, Hong Kong, People's Republic of China

E-mail: engl@indmath.uni-linz.ac.at and zou@math.cuhk.edu.hk

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Abstract. In this paper we investigate the stability and convergence rates of the widely used output least-squares method with Tikhonov regularization for the identification of the conductivity distribution in a heat conduction system. Due to the rather restrictive source conditions and regularity assumptions on the nonlinear parameter-to-solution operator concerned, the existing Tikhonov regularization theory for nonlinear inverse problems is difficult to apply for the convergence rate analysis here. By introducing some new techniques, we are able to relax these regularity requirements and derive a much simpler and easily interpretable source condition but still achieve the same convergence rates as the standard Tikhonov regularization theory does.

1. Introduction

It is well known that the heat conduction of a conductive body Ω can be modelled by the parabolic system

$$\frac{\partial u}{\partial t} - \nabla \cdot (q(x)\nabla u) = f(x,t) \qquad \text{in} \quad \Omega \times (0,T) \tag{1.1}$$

with the initial temperature

$$u(x,0) = u_0(x) \qquad \text{in} \quad \Omega \tag{1.2}$$

and, for example, vanishing boundary temperature

$$u(x,t) = 0 \qquad \text{on} \quad \partial\Omega \times (0,T). \tag{1.3}$$

Here f is a heat source density, and Ω is assumed to be an open bounded and connected domain in R^d ($d \geqslant 1$) with a piecewise smooth boundary $\partial\Omega$.

The identification of the heat conductivity distribution $q(x)$ in the system (1.1)–(1.3) can find a wide range of applications in engineering and industry. For example, one can use some measured temperature data of a newly discovered material to identify the heat conductivity of the material. In addition, the parabolic system (1.1)–(1.3) can serve as the mathematical model for many other processes, such as diffusion processes and population dynamics [5, 6], where the need to identify the diffusivity q also arises.

Such parameter identification problems are usually ill posed in the sense that small perturbations in the measurement data of $u(x,t)$ can have tremendous effects on the parameter $q(x)$ (cf e.g. [1, 3]). Therefore, for the numerical identification process, some type of

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regularization has to be introduced. One most frequently used approach is the so-called output least-squares method with Tikhonov regularization.

Many numerical experiments have demonstrated that, when combined with some appropriate Tikhonov regularization, the output least-squares method performs very well for the identification of various types of heat conductivity distribution: continuous, discontinuous or highly oscillating [2, 8, 9]. It also works satisfactorily with the identification of some nonlinear heat conductivity parameters, namely with q depending on the temperature u [14].

A natural interesting question is whether one can justify this good numerical behaviour and the nice stability achieved by the Tikhonov regularization method mathematically. This justification is important to provide some useful guidance and give practioners certain confidence for their numerical identification process. For ill-posed problems, convergence of any numerical algorithm can be arbitrarily slow. Hence, results that give sufficient conditions for convergence *rates* are not only of theoretical interest but also of practical importance, since they tell the practitioners for which problems fast convergence of numerical algorithms can be expected.

Starting from [16], [4], [11] and [12], there exists a large amount of literature on the convergence theory including stability and convergence rate estimates for the Tikhonov regularization method for nonlinear ill posed problems. Below we briefly summarize the existing general theory and our new contribution in this paper.

Consider the nonlinear ill posed equation

$$F(q) = u \tag{1.4}$$

where $F : K \subset Q \to U$ is a nonlinear mapping between the Hilbert spaces Q and U, and K is some admissible set of the parameters. The output least-squares method with Tikhonov regularization is then formulated as follows:

$$\min_{q\in K} \|F(q) - u^\delta\|_U^2 + \beta\|q - q^*\|_Q^2. \tag{1.5}$$

Here q^* is some *a priori* estimate of the true parameter q, and u^δ is the observed data of u with a measurement error of level δ, that is,

$$\|u - u^\delta\|_U \leqslant \delta.$$

For stability (with fixed $\beta > 0$) and convergence (with $\beta \to 0$) to a solution $q^\dagger$ of (1.4), which is closest to the *a priori* estimate q^*, the 'weak closedness' of F is needed:

$$q_n \rightharpoonup q \qquad \text{and} \qquad F(q_n) \rightharpoonup u \qquad \text{imply} \quad F(q) = u. \tag{1.6}$$

Moreover, if F is Fréchet differentiable and its Fréchet derivative F' is Lipschitz continuous with Lipschitz constant L, and there exists a $w \in U$ such that the so-called 'source condition'

$$q^\dagger - q^* = F'(q^\dagger)^* w \tag{1.7}$$

holds and such a w is small enough, i.e.

$$L\|w\|_U < 1, \tag{1.8}$$

then the regularized minimizers q_β^δ of (1.5) converge to $q^\dagger$ with the rate $\delta^{1/2}$ when β is chosen proportional to the noise level δ:

$$\|q_\beta^\delta - q^\dagger\|_Q = \mathrm{O}(\sqrt{\delta}). \tag{1.9}$$

The source condition (1.7) is usually some type of *a priori* smoothness condition. In the seminorm-regularization case, i.e.

$$\min_{q\in K} \|F(q) - u^\delta\|_U^2 + \beta\|D(q - q^*)\|_Z^2 \tag{1.10}$$

with $D : \mathrm{dom} D \to Z$ being a linear operator and Z a Hilbert space, the source condition (1.7) takes the form (cf [11])

$$D^* D(q^\dagger - q^*) = F'(q^\dagger)^* w, \tag{1.11}$$

while the smallness condition (1.8) becomes

$$L\|w\|_U < \kappa^2 \tag{1.12}$$

with κ being the lower bound given by

$$\|F'(q^\dagger)q\|_U^2 + \|Dq\|_Z^2 \geqslant \kappa^2(\|q\|_Q^2 + \|Dq\|_Z^2), \qquad \forall\, q \in \mathrm{dom} D.$$

This general theory has been applied to some specific inverse problems including parameter identifications (see [4, 11] and [3, section 10.5] for elliptic problems and integral equations and [15] for a parabolic equation identifying the capacity parameter). All these applications are for one-dimensional problems. However, the restrictive conditions required by the general convergence theory do not hold for our currently considered d-dimensional ($d \geqslant 1$) heat conduction problem because of the general weak settings adopted here. Even if one supposes the general theory applies, the source conditions (1.7) and (1.11) will be too complicated to lead to some reasonable geometric interpretation. The main difficulty is that there is no clear understanding of the derivative $F'(q)$ of the parameter-to-solution map and its adjoint $F'(q)^*$, since they often have no physical meanings, unlike $F(q)$ itself. Most importantly, the smallness conditions (1.8) and (1.12) for the source function appear to be extremely restrictive.

The main contribution of this paper is to introduce some new techniques to avoid the use of those restrictive requirements in the general convergence theory, e.g. the Fréchet differentiability of $F(q)$ and the uniform Lipschitz continuity of the Fréchet derivative $F'(q)$. With the new techniques, most surprisingly, we are able to formulate a much simpler source condition and to get rid of the smallness condition for the source function. Moreover, our source condition will use the parameter-to-solution map $F(q)$ itself, instead of its derivative $F'(q)$ and the adjoint $F'(q)^*$, and so can be interpreted much more easily. The key to this is that we use a modified kind of adjoint which is intimately related to the weak form of (1.1). For the first time one can even explicitly find a source function w for the one-dimensional case (section 7). All this still ensures that we preserve the usual convergence rate $\mathrm{O}(\sqrt{\delta})$ under much weaker and more realistic conditions. We will remark that this new theory can be applied to many other inverse problems including elliptic and parabolic problems with nonlinear source terms or nonlinear heat conductivity parameters, and to the case with measurements on a subdomain of Ω.

2. Output least-squares formulation with Tikhonov regularization

In this section we formulate the output least-squares method with Tikhonov regularization for the identification of the heat conductivity in the system (1.1)–(1.3) in our new setting. First by integration by parts we can immediately derive the variational formulation for (1.1)–(1.3).

Find $u \in L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ such that $u(x, 0) = u_0(x)$ for $x \in \Omega$ and

$$\int_\Omega u_t\, \varphi \,\mathrm{d}x + \int_\Omega q(x)\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f(x, t)\, \varphi \,\mathrm{d}x, \qquad \forall\varphi \in H_0^1(\Omega) \tag{2.1}$$

for a.e. $t \in (0, T)$. For our subsequent analysis, we may often use the notation $u(q)$ to denote the solution of (2.1) in the case where we want to emphasize the dependence of u on the parameter q.

Assume that we are given some terminal status observation data $z(x,t)$ of the solution $u(x,t)$, either in gradient form or in pointwise form. Then the *inverse* problem to be studied in this paper is to identify the heat conductivity $q(x)$ using the observation data $z(x,t)$ (cf e.g. [10]).

We now formulate the output least-squares method with Tikhonov regularization for the identification process. Let $\sigma > 0$ be a very small time period, and $Q_\sigma = \Omega \times [T-\sigma, T]$. We will consider the following constrained set of parameters, in which we will search for the desired unknown parameter:

$$K = \{q \in H^1(\Omega);\, \alpha_1 \leqslant q(x) \leqslant \alpha_2 \text{ for a.e. } x \in \Omega\}, \tag{2.2}$$

where α_1 and α_2 are two positive constants. We consider the following two cases of terminal observations.

Case (a). The terminal status observation data of $u(x,t)$ is available in a gradient form: $\nabla z(x,t) = \nabla u(x,t)$, $(x,t) \in Q_\sigma$. This means the distributed measurements of temperature gradients (it can be replaced by the heat flux $-q\nabla u$) are available. Then we use the following regularized output least-squares formulation for the parameter identification process:

$$\min_{q\in K} J_1(q) := \int_{T-\sigma}^{T}\int_\Omega q(x)\,|\nabla(u(q)-z)|^2\,\mathrm{d}x\,\mathrm{d}t + \beta\int_\Omega |\nabla(q-q^*)|^2\,\mathrm{d}x \tag{2.3}$$

with $u(q)$ solving the problem (2.1). Here the energy norm is used as measurement norm.

Case (b). The terminal status observation data of $u(x,t)$ is available in a pointwise form: $z(x,t) = u(x,t)$, $(x,t) \in Q_\sigma$. Then we use the L^2-norm as the measurement norm and formulate the regularized parameter identification problem as

$$\min_{q\in K} J_0(q) := \int_{T-\sigma}^{T}\int_\Omega q(x)\,|u(q)-z|^2\,\mathrm{d}x\,\mathrm{d}t + \beta\int_\Omega |\nabla(q-q^*)|^2\,\mathrm{d}x \tag{2.4}$$

with $u(q)$ solving the problem (2.1).

In both cases (a) and (b) above, $q^* \in H^1(\Omega)$ is an *a priori* estimate of the true parameter q (to be identified). Note that q^* need not be in the constraint set K. As will be discussed below, q^* plays the role of a selection criterion, i.e. if q is not unique, the choice of q^* influences which of the possible parameters is approximated.

Note that, in both cases, we use the unknown parameter q as a weight in the measurement norm, this requiring more accuracy in the residual $u(q) - z$ (resp. $\nabla(u(q) - z)$), where the parameter is large. This approach has given good numerical results (cf [8,9]). As one can see from the proofs in sections 5 and 6, the results about convergence rates remain unchanged if this weight q is omitted or replaced by q^2, although the regularized solutions (the minimizers of (2.3) or (2.4)) are different. The factor q^2 would be appropriate in (2.3) if the available data were the heat flux $-q\nabla u$.

In both cases, we use an H^1-seminorm for regularization. Remarks about the use of other regularization norms will be made in section 8.

Later on, we will always use $q^\dagger \in K$ to denote a solution such that $\nabla u(q^\dagger)(x,t) = \nabla z(x,t)$, $(x,t) \in Q_\sigma$ in case (a), or $u(q^\dagger)(x,t) = z(x,t)$, $(x,t) \in Q_\sigma$ in case (b). We remark that $q^\dagger$ may not exist; even if it exists it may not be unique. We will always assume, however, that such a 'true parameter' $q^\dagger$ exists, i.e. that the exact data ∇z or z are attainable. By some abuse of notation (since we do not use a norm for regularization), we call such a $q^\dagger$ which minimizes $\|\nabla(q^\dagger - q^*)\|$ among all admissible parameters a 'q^*-MNS (minimum-norm solution)' (cf [4]). Under the attainability assumption, a q^*-MNS always exists, which is a consequence of lemma 4.1 below.

In practical applications, the available observation data $z(x,t)$, $(x,t) \in Q_\sigma$ of the solution $u(x,t)$ always contain some measurement error. So the actual available data are of the form

$$z^\delta(x,t) := z(x,t) + \text{noise}.$$

We will denote by $q_\beta^\delta \in K$ a minimizer of the regularized problem (2.3) (or (2.4) respectively) with z replaced by z^δ.

While in [8] the regularized solutions were studied for fixed values of the regularization parameter $\beta > 0$, we are here interested in the convergence behaviour of the regularized solutions as $\beta, \delta \to 0$, where δ is assumed to be the noise level of the data of the form

$$\int_{T-\sigma}^{T} \|\nabla u(q^\dagger) - \nabla z^\delta\|^2 \, \mathrm{d}t \leqslant \delta^2 \tag{2.5}$$

in case (a) and

$$\int_{T-\sigma}^{T} \|u(q^\dagger) - z^\delta\|^2 \, \mathrm{d}t \leqslant \delta^2 \tag{2.6}$$

in case (b). Here and below, " $\cdot$ " stands for the L^2-norm, and the scalar product in $L^2(\Omega)$ or in $L^2(\Omega)^d$ will be denoted by $(\cdot,\cdot)$.

We remark that without information on the noise level, no convergent regularization methods can be constructed (cf [3]).

3. Preliminaries

Unless otherwise specified, we will assume throughout the paper that

$$f \in L^2(0,T; H^{-1}(\Omega)) \qquad \text{and} \qquad u_0 \in L^2(\Omega), \tag{3.1}$$

where $Q_T = \Omega \times (0,T)$, and Ω is a general Lipschitz domain in R^d. With the assumptions, we know from the standard parabolic theory that, for each $q \in K$, there exists a unique solution $u(q)$ to the variational problem (2.1) and that it has the following regularities:

$$u(q) \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)).$$

We first quote a result from [8] about the existence of solutions to the minimization problems (2.3) and (2.4).

Theorem 3.1. *There exists at least one minimizer to the optimization problem (2.3) and (2.4), respectively.*

Now we present some auxiliary results related to the solution of the parabolic system (2.1), which will play an essential role in the subsequent convergence analysis.

As opposed to the general convergence theory of Tikhonov regularization outlined in section 1, we will not need the Fréchet differentiability of the 'forward operator' $q \to u(q)$ and the Lipschitz continuity of the Fréchet derivative, which are not satisfied in the current case due to our weak-smoothness assumptions on the given data and the physical domain. Instead we need only the Gateaux directional differential $u'(q)p$. The estimate of the remainder term $u(p+q) - u(q) - u'(q)p$, which is estimated through the definition of the Fréchet derivative and its Lipschitz continuity in the general theory, will be carried out in a completely different manner here (cf sections 5 and 6).

For any $q \in K$ and $p \in H^1(\Omega)$, $u'(q)\,p \in L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$ satisfies a homogeneous initial condition and solves

$$\int_\Omega (u'(q)\,p)_t \,\varphi \, \mathrm{d}x + \int_\Omega q \,\nabla(u'(q)\,p) \cdot \nabla\varphi \, \mathrm{d}x = -\int_\Omega p \nabla u(q) \cdot \nabla\varphi \, \mathrm{d}x, \qquad \forall \varphi \in H_0^1(\Omega) \tag{3.2}$$

for a.e. $t \in (0, T)$. For the remainder term $r(q) := u(p+q) - u(q) - u'(q)p$, we have the following variational characterization.

Lemma 3.1. *For any $q \in K$ and $p \in H^1(\Omega)$ such that $q + p \in K$, the remainder $r(q) = u(q+p) - u(q) - u'(q)\, p$ belongs to $L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ and solves*

$$\int_\Omega r(q)_t\, \varphi\, \mathrm{d}x + \int_\Omega q\, \nabla r(q) \cdot \nabla\varphi\, \mathrm{d}x = \int_\Omega p\, \nabla(u(q) - u(q+p)) \cdot \nabla\varphi\, \mathrm{d}x, \qquad \forall \varphi \in H_0^1(\Omega) \tag{3.3}$$

for a.e. $t \in (0, T)$.

Proof. By (2.1), $u(q+p)$ satisfies

$$\int_\Omega u(q+p)_t\, \varphi\, \mathrm{d}x + \int_\Omega (q+p)\, \nabla u(q+p) \cdot \nabla\varphi\, \mathrm{d}x = \int_\Omega f\, \varphi\, \mathrm{d}x;$$

subtracting this from (2.1), we obtain

$$\int_\Omega (u(q+p) - u(q))_t\, \varphi\, \mathrm{d}x + \int_\Omega q\, \nabla(u(q+p) - u(q)) \cdot \nabla\varphi\, \mathrm{d}x = -\int_\Omega p\, \nabla u(q+p) \cdot \nabla\varphi\, \mathrm{d}x.$$

Now (3.3) follows by subtracting (3.2) from the above equation. □

We now introduce a linear operator $F(q) : L^2(0, T; L^2(\Omega)^d) \to L^2(\Omega)$ which will appear in our source condition: for any $q \in K$, $F(q)$ is defined by

$$F(q)\, \Phi := -\int_{T-\sigma}^{T} \nabla u(q) \cdot \Phi\, \mathrm{d}t \qquad \forall\, \Phi \in L^2(0, T; L^2(\Omega)^d) \tag{3.4}$$

where $u(q)$ is the solution of (2.1). Using the equation (3.2), we immediately see that for any $p \in H^1(\Omega)$ and any

$$\phi \in H^1(T-\sigma, T; L^2(\Omega)) \cap L^2(T-\sigma, T; H_0^1(\Omega)),$$

the following holds:

$$\begin{aligned}(F(q)\nabla\phi, p) &= -\int_{T-\sigma}^{T} \int_\Omega p\, \nabla u(q) \cdot \nabla\phi\, \mathrm{d}x\, \mathrm{d}t \\ &= \int_{T-\sigma}^{T} \int_\Omega \{(u'(q)\, p)_t\, \phi + q\, \nabla(u'(q)\, p) \cdot \nabla\phi\}\, \mathrm{d}x\, \mathrm{d}t.\end{aligned} \tag{3.5}$$

As we will explain in section 8, (3.5) is used to define a problem-adapted adjoint for our source condition.

In our analysis below, we will make use of an adjoint operator ∇^* of ∇ defined by

$$(\nabla^* w, \varphi)_{L^2(\Omega)} = (w, \nabla\varphi)_{L^2(\Omega)^d} \qquad \forall w \in L^2(\Omega)^d, \quad \varphi \in H^1(\Omega). \tag{3.6}$$

Note that we consider ∇ as a unbounded densely defined operator from $L^2(\Omega)$ into $L^2(\Omega)^d$ and thus ∇^* is also a unbounded operator from $L^2(\Omega)^d$ into $L^2(\Omega)$. As in (1.11), we will also use $\nabla^*\nabla\psi$, which would normally be defined only for $\psi \in H^2(\Omega)$ fulfilling appropriate boundary conditions; e.g., for $\Omega = (0, 1)$, the domain of $\nabla^*\nabla$ would be

$$\{\varphi \in H^2(0, 1);\, \varphi'(0) = \varphi'(1) = 0\}.$$

However, we will only need a weak form of $\nabla^*\nabla$. Our source condition assumes the existence of an H^1-function ϕ (cf sections 5 and 6) such that

$$F(q^\dagger)\nabla\phi = \nabla^*\nabla(q^\dagger - q^*) \tag{3.7}$$

holds; normally, this would require that $q^\dagger - q^* \in H^2(\Omega)$. However, our proofs need only the following weak form:

$$(F(q^\dagger)\nabla\phi, \psi) = (\nabla(q^\dagger - q^*), \nabla\psi) \qquad \forall\psi \in H^1(\Omega). \tag{3.8}$$

This condition makes sense also for $q^\dagger - q^* \in H^1(\Omega)$. Such considerations are important since we aim at conditions for convergence that need as little *a priori* smoothness as possible.

Note that the test function ψ in (3.8) can be a constant, and this implies

$$(F(q^\dagger)\nabla\phi, 1) = 0. \tag{3.9}$$

By analysis of which function ψ (3.8) is actually needed in our proofs, we will discuss a variant of (3.8) that does not imply (3.9) in section 7.

4. Stability and convergence without rates

In this section we study the stability and the convergence of the Tikhonov regularization methods (2.3) and (2.4) without rates. In sections 5 and 6, we will derive the convergence rates.

We now first show that solving the regularized problems (2.3) and (2.4), respectively, is indeed a regularization in the sense of continuous dependence of the solutions q_β^δ on the data ∇z^δ and z^δ, respectively. Since neither the regularized solutions nor the q^*-MNS are in general unique, convergence here is always to be understood in a multi-valued sense.

The following lemma from [8] replaces the weak closedness used in the general convergence theory.

Lemma 4.1. *For any sequence $\{q_n\}$ in K which converges to some $q \in K$ in $L^2(\Omega)$ as $n \to \infty$, we have (for a subsequence) $u(q_n) \rightharpoonup u(q)$ in $L^2(0, T;\ H_0^1(\Omega))$ and*

$$\lim_{n\to\infty} \int_{T-\sigma}^T \int_\Omega q_n\, |\nabla(u(q_n) - z)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_{T-\sigma}^T \int_\Omega q\, |\nabla(u(q) - z)|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Now, for the energy-norm formulation (2.3) of the regularized problem, we obtain the following.

Theorem 4.1. *For any $\beta > 0$, let $\{\nabla z_n\}$ be a sequence such that $\nabla z_n \to \nabla z^\delta$ in $L^2(T-\sigma, T; L^2(\Omega))$, and $\{q_n\}$ be minimizers of the problem (2.3) with ∇z^δ replaced by ∇z_n. Then there exists a subsequence of $\{q_n\}$ that converges in $H^1(\Omega)$, and the limit of every such convergent subsequence is a minimizer of (2.3).*

Proof. By definition of $\{q_n\}$, we have

$$\int_{T-\sigma}^T \int_\Omega q_n\, |\nabla u(q_n) - \nabla z_n|^2 \,\mathrm{d}x\,\mathrm{d}t + \beta \int_\Omega |\nabla q_n - \nabla q^*|^2 \,\mathrm{d}x$$
$$\leqslant \int_{T-\sigma}^T \int_\Omega q\, |\nabla u(q) - \nabla z_n|^2 \,\mathrm{d}x\,\mathrm{d}t + \beta \int_\Omega |\nabla q - \nabla q^*|^2 \,\mathrm{d}x, \qquad \forall q \in K,$$

from which we know that $\{"\nabla q_n"\}$ and $\{\int_{T-\sigma}^T "\nabla u(q_n)"^2 \,\mathrm{d}t\}$ are bounded. Since $q_n \in K$, the L^2-norms of q_n are also bounded, so $\{q_n\}$ is bounded in $H^1(\Omega)$. Hence there exists a subsequence (again denoted by $\{q_n\}$) such that $q_n \to \bar{q}$ in $L^2(\Omega)$ and

$$q_n \rightharpoonup \bar{q} \quad \text{in} \quad H^1(\Omega) \quad \text{and} \quad u(q_n) \rightharpoonup \bar{u} \quad \text{in} \quad L^2(T-\sigma, T; H_0^1(\Omega)).$$

Using lemma 4.1, we obtain $\bar{u} = u(\bar{q})$ and

$$\lim_{n\to\infty} \int_{T-\sigma}^{T} \int_{\Omega} q_n \, |\nabla u(q_n) - \nabla z_n|^2 \, \mathrm{d}x \, \mathrm{d}t = \int_{T-\sigma}^{T} \int_{\Omega} \bar{q} \, |\nabla u(\bar{q}) - \nabla z^{\delta}| \, \mathrm{d}x \, \mathrm{d}t. \tag{4.1}$$

Using this with $\nabla z_n \to \nabla z^{\delta}$, we derive

$$\begin{aligned} J_1(\bar{q}) &= \int_{T-\sigma}^{T} \int_{\Omega} \bar{q} \, |\nabla u(\bar{q}) - \nabla z^{\delta}|^2 \, \mathrm{d}x \, \mathrm{d}t + \beta \int_{\Omega} |\nabla \bar{q} - \nabla q^*|^2 \, \mathrm{d}x \\ &\leqslant \liminf \left\{ \int_{T-\sigma}^{T} \int_{\Omega} q_n \, |\nabla u(q_n) - \nabla z_n|^2 \, \mathrm{d}x \, \mathrm{d}t + \beta \int_{\Omega} |\nabla q_n - \nabla q^*|^2 \, \mathrm{d}x \right\} \\ &\leqslant \limsup \left\{ \int_{T-\sigma}^{T} \int_{\Omega} q_n \, |\nabla u(q_n) - \nabla z_n|^2 \, \mathrm{d}x \, \mathrm{d}t + \beta \int_{\Omega} |\nabla q_n - \nabla q^*|^2 \, \mathrm{d}x \right\} \\ &\leqslant J_1(q) \qquad \forall q \in K, \end{aligned} \tag{4.2}$$

therefore $\bar{q}$ is a minimizer of (2.3).

We now prove that q_n converges to $\bar{q}$ strongly in $H^1(\Omega)$ by contradiction. Assume this is not true; then, since $q_n \rightharpoonup \bar{q}$ in $H^1(\Omega)$ and $q_n \to \bar{q}$ in $L^2(\Omega)$, $\nabla q_n \not\to \nabla \bar{q}$. Since $\nabla q_n \rightharpoonup \nabla \bar{q}$ in $L^2(\Omega)$, by the weak lower semicontinuity of the norm, we have

$$\|\nabla \bar{q} - \nabla q^*\| \leqslant \liminf_{n\to\infty} \|\nabla q_n - \nabla q^*\|.$$

Hence,

$$\mu := \limsup_{n\to\infty} \|\nabla q_n - \nabla q^*\|^2 > \|\nabla \bar{q} - \nabla q^*\|^2,$$

which implies that there exists a subsequence $\{q_m\}$ such that

$$\lim_{m\to\infty} \|\nabla q_m - \nabla q^*\|^2 = \mu; \tag{4.3}$$

but from (4.2), we know by taking $q = \bar{q}$ that

$$\begin{aligned} \lim_{m\to\infty} &\left\{ \int_{T-\sigma}^{T} \int_{\Omega} q_m \, |\nabla u(q_m) - \nabla z_m|^2 \, \mathrm{d}x \, \mathrm{d}t + \beta \int_{\Omega} |\nabla q_m - \nabla q^*|^2 \, \mathrm{d}x \right\} \\ &= \int_{T-\sigma}^{T} \int_{\Omega} \bar{q} \, |\nabla u(\bar{q}) - \nabla z^{\delta}|^2 \, \mathrm{d}x \, \mathrm{d}t + \beta \int_{\Omega} |\nabla \bar{q} - \nabla q^*|^2 \, \mathrm{d}x. \end{aligned}$$

Combining this with (4.3), we obtain

$$\begin{aligned} \lim_{m\to\infty} &\int_{T-\sigma}^{T} \int_{\Omega} q_m \, |\nabla u(q_m) - \nabla z_m|^2 \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_{T-\sigma}^{T} \int_{\Omega} \bar{q} \, |\nabla u(\bar{q}) - \nabla z^{\delta}|^2 \, \mathrm{d}x \, \mathrm{d}t + \beta \left(\int_{\Omega} |\nabla \bar{q} - \nabla q^*|^2 \, \mathrm{d}x - \mu \right) \\ &< \int_{T-\sigma}^{T} \int_{\Omega} \bar{q} \, |\nabla u(\bar{q}) - \nabla z^{\delta}|^2 \, \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

which is a contradiction to (4.1). Thus, $q_n \to \bar{q}$ in $H^1(\Omega)$. □

For the L^2-norm formulation (2.4), we have the following stability result whose proof is analogous to that of theorem 4.1.

Theorem 4.2. *For any $\beta > 0$, let $\{z_n\}$ be a sequence in $L^2(T - \sigma, T; H_0^1(\Omega))$ such that $z_n \to z^{\delta}$ in $L^2(\Omega)$, and $\{q_n\}$ be minimizers of (2.4) with z^{δ} replaced by z_n. Then there exists a subsequence of $\{q_n\}$ that converges in $H^1(\Omega)$, and the limit of every such convergent subsequence is a minimizer of (2.4).*

After these stability results, we end the section with the following general convergence results of the Tikhonov regularization method for our problem.

Theorem 4.3. *For any positive sequence $\{\delta_k\}$, let $\beta_k := \beta(\delta_k)$ be such that $\beta_k \to 0$, $\delta_k^2/\beta_k \to 0$ as $k \to \infty$, and $z^{\delta_k} \in L^2(T-\sigma, T;\ H_0^1(\Omega))$ satisfy (2.5) ((2.6)). Then the sequence $\{q_{\beta_k}^{\delta_k}\}$ formed by solutions of (2.3) ((2.4)), with z and β replaced by z^{δ_k} and β_k, has a subsequence which converges in $H^1(\Omega)$. All limits of such convergent subsequences are minimizers of (2.3) ((2.4)) for $\beta = 0$ and they minimize "$\nabla q - \nabla q^*$" among all such minimizers, i.e. they are q^*-MNS of the problem for case (a) or (b) respectively.*

Proof. We prove this only for the L^2-norm formulation, i.e. case (b). The proof of case (a) is similar.

As $q_{\beta_k}^{\delta_k}$ is a minimizer, we have $J_0(q_{\beta_k}^{\delta_k}) \leqslant J_0(q)\ \forall q \in K$. By taking $q = q^\dagger$ and using (2.6) we obtain, with α_1, α_2 as in (2.2),

$$\alpha_1 \int_{T-\sigma}^{T} \int_\Omega |u(q_{\beta_k}^{\delta_k}) - z^{\delta_k}|^2 \,\mathrm{d}x\,\mathrm{d}t + \beta_k \|\nabla q_{\beta_k}^{\delta_k} - \nabla q^*\|^2 \leqslant \alpha_2\, \delta_k^2 + \beta_k \|\nabla q^\dagger - \nabla q^*\|^2,$$

thus

$$\int_{T-\sigma}^{T} \int_\Omega |u(q_{\beta_k}^{\delta_k}) - z^{\delta_k}|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0 \qquad \text{as} \quad k \to \infty,$$

which with (2.6) implies

$$u(q_{\beta_k}^{\delta_k}) \to z \qquad \text{in} \quad L^2(T-\sigma, T; L^2(\Omega)) \tag{4.4}$$

and

$$\limsup_{k\to\infty} \|\nabla q_{\beta_k}^{\delta_k} - \nabla q^*\| \leqslant \|\nabla q^\dagger - \nabla q^*\|. \tag{4.5}$$

Therefore $\{q_{\beta_k}^{\delta_k}\}$ is bounded in $H^1(\Omega)$ and there exists a subsequence, again denoted by $\{q_{\beta_k}^{\delta_k}\}$, such that

$$q_{\beta_k}^{\delta_k} \rightharpoonup \bar q \qquad \text{in} \quad H^1(\Omega), \qquad q_{\beta_k}^{\delta_k} \to \bar q \qquad \text{in} \quad L^2(\Omega) \qquad \text{as} \quad k \to \infty.$$

By lemma 4.1,

$$u(q_{\beta_k}^{\delta_k}) \rightharpoonup u(\bar q) \qquad \text{in} \quad L^2(0, T;\ H_0^1(\Omega)),$$

which with (4.4) shows that $u(\bar q) = z$ for a.e. $t \in [T-\sigma, T]$, so that $\bar q$ is a minimizer of (2.4) with $\beta = 0$. By the weak lower semicontinuity of the norm and the fact that $\nabla q_{\beta_k}^{\delta_k} \rightharpoonup \nabla \bar q$ in $L^2(\Omega)$,

$$\text{"}\nabla \bar q - \nabla q^*\text{"} \leqslant \liminf_{k\to\infty} \|\nabla q_{\beta_k}^{\delta_k} - \nabla q^*\|,$$

which implies with (4.5) that $\bar q$ is a q^*-MNS.

We next show that $q_{\beta_k}^{\delta_k}$ converges to $\bar q$ strongly in $H^1(\Omega)$. Similar to (4.5), by using $J_0(q_{\beta_k}^{\delta_k}) \leqslant J_0(\bar q)$ we obtain that

$$\limsup_{n\to\infty} \|\nabla q_{\beta_k}^{\delta_k} - \nabla q^*\| \leqslant \|\nabla \bar q - \nabla q^*\|.$$

This along with the identity

$$\|\nabla q_{\beta_k}^{\delta_k} - \nabla \bar q\|^2 = \|\nabla q_{\beta_k}^{\delta_k} - \nabla q^*\|^2 + \|\nabla q^* - \nabla \bar q\|^2 + 2\,(\nabla q_{\beta_k}^{\delta_k} - \nabla q^*, \nabla q^* - \nabla \bar q)$$

yields $\limsup_{k\to\infty} \|\nabla q_{\beta_k}^{\delta_k} - \nabla \bar q\|^2 \leqslant 0$. Hence, $\|\nabla q_{\beta_k}^{\delta_k} - \nabla \bar q\| \to 0$ as $k \to \infty$, which completes the proof. □

5. Convergence rates for energy-norm formulation

For the convergence rate in the energy-norm formulation (2.3), the following theorem summarizes our main result.

Theorem 5.1. *Assume that there exists a function*

$$\phi \in H_0^1(T-\sigma, T; L^2(\Omega)) \cap L^2(T-\sigma, T; H_0^1(\Omega))$$

such that the following source condition holds (in the weak sense, see (3.8)):

$$F(q^\dagger)\,\nabla\phi = \nabla^*\nabla(q^\dagger - q^*) \tag{5.1}$$

with $F(q^\dagger)$ defined by (3.4). Then, with $\beta \sim \delta$, we have

$$\|\nabla q_\beta^\delta - \nabla q^\dagger\| = \mathrm{O}(\sqrt{\delta}) \tag{5.2}$$

and

$$\int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)\|^2\,\mathrm{d}t = \mathrm{O}(\delta^2). \tag{5.3}$$

Proof. As q_β^δ is a minimizer of (2.3), we have $J_1(q_\beta^\delta) \leqslant J_1(q^\dagger)$, which with (2.5) implies (with α_1, α_2 as in (2.2))

$$\alpha_1 \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2\,\mathrm{d}t + \beta\,\|\nabla(q_\beta^\delta - q^*)\|^2 \leqslant \alpha_2\,\delta^2 + \beta\,\|\nabla(q^\dagger - q^*)\|^2,$$

from which we obtain

$$\begin{aligned}
\alpha_1 \int_{T-\sigma}^{T} &\|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2\,\mathrm{d}t + \beta\,\|\nabla(q^\dagger - q_\beta^\delta)\|^2 \\
&\leqslant \alpha_2\,\delta^2 + \beta\|\nabla(q^\dagger - q^*)\|^2 + \beta\{\|\nabla(q^\dagger - q_\beta^\delta)\|^2 - \|\nabla(q_\beta^\delta - q^*)\|^2\} \\
&= \alpha_2\,\delta^2 + 2\beta\,(\nabla(q^\dagger - q^*), \nabla(q^\dagger - q_\beta^\delta)).
\end{aligned} \tag{5.4}$$

Using (5.1) and the relation (3.5), we deduce for the last term in (5.4) that

$$\begin{aligned}
(\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta)) &= (F(q^\dagger)\,\nabla\phi,\ q^\dagger - q_\beta^\delta) \\
&= \int_{T-\sigma}^{T}\int_\Omega \{(u'(q^\dagger)(q^\dagger - q_\beta^\delta))_t\,\phi + q^\dagger\,\nabla(u'(q^\dagger)(q^\dagger - q_\beta^\delta))\cdot\nabla\phi\}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Now let

$$r_\beta^\delta := u(q_\beta^\delta) - u(q^\dagger) - u'(q^\dagger)(q_\beta^\delta - q^\dagger); \tag{5.5}$$

by lemma 3.1, $r_\beta^\delta \in H_0^1(\Omega)$. Using this notation, we obtain

$$\begin{aligned}
\beta\,(\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta)) &= -\beta\int_{T-\sigma}^{T}\int_\Omega (u(q_\beta^\delta) - u(q^\dagger))_t\,\phi\,\mathrm{d}x\,\mathrm{d}t \\
&\quad +\beta\int_{T-\sigma}^{T}\int_\Omega \{(r_\beta^\delta)_t\,\phi + q^\dagger\,\nabla r_\beta^\delta\cdot\nabla\phi\}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\beta\int_{T-\sigma}^{T}\int_\Omega q^\dagger\,\nabla(u(q_\beta^\delta) - u(q^\dagger))\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t =: (\mathrm{I})_1 + (\mathrm{I})_2 + (\mathrm{I})_3.
\end{aligned} \tag{5.6}$$

We next estimate $(\mathrm{I})_1$, $(\mathrm{I})_2$ and $(\mathrm{I})_3$. First for $(\mathrm{I})_1$, using integration by parts with respect to t and the boundary condition, we derive

$$(\mathrm{I})_1 = \beta\int_{T-\sigma}^{T}\int_\Omega (u(q_\beta^\delta) - u(q^\dagger))\phi_t\,\mathrm{d}x\,\mathrm{d}t;$$

applying the Cauchy–Schwarz, the Poincaré and triangle inequalities, we obtain that

$$
\begin{aligned}
|(\mathrm{I})_1| &\leqslant C\,\beta \int_{T-\sigma}^{T} \|\nabla(u(q_\beta^\delta) - u(q^\dagger))\|\,\|\phi_t\|\,\mathrm{d}t \\
&\leqslant C\,\beta \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|\,\|\phi_t\|\,\mathrm{d}t + C\beta \int_{T-\sigma}^{T} \|\nabla z^\delta - \nabla u(q^\dagger)\|\,\|\phi_t\|\,\mathrm{d}t.
\end{aligned}
$$

Using (2.5) and Young's inequality in the form

$$
a \cdot b \leqslant \varepsilon\, a^2 + \frac{b^2}{4\,\varepsilon} \tag{5.7}
$$

for any $\varepsilon > 0$ yields

$$
|(\mathrm{I})_1| \leqslant C\varepsilon \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2\,\mathrm{d}t + \frac{C\beta^2}{2\varepsilon} \int_{T-\sigma}^{T} \|\phi_t\|^2\,\mathrm{d}t + C\,\varepsilon\,\delta^2.
$$

To estimate $(\mathrm{I})_2$, we use (3.3) to obtain

$$
(\mathrm{I})_2 = \beta \int_{T-\sigma}^{T} \int_\Omega (q_\beta^\delta - q^\dagger)(\nabla u(q^\dagger) - \nabla u(q_\beta^\delta)) \cdot \nabla\phi\,\mathrm{d}x\,\mathrm{d}t.
$$

Then using the Cauchy–Schwarz and Young's inequalities, we obtain

$$
\begin{aligned}
|(\mathrm{I})_2| &\leqslant \beta \int_{T-\sigma}^{T} \|\nabla u(q^\dagger) - \nabla z^\delta\|\,\|(q_\beta^\delta - q^\dagger)\nabla\phi\|\,\mathrm{d}t \\
&\quad + \beta \int_{T-\sigma}^{T} \|\nabla z^\delta - \nabla u(q_\beta^\delta)\|\,\|(q_\beta^\delta - q^\dagger)\nabla\phi\|\,\mathrm{d}t \\
&\leqslant \varepsilon\,\delta^2 + \frac{2\beta^2\alpha_2^2}{\varepsilon} \int_{T-\sigma}^{T} \|\nabla\phi\|^2\,\mathrm{d}t + \varepsilon \int_{T-\sigma}^{T} \|\nabla z^\delta - \nabla u(q_\beta^\delta)\|^2\,\mathrm{d}t,
\end{aligned}
$$

where we have used the boundedness of q_β^δ and $q^\dagger$ in K.

Finally we estimate $(\mathrm{I})_3$: we have

$$
\begin{aligned}
|(\mathrm{I})_3| &\leqslant \alpha_2\beta \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|\,\|\nabla\phi\|\,\mathrm{d}t + \alpha_2\beta \int_{T-\sigma}^{T} \|\nabla z^\delta - \nabla u(q^\dagger)\|\,\|\nabla\phi\|\,\mathrm{d}t \\
&\leqslant \varepsilon\alpha_2^2\delta^2 + \frac{\beta^2}{2\varepsilon} \int_{T-\sigma}^{T} \|\nabla\phi\|^2\,\mathrm{d}t + \varepsilon\alpha_2^2 \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2\,\mathrm{d}t.
\end{aligned}
$$

Substituting the above estimates for $(\mathrm{I})_1$, $(\mathrm{I})_2$ and $(\mathrm{I})_3$ into (5.4) and (5.6) we obtain

$$
\begin{aligned}
&\alpha_1 \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2\,\mathrm{d}t + \beta\|\nabla(q^\dagger - q_\beta^\delta)\|^2 \\
&\qquad \leqslant 2\varepsilon(C + 1 + \alpha_2^2) \int_{T-\sigma}^{T} \|\nabla u(q_\beta^\delta) - \nabla z^\delta\|^2\,\mathrm{d}t \\
&\qquad\quad + \frac{C\beta^2}{\varepsilon} \int_{T-\sigma}^{T} \|\phi_t\|^2\,\mathrm{d}t + 2\varepsilon(C + 1 + \alpha_2^2)\delta^2 + \beta^2 \int_{T-\sigma}^{T} \|\nabla\phi\|^2\,\mathrm{d}t \cdot \left[\frac{4\alpha_2^2}{\varepsilon} + \frac{1}{\varepsilon}\right].
\end{aligned}
$$

Now (5.2), (5.3) follows immediately by taking ε such that $4\varepsilon(C+1+\alpha_2^2) = \alpha_1$ and $\beta \sim \delta$. $\square$

Remark 5.1. The source condition (5.1) will be discussed in section 7. Here we only remark that no smallness condition on the source function ϕ is needed to ensure the same convergence rate as in the existing general theory. This is one of the big advantages of the new theory over the existing one. This advantage also carries over to the L^2-norm formulation in section 6.

6. Convergence rates for L^2-norm formulation

We now carry over the convergence results of section 5 to the L^2-norm formulation (2.4). The next theorem summarizes our main results.

Theorem 6.1. *Assume that there exists a function*

$$\phi \in H_0^1(T-\sigma, T; L^2(\Omega)) \cap L^2(T-\sigma, T; H_0^1(\Omega)) \tag{6.1}$$

such that (5.1) holds. Furthermore, assume that

$$\phi \in L^2(T-\sigma, T; W^{1,\infty}(\Omega)) \qquad \text{and} \qquad \Delta\phi \in L^2(T-\sigma, T;\ L^2(\Omega)). \tag{6.2}$$

Then, with $\beta \sim \delta$, we have the estimates (5.2) and

$$\int_{T-\sigma}^{T} \|u(q_\beta^\delta) - u(q^\dagger)\|^2 \,\mathrm{d}t = \mathrm{O}(\delta^2). \tag{6.3}$$

Proof. Using the same techniques as used in deriving (5.4)–(5.6), we obtain

$$\alpha_1 \int_{T-\sigma}^{T} \|u(q_\beta^\delta) - z^\delta\|^2 \,\mathrm{d}t + \beta\, \|\nabla(q^\dagger - q_\beta^\delta)\|^2 \leqslant \alpha_2\, \delta^2 + 2\beta\, (\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta)) \tag{6.4}$$

and

$$\begin{aligned}
\beta\, (\nabla(q^\dagger - q^*),\ \nabla(q^\dagger - q_\beta^\delta)) &= -\beta \int_{T-\sigma}^{T} \int_\Omega (u(q_\beta^\delta) - u(q^\dagger))_t\, \phi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \beta \int_{T-\sigma}^{T} \int_\Omega \{(r_\beta^\delta)_t\, \phi + q^\dagger\, \nabla r_\beta^\delta \cdot \nabla\phi\,\} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \beta \int_{T-\sigma}^{T} \int_\Omega q^\dagger\, \nabla(u(q_\beta^\delta) - u(q^\dagger)) \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t =: (\mathrm{II})_1 + (\mathrm{II})_2 + (\mathrm{II})_3.
\end{aligned} \tag{6.5}$$

Note that (6.5) is still the same as (5.6), but we now have to estimate the terms differently, since we have to aim at $\int_{T-\sigma}^{T} \|u(q_\beta^\delta) - z^\delta\|^2 \,\mathrm{d}t$ instead of $\int_{T-\sigma}^{T} \|\nabla(u(q_\beta^\delta) - z^\delta)\|^2 \,\mathrm{d}t$.

First the term $(\mathrm{II})_1$ can be bounded readily by means of integration by parts with respect to t and Young's inequality (5.7):

$$|(\mathrm{II})_1| \leqslant \varepsilon \int_{T-\sigma}^{T} \|u(q_\beta^\delta) - z^\delta\|^2 \,\mathrm{d}t + \frac{\beta^2}{2\varepsilon} \int_{T-\sigma}^{T} \|\phi_t\|^2 \,\mathrm{d}t + \varepsilon\delta^2,$$

while $(\mathrm{II})_3$ can be estimated similarly using integration by parts with respect to x and the Cauchy–Schwarz and Young inequalities:

$$\begin{aligned}
|(\mathrm{II})_3| &= \beta \left| \int_{T-\sigma}^{T} \int_\Omega (u(q_\beta^\delta) - u(q^\dagger))\, \nabla \cdot (q^\dagger \nabla\phi) \,\mathrm{d}x\,\mathrm{d}t \right| \\
&\leqslant \varepsilon \int_{T-\sigma}^{T} \|u(q_\beta^\delta) - z^\delta\|^2 \,\mathrm{d}t + \varepsilon\, \delta^2 + \frac{\beta^2}{2\varepsilon} \int_{T-\sigma}^{T} \|\nabla \cdot (q^\dagger \nabla\phi)\|^2 \,\mathrm{d}t.
\end{aligned}$$

What remains is to bound $(\mathrm{II})_2$. As for the treatment of $(\mathrm{I})_2$ in the proof of theorem 5.1 we obtain with integration by parts with respect to x that

$$\begin{aligned}
(\mathrm{II})_2 &= -\beta \int_{T-\sigma}^{T} \int_\Omega (q_\beta^\delta - q^\dagger)(\nabla u(q_\beta^\delta) - \nabla u(q^\dagger)) \cdot \nabla\phi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\beta \int_{T-\sigma}^{T} \int_\Omega (u(q_\beta^\delta) - u(q^\dagger)) \nabla \cdot ((q_\beta^\delta - q^\dagger)\nabla\phi) \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Using Young's inequality again, we obtain

$$|(\mathrm{II})_2| \leqslant \varepsilon\, \delta^2 + \frac{\beta^2}{2\varepsilon}\int_{T-\sigma}^{T} \|\nabla\cdot((q_\beta^\delta - q^\dagger)\nabla\phi)\|^2\,\mathrm{d}t + \varepsilon\int_{T-\sigma}^{T} \|u(q_\beta^\delta) - z^\delta\|^2\,\mathrm{d}t.$$

The second term in this bound can be further estimated:

$$\begin{aligned}
\frac{\beta^2}{2\varepsilon}&\int_{T-\sigma}^{T}\int_\Omega |\nabla\cdot((q_\beta^\delta - q^\dagger)\nabla\phi)|^2\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{\beta^2}{2\varepsilon}\int_{T-\sigma}^{T}\int_\Omega |\nabla(q_\beta^\delta - q^\dagger)\cdot\nabla\phi + (q_\beta^\delta - q^\dagger)\Delta\phi|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant \beta\,\|\nabla(q_\beta^\delta - q^\dagger)\|^2\left\{\frac{\beta}{\varepsilon}\int_{T-\sigma}^{T}\|\nabla\phi\|^2_{L^\infty(\Omega)}\,\mathrm{d}t\right\} + \frac{\beta^2}{\varepsilon}\int_{T-\sigma}^{T}\|\Delta\phi\|^2_{L^2(\Omega)}\,\mathrm{d}t.
\end{aligned}$$

Now the desired results of theorem 6.1 follow immediately by substituting $(\mathrm{II})_1$, $(\mathrm{II})_2$ and $(\mathrm{II})_3$ into (6.5), taking ε such that $6\varepsilon = \alpha_1$, and then taking β small enough so that $\frac{\beta}{\varepsilon}\int_{T-\sigma}^{T}\|\nabla\phi\|^2_{L^\infty(\Omega)}\,\mathrm{d}t < 1$. □

7. Discussion of the source condition

First, we note that due to theorem 4.3, the parameter $q^\dagger$, to which the regularized solutions in theorems 5.1 and 6.1 converge, must be a q^*-minimum norm solution $q^\dagger$. There might be more than one q^*-MNS, but only one (up to a constant, cf (5.2)) can fulfil the source conditions of theorems 5.1 and 6.1, respectively.

The crucial assumption in our results on the convergence rates (theorems 5.1 and 6.1) is the existence of a source function

$$\phi \in H_0^1(T-\sigma, T; L^2(\Omega)) \cap L^2(T-\sigma, T; H_0^1(\Omega)) \tag{7.1}$$

(with the additional assumption (6.2) for theorem 6.1) such that

$$\int_{T-\sigma}^{T} \nabla u(q^\dagger)\cdot\nabla\phi\,\mathrm{d}t = \nabla^*\nabla(q^* - q^\dagger) \tag{7.2}$$

holds in the weak form (3.8), i.e.

$$\int_{T-\sigma}^{T} (p\,\nabla\phi,\ \nabla u(q^\dagger))\,\mathrm{d}t = (\nabla(q^* - q^\dagger),\ \nabla p), \qquad \forall\, p \in H^1(\Omega). \tag{7.3}$$

This requires only the regularity that q^* and $q^\dagger$ are both in $H^1(\Omega)$, so the source condition does not impose or require any more regularity on q^* and $q^\dagger$. This is quite remarkable.

As mentioned in section 3, (7.3) (or (3.8)) has an extra implication for constant functions (see (3.9)). The following alternative is possible. As the proofs of theorems 5.1 and 6.1 show, (7.3) is actually needed only for $\psi = q^\dagger - q_\beta^\delta$. Hence, if we assume we know $q^\dagger$ e.g. on an open subset Γ of $\partial\Omega$, then we can incorporate this knowledge into the admissible set K (see (2.2)), which would imply that $q^\dagger - q_\beta^\delta = 0$ on Γ. Therefore, (7.3) would have to be required only for p in the space $\{p \in H^1(\Omega);\ p = 0 \text{ on } \Gamma\}$. If we do this, the Poincaré inequality then implies with theorems 5.1 and 6.1 the full H^1-norm error estimate

$$\|q_\beta^\delta - q^\dagger\|_{H^1(\Omega)} = \mathrm{O}(\sqrt{\delta}).$$

Comparing with the usual source condition (1.11), another advantage of the new source condition (7.2) lies in its simple and clear interpretation. The derivative $u'(q^\dagger)$ and its adjoint $u'(q^\dagger)^*$ used in (1.11) are usually very difficult to comprehend. Instead, the new condition (7.2) uses the parameter-to-solution $u(q^\dagger)$ itself, which has a direct physical meaning. The condition (7.2) indicates that one has to know the true parameter $q^\dagger$ in a region where $u(q^\dagger)$ is constant ($\nabla u = 0$) near the terminal status, which is compatible with the fact that in such regions the true $q^\dagger$ is not identifiable. Except in the region where ∇u vanishes, one does not necessarily require very accurate information on the true parameter, including locations near isolated critical points of the solution $u(q^\dagger)$.

So far, source conditions could at best be interpreted in the sense that they gave some information on smoothness and boundary behaviour of the exact solutions needed for convergence rates (cf [3]). Therefore, we consider it also remarkable that we can now even construct functions ϕ such that our new source condition is fulfilled at least for the one-dimensional case under a quite natural condition to be explained in remark 7.1.

Lemma 7.1. *Let $\Omega = (0,1)$ and u be a given function defined on $\Omega \times (T-\sigma, T)$. Assume that $\psi_1, \psi_2 \in H^1(T-\sigma, T)$ and $\phi_1 \in H^1(\Omega)$ are three arbitrary functions. Define*

$$w(x) := u_1(x)\{(1-2x)\phi_1(x) + (x-x^2)\,\phi_1'(x)\},$$

$$u_i(x) := \int_{T-\sigma}^{T} (T-t)(T-\sigma-t)\,u_x(x,t)\,\psi_i(t)\,\mathrm{d}t, \qquad i = 1, 2.$$

Moreover, we assume that

$$\frac{q-w}{u_2} \in L^2(0,1) \qquad \text{and} \qquad \int_0^1 \frac{q(x)-w(x)}{u_2(x)}\,\mathrm{d}x = 0 \tag{7.4}$$

for some given function $q(x)$. Then the equation

$$\int_{T-\sigma}^{T} u_x(x,t)\,\phi_x(x,t)\,\mathrm{d}t = q(x) \qquad \forall x \in (0,1) \tag{7.5}$$

has the following solutions:

$$\phi(x,t) := (T-t)(T-\sigma-t)\left\{(x-x^2)\,\phi_1(x)\psi_1(t) + \psi_2(t)\int_0^x \frac{q(\xi)-w(\xi)}{u_2(\xi)}\,\mathrm{d}\xi\right\}, \tag{7.6}$$

and (7.1) holds.

Proof. By definition,

$$\phi(x,T) = \phi(x,T-\sigma) = 0 \qquad \forall x \in \Omega$$

and

$$\phi(0,t) = \phi(1,t) = 0 \qquad \forall t \in (T-\sigma,\ T),$$

so ϕ fulfils the boundary conditions required by (7.1) for both t and x. To show that ϕ satisfies (7.5), we make the simple calculation

$$\phi_x(x,t) = (T-t)(T-\sigma-t)\Big\{(1-2x)\phi_1(x)\,\psi_1(t) + (x-x^2)\phi_1'(x)\psi_1(t)$$
$$+\psi_2(t)\frac{q(x)-w(x)}{u_2(x)}\Big\},$$

from which and the definitions of w and u_2 we derive

$$\int_{T-\sigma}^{T} u_x(x,t)\,\phi_x(x,t)\,\mathrm{d}t = \{(1-2x)\phi_1(x) + (x-x^2)\phi_1'(x)\}u_1(x) + \{q(x)-w(x)\} = q(x).$$

Finally, we can easily verify that

$$\phi_x, \phi_t \in L^2(T-\sigma, T;\ L^2(\Omega)),$$

which completes the proof of lemma 7.1. □

Remark 7.1. Lemma 7.1 provides a general way of constructing solutions ϕ for the equation (7.2) in the case $d = 1$. For any $\psi_1, \psi_2 \in H^1(T-\sigma, T)$ with $\psi_1 \neq \psi_2$ satisfying (with u_i defined by lemma 7.1)

$$\int_0^1 \frac{(1-2x)u_1(x)}{u_2(x)} \neq 0, \tag{7.7}$$

we can find a constant γ_0 such that

$$\gamma_0 \int_0^1 \frac{(1-2x)\,u_1(x)}{u_2(x)}\,\mathrm{d}x = \int_0^1 \frac{q(x)}{u_2(x)}\,\mathrm{d}x;$$

then choose $\phi_1(x) \equiv \gamma_0$. This results in

$$\int_0^1 \frac{q(x) - w(x)}{u_2(x)}\,\mathrm{d}x = 0,$$

and ϕ as defined in (7.6) is the required solution of (7.2) provided that $\frac{q-w}{u_2} \in L^2(0,1)$, which is the only remaining condition. This condition seems natural: it contains an expression involving u_x in the denominator; if u_x vanishes in a subregion of Ω near the final time $t = T$, it would be impossible to determine q there.

8. Variants and extensions

The results achieved in this paper are still in the infinite-dimensional setting, unrealistically assuming that we have distributed measurements of u or ∇u. Of course, in reality only discrete measurements are available, which requires us to extend the theory by adding discretization in solving the forward problem and in approximating the unknown parameter. For fixed $\beta > 0$, this has been considered in [8]; for $\beta \to 0$ we could proceed by combining our arguments with those in [12, 13].

Both (2.3) and (2.4) involve gradients in the regularization term. Instead, we could also consider L^2-regularization, i.e.

$$\min_{q\in K} J(q) := \int_{T-\sigma}^{T} \int_\Omega q(x)|(\nabla)(u(q) - z^\delta)|^2\,\mathrm{d}x\,\mathrm{d}t + \beta \int_\Omega |q - q^*|^2\,\mathrm{d}x. \tag{8.1}$$

Results corresponding to theorems 5.1 and 6.1 can be proven in a completely analogous way, where $\nabla^*\nabla(q^\dagger - q^*)$ in (5.1) is replaced by $(q^\dagger - q^*)$, and "$\nabla q_\beta^\delta - \nabla q^\dagger$" in (5.2) is replaced by "$q_\beta^\delta - q^\dagger$". For the analogue of theorem 6.1, we would need to incorporate an *a priori* bound on $|\nabla q|$ to take care of the term β"$\nabla(q_\beta^\delta - q^\dagger)$"^{2} on the right-hand side of the estimate for $(\mathrm{II})_2$ in the proof.

The existence of minimizers of (8.1) cannot be guaranteed by theorem 3.1. This difficulty could either be resolved by using some weak-closedness argument or circumvented by incorporating a tolerance η into the minimization, i.e. replacing minimizers of (8.1) by elements $q_{\beta,\eta}^\delta \in K$ such that

$$J(q_{\beta,\eta}^\delta) \leqslant \inf_{q\in K} J(q) + \eta;$$

as long as $\eta = \mathrm{O}(\delta^2)$, all proofs carry over. This can of course also be done for (2.3) and (2.4).

The main purpose of this paper was, instead of applying the general theory of Tikhonov regularization to our problem, to develop a problem-adapted new approach which allows us to prove results about convergence rates under much weaker conditions. Although we developed this theory for a specific problem, there is a general structure behind our new approach which we will outline now; we proceed only in a formal way without precise assumptions and omit, for simplicity of notation, the integration over $[T-\sigma, T]$ in our observations (cf e.g. (2.3)–(2.6)).

Our direct (forward) problem corresponding to (2.1) has the abstract form

$$\langle u_t, \psi\rangle + B(N(q,u),\psi) = \langle f,\psi\rangle \qquad \forall\psi\in H, \tag{8.2}$$

where H is a Hilbert space with inner product $\langle , \rangle$, $B(\cdot,\cdot)$ is a bilinear form on H and N is possibly nonlinear.

The linearization $u'(q)$ of u with respect to q has the weak form

$$\langle (u'(q)p)_t, \psi\rangle + B(N_u(q,u(q))u'(q)p,\psi) = -B(pN_q(q,u(q)),\psi) \tag{8.3}$$

as can be seen by implicit differentiation of (8.2). The classical source condition corresponding to (1.7) would be to require the existence of a ϕ with

$$q^\dagger - q^* = u'(q^\dagger)^*\phi \tag{8.4}$$

or in weak form

$$\langle q^\dagger - q^*, \psi\rangle = \langle \phi, u'(q^\dagger)\psi\rangle \tag{8.5}$$

for all ψ. This is used in a crucial step of the convergence-rates proof (cf [4] or [3, theorem 10.4]) as follows:

$$\langle q^\dagger - q^*, q^\dagger - q^\delta_\beta\rangle = \langle u'(q^\dagger)^*\phi, q^\dagger - q^\delta_\beta\rangle = \langle \phi, u'(q^\dagger)(q^\dagger - q^\delta_\beta)\rangle.$$

In the analogous step in the proofs of theorems 5.1 and 6.1, we used (5.1) instead, which corresponds to

$$\langle q^\dagger - q^*, \psi\rangle = -B(\psi N_q(q^\dagger, u(q^\dagger)),\phi) = \langle (u'(q^\dagger)\psi)_t, \phi\rangle + B(N_u(q^\dagger, u(q^\dagger))u'(q^\dagger)\psi,\phi)$$

with $\psi = q^\dagger - q^\delta_\beta$, where the first equality is the abstract version of the source condition requiring the existence of a ϕ such that

$$\langle q^\dagger - q^*, \psi\rangle = -B(\psi N_q(q^\dagger, u(q^\dagger)),\phi) \tag{8.6}$$

holds for all ψ, or in the case of seminorm-regularization (1.10),

$$\langle D(q^\dagger - q^*), D\psi\rangle = -B(\psi N_q(q^\dagger, u(q^\dagger)),\phi). \tag{8.7}$$

The coupling of the source condition with the weak form (8.3) of the linearized equation is crucial. For our concrete problem, where the bilinear form B is

$$B(v,\psi) := (v, \nabla\psi) \tag{8.8}$$

and the nonlinear operator N is

$$N(q,u) := q\nabla u, \tag{8.9}$$

this can be seen in (3.5): the last expression there, considered as a linear operator acting on p, can be viewed as an adjoint of $u'(q)$, not with respect to the standard inner product but with respect to the bilinear form (3.2).

To proceed with the convergence-rate proof on this abstract level would now require specific assumptions about the interplay of the bilinear form B with the nonlinearity N and the operator D. For the case of B as in (8.8) and $N(q,u) := q(u)\nabla u$, i.e. the identification of conductivity parameters q depending nonlinearly on u, see [14]. Also the theory can be naturally adapted to the elliptic problems and the case with measurements on a subdomain of Ω.

We close by conjecturing that such a problem-adapted adjoint might also be of use for constructing efficient iterative regularization methods for certain nonlinear ill posed problems, since adjoints appear in virtually all such methods [7].

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