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A mortar element method for elliptic problems with discontinuous coefficients

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This paper proposes a mortar finite element method for solving the two-dimensional second-order elliptic problem with jumps in coefficients across the interface between two subregions. Non-matching finite element grids are allowed on the interface, so independent triangulations can be used in different subregions. Explicitly realizable mortar conditions are introduced to couple the individual discretizations. The same optimal L^2-norm and energy-norm error estimates as for regular problems are achieved when the interface is of arbitrary shape but smooth, though the regularity of the true solution is low in the whole physical domain.

Keywords: Mortar element method; jumps in coefficients; mortar condition.

1. Introduction

This paper is concerned with a mortar finite element method for solving the following two-dimensional elliptic interface problem:

$$-\nabla \cdot (\beta \nabla u) = f \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

$$[u] = 0, \quad [\beta \partial_{\mathbf{n}} u] = g \quad \text{on} \quad \Gamma, \tag{1.3}$$

where Ω is a convex polygon in R^2. We assume that the coefficient function $\beta(x)$ is discontinuous across an arbitrary but C^2-smooth interface $\Gamma \subset \Omega$. Here Γ is the boundary of an open domain $\Omega_1 \subset\subset \Omega$. Let $\Omega_2 = \Omega \setminus \bar{\Omega}_1$ (see Fig. 1). Equations (1.3) are called the jump conditions on the interface Γ, with $[v]$ meaning the jump of a function v across Γ, with $\mathbf{n}$ the unit outward normal to the boundary $\partial\Omega_1$. For definiteness, we let $[v](x) = v_1(x) - v_2(x)$ for $x \in \Gamma$, with v_1 and v_2 being the restrictions of v on Ω_1 and Ω_2 respectively. Moreover, we assume that the coefficient function $\beta(x)$ is positive and

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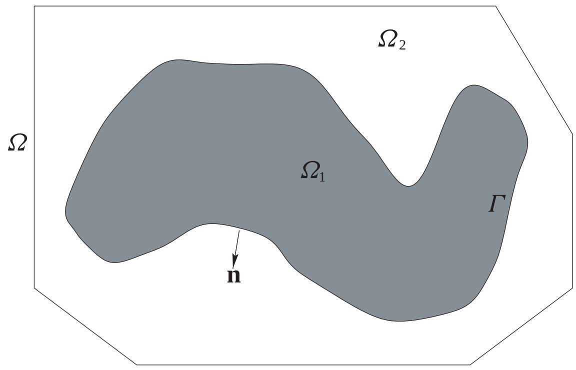

FIG. 1. Domain Ω, its subregions Ω_1, Ω_2 and interface Γ.

piecewise smooth, i.e.

$$\beta(x) = \begin{cases} \beta_1(x) & \text{for } x \in \Omega_1, \\ \beta_2(x) & \text{for } x \in \Omega_2, \end{cases}$$

where $\beta_1(x) \in C^2(\bar{\Omega}_1)$ and $\beta_2(x) \in C^2(\bar{\Omega}_2)$, and there exist two positive constants β_1 and β_2 such that

$$C_0\beta_1 \leqslant \beta_1(x) \leqslant C_1\beta_1, \quad \forall x \in \Omega_1; \quad C_0\beta_2 \leqslant \beta_2(x) \leqslant C_1\beta_2, \quad \forall x \in \Omega_2.$$

Here C_0 and C_1 are two positive constants independent of β_1 and β_2. This means that $\beta(x)$ is of size β_1 in domain Ω_1 and of size β_2 in domain Ω_2, and that β_1 and β_2 may differ greatly in magnitude.

Such interface problems are often encountered in material sciences and fluid dynamics. It is the case when two distinct materials or fluids with different conductivities or densities or diffusions are involved. Much attention has been paid to numerical solutions of interface problems in recent years. The conforming finite element methods (Bramble & King, 1996; Chen & Zou, 1998; Xu, 1982) were used for such problems when the interfaces are of arbitrary shape but smooth, while the finite element/finite difference methods with uniform grids were also widely applied for solving such interface problems: see, for example, LeVeque & Li (1994); Li (1998). We refer to Chen & Zou (1998) and the references therein for more detailed elaborations on many existing finite element methods for the elliptic and parabolic interface problems, and to Xu & Zou (1998) for a survey on non-overlapping domain decomposition methods for elliptic interface problems.

Most existing methods are basically conforming finite element methods and require the triangulations in different subregions to be matching on the interface. This may pose serious restrictions when the physical solutions of the interface problems are of different scales in different subregions. Mortar element methods seem to be a good alternative to relax such restrictions. To our knowledge, there has been no study concerned with the mortar element method for solving interface problems with interfaces of arbitrary shape. The purpose of this paper is to propose a mortar finite element method for solving the elliptic interface problem (1.1)–(1.3). This method allows non-matching finite element grids on the interface Γ, so independent triangulations can be used in the subregions Ω_1 and Ω_2. Explicitly realizable mortar conditions are introduced to couple the individual discretizations in two subregions. It seems to be the first time that the same optimal L^2-norm and energy-norm error estimates as for regular problems are achieved with mortar finite element methods for the interface problems with interfaces of arbitrary shape, though the regularity of the true solution for this case is low in the whole physical domain. The derivation of such optimal error estimates is very tricky and technical and we need many new technical tools to manage them (see Section 3). For related work, (see Bernardi *et al.*, 1990a,b) for the basic ideas of the mortar element methods, (Achdou, 1995; Belgacem, 1999; Belgacem & Maday, 1997; Du & Gunzburger, 2000; Marcinkowski, 1996) for the recent advance on the mortar element methods for PDEs with smooth coefficients, (Cao & Cunzburger, 1998) for the use of a least-squares finite element method for solving the elliptic interface problems and (Chen *et al.*, 2000) for solving the Maxwell equations with jumps in coefficients across some polyhedral interface.

An efficient numerical method for the interface problem should make full use of the basic feature of the problem: even though the interface is sufficiently smooth, the solution of the interface problem is only smooth in the individual subregions occupied by different materials or fluids, but has much lower regularities in the whole domain. For example, if $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, then the solution u of the problem (1.1)–(1.3) is H^2-regular locally but only H^1-regular globally, namely

$$u \in H_0^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \equiv X. \tag{1.4}$$

Here and in what follows, for each integer $m \geqslant 0$ and real p with $1 \leqslant p \leqslant \infty$, we use $W^{m,p}(\Omega)$ to denote the standard Sobolev space of real functions with their weak derivatives of order up to m in the Lebesgue space $L^p(\Omega)$, $\|\cdot\|_{m,p,\Omega}$ and $|\cdot|_{m,p,\Omega}$ to denote its norm and semi-norm (Grisvard, 1985). When $p = 2$, we write $W^{m,2}(\Omega) = H^m(\Omega)$, and denote its norm and semi-norm by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$. For a fractional number s, the Sobolev space $H^s(\Omega)$ is defined by the standard interpolation theory (Bergh & Löfstrom, 1976). For the space X defined in (1.4), we use its norm of the form

$$\|v\|_X = \|v\|_{1,\Omega} + \|v\|_{2,\Omega_1} + \|v\|_{2,\Omega_2} \quad \forall\, v \in X.$$

The following *a priori* estimate for the solution of (1.1)–(1.3) will be frequently used later in our analysis (Chen & Zou, 1998):

$$\|u\|_X \lesssim \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\Gamma)}. \tag{1.5}$$

Here and in what follows, for any two non-negative numbers x and y, $x \lesssim y$ means that $x \leqslant Cy$ for some constant C independent of the mesh size h and the related parameters (e.g. the constant in (1.5) is independent of f, g and u), and $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

2. A mortar finite element method

By integration by parts, we can easily derive the weak formulation of the interface problem (1.1)–(1.3): Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) + \langle g, v \rangle, \quad \forall v \in H_0^1(\Omega). \tag{2.1}$$

Here $(\cdot, \cdot)$ denotes the inner product in $L^2(\Omega)$ and $\langle \cdot, \cdot \rangle$ the dual form between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ (or the inner product in $L^2(\Gamma)$ if functions are smooth enough). $a(\cdot, \cdot)$ is the bilinear form with $a(u, v) = a_1(u, v) + a_2(u, v)$ and

$$a_i(u, v) = \int_{\Omega_i} \beta_i(x) \nabla u \cdot \nabla v \, dx, \quad i = 1, 2.$$

We now derive a mortar finite element method for solving (1.1)–(1.3) or (2.1). We first introduce two triangulations: $\mathcal{T}_{h_1}$ for the domain Ω_1 and $\mathcal{T}_{h_2}$ for the domain Ω_2. To do so, we choose m_1 points on the interface Γ: $P_1^1, P_2^1, \ldots, P_{m_1}^1$; then connect all neighbouring pairs $\{P_i^1, P_{i+1}^1\}$ to obtain a closed polygonal curve approximating Γ and a polygonal domain Ω_{h_1} approximating Ω_1. We assume the line segments $\{\tilde{e}_i^1\}_{i=1}^{m_1}$ with $\tilde{e}_i^1 = P_i^1 P_{i+1}^1$ ($P_{m_1+1}^1 = P_1^1$) are of size h_1, that means, $|\tilde{e}_i^1| \eqsim h_1$, $i = 1, \ldots, m_1$. We further triangulate Ω_{h_1} by a finite set of open triangles $\tilde{\mathcal{T}}_{h_1} = \{K\}$, which is assumed to be quasi-uniform with mesh size h_1. The triangulation $\mathcal{T}_{h_1}$ is then only the slight modification of $\tilde{\mathcal{T}}_{h_1}$ by changing those triangles with one of their edges being $\tilde{e}_j^1$ (for some $1 \leqslant j \leqslant m_1$) into the curved triangles with two original edges unchanged but the third edge $\tilde{e}_j^1$ replaced by the curved segment $e_j^1 = \widehat{P_j^1 P_{j+1}^1}$, where $\widehat{P_j^1 P_{j+1}^1}$ denotes the curved segment on the interface Γ with two endpoints P_j^1 and P_{j+1}^1. This generates a triangulation $\mathcal{T}_{h_1}$ of Ω_1 satisfying $\bar{\Omega}_1 = \bigcup_{K \in \mathcal{T}_{h_1}} \bar{K}$.

Furthermore, we choose another set of points on Γ: $P_1^2, P_2^2, \ldots, P_{m_2}^2$ such that the line segments $\{\tilde{e}_i^2\}_{i=1}^{m_2}$ with $\tilde{e}_i^2 = P_i^2 P_{i+1}^2$ ($P_{m_2+1}^2 = P_1^2$) are of size h_2. We then repeat the same process for constructing $\mathcal{T}_{h_1}$ to generate a triangulation $\mathcal{T}_{h_2}$ of the domain Ω_2 satisfying $\bar{\Omega}_2 = \bigcup_{K \in \mathcal{T}_{h_2}} \bar{K}$. As before, $e_i^2 = \widehat{P_i^2 P_{i+1}^2}$ denotes the curved segment on the interface Γ with two endpoints P_i^2 and P_{i+1}^2. We also define $\Gamma_{h_1} = \{e_i^1\}_{i=1}^{m_1}$ and $\Gamma_{h_2} = \{e_i^2\}_{i=1}^{m_2}$, which are two independent triangulations of the interface Γ. Since Γ is C^2-smooth, it is easy to see that the two triangulations are quasi-uniform with respect to the mesh sizes h_1 and h_2 respectively, that is $|e_i^1| \eqsim h_1$ for $i = 1, 2, \ldots, m_1$, and $|e_i^2| \eqsim h_2$ for $i = 1, 2, \ldots, m_2$.

Since the interface Γ is of class C^2, there exists a positive constant h_0 such that for $0 < h_1 \leqslant h_0$, one can introduce a local coordinate system (x_1^j, x_2^j) for each curved segment $e_j^1 \in \Gamma_{h_1}$. We take the x_1^j-axis along the line segment $\tilde{e}_1^j$ and the x_2^j-axis along the normal to $\tilde{e}_j^1$ (Chen & Zou, 1998). Then the curved segment e_j^1 can be parametrized as follows:

$$e_j^1 = \left\{ (x_1^j, x_2^j);\ x_2^j = \phi_j^1(x_1^j),\ x_1^j \in [0, s_j^{h_1}] \right\} \tag{2.2}$$

where $s_j^{h_1}$ is the length of $\tilde{e}_j^1$. We know (Chen & Zou, 1998) that $\phi_j^1 \in C^2[0, s_j^{h_1}]$ for $j = 1, \ldots, m_1$, and

$$|\phi_j^1(x_1^j)| \lesssim h_1^2, \quad \forall x_1^j \in [0, s_j^{h_1}], \tag{2.3}$$

$$|\dot{\phi}_j^1(x_1^j)| \lesssim h_1, \quad \forall x_1^j \in [0, s_j^{h_1}] \tag{2.4}$$

where $\dot{\phi}_j^1(x_1^j)$ denotes the first-order derivative of the function $\phi_j^1(x_1^j)$. For the curved segments $e_j^2 \in \Gamma_{h_2}$, we have similar results, and the related restricted parameter is still denoted as h_0. From now on we assume that the mesh sizes $h_1, h_2 \in (0, h_0)$.

We next introduce some finite element spaces associated with the triangulations constructed above. Let V_{h_i} $(i = 1, 2)$ be the piecewise linear finite element spaces on Ω_i:

$$V_{h_1} = \{v \in C^0(\bar{\Omega}_1);\ v|_K \in P_1(K), \quad \forall K \in \mathcal{T}_{h_1}\}, \tag{2.5}$$

$$V_{h_2} = \{v \in C^0(\bar{\Omega}_2);\ v|_K \in P_1(K), \quad \forall K \in \mathcal{T}_{h_2} \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial\Omega\}. \tag{2.6}$$

Here we adopt the convention that for any function v_h in V_{h_i} $(i = 1, 2)$, its value on any element $K \in \mathcal{T}_{h_i}$ (including the elements with a curved edge) is uniquely defined by the linear function determined by the values of v_h at the three vertices of K.

Furthermore, we define W_{h_i} and $\bar{W}_{h_i}$ $(i = 1, 2)$ to be the piecewise linear and piecewise constant finite element spaces on Γ_{h_i} respectively, i.e.

$$W_{h_i} = \{v \in C^0(\Gamma);\ v|_e \in P_1(e), \quad \forall e \in \Gamma_{h_i}\}, \tag{2.7}$$

$$\bar{W}_{h_i} = \{v \in L^2(\Gamma);\ v|_e = \text{constant}, \quad \forall e \in \Gamma_{h_i}\} \tag{2.8}$$

where $P_1(K)$ is the space of linear polynomials on K and $P_1(e)$ is the space of linear polynomials (according to the arc length parameter) on the curved segment e.

Also, we define a transfer operator $E_{h_i}\colon C^0(\bar{\Omega}_i) \to W_{h_i}$ $(i = 1, 2)$ by

$$(E_{h_i} v)(P_j^i) = v(P_j^i), \ \ j = 1, 2, \ldots, m_i; \quad \forall v \in C^0(\bar{\Omega}_i), \tag{2.9}$$

and the L^2-orthogonal projection operator $Q_{h_i} : L^2(\Gamma) \to \bar{W}_{h_i}$ $(i = 1, 2)$ by

$$\langle Q_{h_i} v, w\rangle = \langle v, w\rangle, \quad \forall v \in L^2(\Gamma),\ w \in \bar{W}_{h_i}. \tag{2.10}$$

With the above preparations, we now state the mortar finite element space as

$$V_h = \{v_h = (v_{h_1}, v_{h_2}) \in V_{h_1} \times V_{h_2};\ \ Q_{h_1} E_{h_2} v_{h_2} = Q_{h_1} E_{h_1} v_{h_1}\}. \tag{2.11}$$

The condition $Q_{h_1} E_{h_2} v_{h_2} = Q_{h_1} E_{h_1} v_{h_1}$ in (2.11) is called the mortar condition, which provides a connection between v_{h_1} and v_{h_2} to ensure that they are weakly continuous across the interface Γ. This mortar condition can be replaced by the condition $Q_{h_2} E_{h_2} v_{h_2} = Q_{h_2} E_{h_1} v_{h_1}$ without any effect on the subsequent convergence results.

Assume that $h_1 \geqslant h_2$ and m_1 is an odd number. Then the *mortar finite element method* for solving (1.1)–(1.3) is formulated as follows: Find $u_h = (u_{h_1}, u_{h_2}) \in V_h$ such that

$$a(u_h, v_h) = \sum_{i=1}^{2} \int_{\Omega_i} f\, v_{h_i}\, \mathrm{d}x + \int_{\Gamma} g\, v_{h_2}\, \mathrm{d}s, \quad \forall v_h = (v_{h_1}, v_{h_2}) \in V_h, \tag{2.12}$$

where $a(u_h, v_h) = a_1(u_{h_1}, v_{h_1}) + a_2(u_{h_2}, v_{h_2})$.

REMARK 2.1 In the case that $h_1 \leqslant h_2$, the second term in the right-hand side of (2.12) should be replaced by $\int_\Gamma g v_{h_1}\,\mathrm{d}s$ in order to achieve the optimal H^1-norm error estimate, see Section 4.

REMARK 2.2 The mortar condition $Q_{h_1}E_{h_2}v_{h_2} = Q_{h_1}E_{h_1}v_{h_1}$ in (2.11) can be described in an explicit form. To see this, by the definitions (2.8) and (2.10), the condition can be written as

$$(E_{h_1}v_{h_1})(M_i) = \frac{1}{|e_i^1|}\int_{e_i^1} E_{h_2}v_{h_2}\,\mathrm{d}s, \quad i = 1, 2, \ldots, m_1, \tag{2.13}$$

where M_i is the midpoint of the curved segment e_i^1. Noting that $E_{h_1}v_{h_1}$ is a linear function on e_i^1, it follows from (2.9) that (2.13) is equivalent to

$$v_{h_1}(P_i^1) + v_{h_1}(P_{i+1}^1) = \frac{2}{|e_i^1|}\int_{e_i^1} E_{h_2}v_{h_2}\,\mathrm{d}s,\ i = 1, 2, \ldots, m_1. \tag{2.14}$$

As m_1 is an odd number, for any given $v_{h_2} \in V_{h_2}$, the mortar condition (2.14) determines the nodal values $\{v_{h_1}(P_j^1)\}_{j=1}^{m_1}$ of v_{h_1} on Γ uniquely. In fact, using (2.14) one can easily express each value $v_{h_1}(P_j^1)$ explicitly in terms of the average values of $E_{h_2}v_{h_2}$ on each e_i^1, $i = 1, 2, \ldots, m_1$. So the nodal values $\{v_{h_2}(P_j^2)\}_{j=1}^{m_2}$ of v_{h_2} on the interface Γ can be chosen arbitrarily in advance (Master), then the nodal values of v_{h_1} on Γ are uniquely determined (Slave).

The following lemma guarantees the unisolvability of problem (2.12).

LEMMA 2.1 The mortar finite element problem (2.12) is unisolvable.

Proof. Since m_1 is an odd number, it is easy to see from (2.14) that the mortar space V_h is a nonempty subspace of the product space $V_{h_1} \times V_{h_2}$. Hence, the unisolvability of the problem (2.12) follows if we can verify that $a(v_h, v_h) = 0$ with $v_h = (v_{h_1}, v_{h_2}) \in V_h$ implies $v_{h_1} = 0$ in Ω_1 and $v_{h_2} = 0$ in Ω_2. This can be done easily. From $a_{h_2}(v_{h_2}, v_{h_2}) = 0$ and $v_{h_2}|_{\partial\Omega} = 0$ we know $v_{h_2} = 0$ in Ω_2. Using the mortar condition (2.14) we then have $v_{h_1}(P_j^1) = 0$, $j = 1, 2, \ldots, m_1$, which together with $a_{h_1}(v_{h_1}, v_{h_1}) = 0$ yields $v_{h_1} = 0$ in Ω_1 immediately. □

We end this section with a remark on a possible solver for the linear algebraic system of equations corresponding to the mortar finite element method (2.12). Note that (2.12) is equivalent to the following saddle-point system:

Find $(u_h, \eta_{h_1}) \in (V_{h_1} \times V_{h_2}) \times \bar{W}_{h_1}$ such that

$$a(u_h, v_h) + \langle E_{h_1}v_{h_1} - E_{h_2}v_{h_2}, \eta_{h_1}\rangle = \sum_{i=1}^{2}(f, v_{h_i}) + \langle g, v_{h_2}\rangle \quad \forall\, v_h \in V_{h_1} \times V_{h_2}$$

$$\langle E_{h_1}u_{h_1} - E_{h_2}u_{h_2}, \zeta_{h_1}\rangle = 0 \quad \forall\, \zeta_{h_1} \in \bar{W}_{h_1}.$$

There are many recent investigations on iterative methods for solving such saddle-point systems, see, for example, the preconditioned Uzawa-type iterative methods (Elman & Golub, 1994; Hu & Zou, 2001; Rusten & Winther, 1992).

3. Some discrete operators and their approximation properties

In this section, we introduce some discrete operators and present their approximation properties, which will be used in the subsequent error analysis for the mortar element method (2.12).

Before we proceed, we first give some useful estimates for the following elliptic problem:

$$-\Delta u_1 + u_1 = 0 \quad \text{in } \Omega_1; \quad u_1 = g_1 \quad \text{on } \Gamma. \tag{3.1}$$

LEMMA 3.1 The following *a priori* estimates hold for the solution u_1 of (3.1):

$$\|u_1\|_{1,\Omega_1} \lesssim \|g_1\|_{H^{1/2}(\Gamma)}, \quad \|u_1\|_{0,\Omega_1} \lesssim \|g_1\|_{H^{-1/2}(\Gamma)} \tag{3.2}$$

if $g_1 \in H^{1/2}(\Gamma)$, and

$$\|u_1\|_{3/2,\Omega_1} \lesssim \|g_1\|_{H^1(\Gamma)}, \quad \|u_1\|_{3/2+\epsilon,\Omega_1} \lesssim \|g_1\|_{H^{1+\epsilon}(\Gamma)} \tag{3.3}$$

if $g_1 \in H^1(\Gamma)$ or $g_1 \in H^{1+\epsilon}(\Gamma)$ for any $0 < \epsilon \leqslant 1/2$.

Proof. (3.3) and the first estimate of (3.2) are well known (Grisvard, 1985). The second inequality of (3.2) can be proved by the duality argument, see Huang & Zou (2000) for details. □

3.1 *Interpolation and H^1-norm projection operators*

Let I_{h_1} and I_{h_2} be the piecewise linear nodal value interpolation operators associated with the finite element spaces V_{h_1} and V_{h_2}. Then we have the following lemma.

LEMMA 3.2 For any mesh sizes $h_1, h_2 \in (0, h_0)$ and $1 < s \leqslant 2$,

$$\left\| v_i - I_{h_i} v_i \right\|_{0,\Omega_i} + h_i \left\| v_i - I_{h_i} v_i \right\|_{1,\Omega_i} \lesssim h_i^s \, \|v_i\|_{s,\Omega_i}, \quad \forall v_i \in H^s(\Omega_i),\ i = 1, 2. \tag{3.4}$$

Proof. The proof follows basically the standard techniques used for the H^1- and L^2-norm error estimates of finite element methods (Brenner & Scott, 1994; Ciarlet, 1978). The crucial step here is to derive the required estimates corresponding to those curved elements near the interface. For completeness, we give a simple proof for $i = 1$, the case with $i = 2$ can be proved in the same manner. For any $v_1 \in H^s(\Omega_1)$, by the extension theorem for Sobolev spaces (Grisvard, 1985), there exists an extension operator $E_1 : H^s(\Omega_1) \to H^s(R^2)$ such that $E_1 v_1 = v_1$ in Ω_1 and

$$\|E_1 v_1\|_{s,R^2} \lesssim \|v_1\|_{s,\Omega_1}. \tag{3.5}$$

If $K \in \mathcal{T}_{h_1}$ is a triangle, by the standard interpolation error estimates we have (Brenner & Scott, 1994; Ciarlet, 1978)

$$\left\| v_1 - I_{h_1} v_1 \right\|_{0,K}^2 + h_1^2 \left\| v_1 - I_{h_1} v_1 \right\|_{1,K}^2 \lesssim h_1^{2s} \, \|v_1\|_{s,K}^2 . \tag{3.6}$$

Now consider a curved element $K \in T_{h_1}$ with a curved segment $e_j^1 = \widehat{P_j^1 P_{j+1}^1}$ as one of its edges. We know from (2.3) that the largest distance between e_j^1 and $\tilde{e}_j^1$ is of order $O(h_1^2)$, so we can construct a shape-regular triangle $\tilde{K}$ of size h_1 such that $K \subset \tilde{K}$. Then, similar to (3.6), we have

$$\left\|v_1 - I_{h_1}v_1\right\|_{0,K}^2 + h_1^2 \left\|v_1 - I_{h_1}v_1\right\|_{1,K}^2 \lesssim \left\|E_1v_1 - I_{h_1}E_1v_1\right\|_{0,\tilde{K}}^2 + h_1^2 \left\|E_1v_1 - I_{h_1}E_1v_1\right\|_{1,\tilde{K}}^2 \lesssim h_1^{2s} \left\|E_1v_1\right\|_{s,\tilde{K}}^2 . \tag{3.7}$$

Summing all the estimates (3.6) and (3.7) over $K \in \mathcal{T}_{h_1}$ and using (3.5), we obtain

$$\left\|v_1 - I_{h_1}v_1\right\|_{0,\Omega_1}^2 + h_1^2 \left\|v_1 - I_{h_1}v_1\right\|_{1,\Omega_1} \lesssim h_1^{2s} \left\|E_1v_1\right\|_{s,R^2}^2 \lesssim h_1^{2s} \left\|v_1\right\|_{s,\Omega_1}^2 .$$

□

LEMMA 3.3 For any $v_1 \in H_0^1(\Omega_1) \cap H^2(\Omega_1)$, we have

$$\|I_{h_1}v_1\|_{L^2(\Gamma)} \lesssim h_1^2\|v_1\|_{2,\Omega_1}. \tag{3.8}$$

Proof. For any $e_j^1 \in \Gamma_{h_1}$, using (2.2)–(2.4), the inverse inequality (Babuska & Aziz, 1972) and Lemma 3.2, we have

$$\begin{aligned}
\|I_{h_1}v_1\|_{L^2(e_j^1)}^2 &= \int_0^{s_j^{h_1}} [(I_{h_1}v_1)(x_1^j, \phi_j^1(x_1^j)) - (I_{h_1}v_1)(x_1^j, 0)]^2 \sqrt{1 + (\dot{\phi}_j^1(x_1^j))^2}\, \mathrm{d}x_1^j \\
&\lesssim |I_{h_1}v_1|_{1,\infty,K_j}^2 \max_{0 \leqslant x_1^j \leqslant s_j^{h_1}} |\phi_j^1(x_1^j)|^2 h_1 \\
&\lesssim h_1^3 |I_{h_1}v_1|_{1,K_j}^2 \lesssim h_1^5|v_1|_{2,K_j}^2 + h_1^3|v_1|_{1,K_j}^2
\end{aligned} \tag{3.9}$$

where $K_j \in \mathcal{T}_{h_1}$ is the curved triangle with e_j^1 being one of its edges. Summing the estimate (3.9) over j, we obtain

$$\|I_{h_1}v_1\|_{L^2(\Gamma)}^2 \lesssim h_1^5|v_1|_{2,\tilde{\Omega}_1}^2 + h_1^3|v_1|_{1,\tilde{\Omega}_1}^2, \tag{3.10}$$

where $\tilde{\Omega}_1$ denotes the union of all those curved elements K_j near the interface Γ. We next estimate the term $|v_1|_{1,\tilde{\Omega}_1}$. Consider the neighbourhood $N_{h_1}(\Gamma)$ of the interface Γ of width h_1, i.e.,

$$N_{h_1}(\Gamma) = \{x \in \bar{\Omega}_1;\ \mathrm{dist}(x, \Gamma) \leqslant h_1\},$$

we see that $\tilde{\Omega}_1 \subset N_{h_1}(\Gamma)$. For any $(x_1, x_2) \in N_{h_1}(\Gamma)$, let y_2 be the distance from this point to the interface Γ with the corresponding projection point on Γ having arc length y_1. When h_1 is appropriately small, the mapping from (x_1, x_2) to (y_1, y_2) is a C^2-diffemorphism, we denote it by Φ, that is $\Phi(x_1, x_2) = (y_1, y_2)$. For any $w_1 \in H^1(N_{h_1}(\Gamma))$, define $\hat{w}_1(y_1, y_2) = w_1(\Phi^{-1}(y_1, y_2)) \in H^1(R_1)$, where $R_1 = [0, s_\Gamma] \times [0, h_1]$ with s_Γ being the length of the interface Γ. Therefore, by the Cauchy–Schwartz inequality we easily have

$$|\hat{w}_1(y_1, y_2)|^2 = \left|\hat{w}_1(y_1, 0) + \int_0^{y_2} \partial_t \hat{w}_1(y_1, t)\, \mathrm{d}t\right|^2 \lesssim \hat{w}^2(y_1, 0) + y_2 \int_0^{y_2} |\partial_t \hat{w}_1|^2(y_1, t)\, \mathrm{d}t.$$

Then integrating both sides over the domain R_1 yields

$$\begin{aligned}\|\hat{w}_1\|_{0,R_1}^2 &\lesssim h_1\|\hat{w}\|^2_{L^2([0,s_\Gamma]\times\{0\})} + \|\partial_{y_2}\hat{w}_1\|^2_{0,R_1}\int_0^{h_1} y_2\,\mathrm{d}y_2\\ &\lesssim h_1\|\hat{w}\|^2_{L^2([0,s_\Gamma]\times\{0\})} + h_1^2\|\partial_{y_2}\hat{w}_1\|^2_{0,R_1}. \end{aligned}\tag{3.11}$$

Since Φ is a C^2-diffemorphism, we also have $\|\hat{w}\|_{L^2([0,s_\Gamma]\times\{0\})} \eqsim \|w_1\|_{L^2(\Gamma)}$ and $\|w_1\|_{s,N_{h_1}(\Gamma)} \eqsim \|\hat{w}_1\|_{s,R_1}$ for $s = 0, 1$ (Grisvard, 1985), thus the estimate (3.11) can be rewritten as

$$\|w_1\|_{0,N_{h_1}(\Gamma)} \lesssim h_1^{1/2}\|w_1\|_{L^2(\Gamma)} + h_1\|w_1\|_{1,N_{h_1}(\Gamma)} \lesssim h_1^{1/2}\|w_1\|_{1,\Omega_1}. \tag{3.12}$$

This implies by letting $w_1 = \partial_1 v_1$ and $w_1 = \partial_2 v_1$ that

$$|v_1|_{1,\tilde{\Omega}_1} \lesssim h_1^{1/2}\,\|v_1\|_{2,\Omega_1}, \tag{3.13}$$

which together with (3.10) leads to the desired estimate. □

We next introduce two elliptic projection operators $P_{h_1} : H^1(\Omega_1) \to V_{h_1}$ and $P_{h_2} : H^1_*(\Omega_2) \to V_{h_2}$ with $H^1_*(\Omega_2) = \{v_2 \in H^1(\Omega_2);\ v_2 = 0 \text{ on } \partial\Omega\}$. For any $v_1 \in H^1(\Omega_1)$ and $v_2 \in H^1_*(\Omega_2)$, $P_{h_i}v_i \in V_{h_i}$ $(i = 1, 2)$ is defined by

$$(P_{h_i}v_i, w)_{1,\Omega_i} = (v_i, w)_{1,\Omega_i} \quad \forall w \in V_{h_i} \tag{3.14}$$

where the scalar products $(\cdot,\cdot)_{1,\Omega_i}$ for $i = 1, 2$ are given by

$$(v, w)_{1,\Omega_i} = \int_{\Omega_i} (\nabla v\cdot\nabla w + vw)\,\mathrm{d}x, \quad \forall\, v, w \in H^1(\Omega_i).$$

LEMMA 3.4 Operators P_{h_i}, $i = 1, 2$, possess the following approximation properties:

$$\left\|v_i - P_{h_i}v_i\right\|_{0,\Omega_i} + h_i\left\|v_i - P_{h_i}v_i\right\|_{1,\Omega_i} \lesssim h_i^2\|v_i\|_{2,\Omega_i}, \quad \forall v_i \in H^2(\Omega_i), \tag{3.15}$$

$$\left\|v_i - P_{h_i}v_i\right\|_{H^{-1/2}(\Gamma)} \lesssim h_i^2\|v_i\|_{2,\Omega_i}, \quad \forall v_i \in H^2(\Omega_i). \tag{3.16}$$

Proof. (3.15) can be obtained using Lemma 3.2 and the standard finite element analysis as used for deriving the H^1- and L^2-norm error estimates (Brenner & Scott, 1994; Ciarlet, 1978). With the help of (3.15), (3.16) can be shown by the standard duality argument, see Huang & Zou (2000) for details. □

3.2 *Extension and modified H^1-norm projection operators*

We now construct an important extension operator $F_{h_1} : W_{h_1} \to V_{h_1}$. For any $\alpha_{h_1} \in W_{h_1}$, $F_{h_1}\alpha_{h_1} \in V_{h_1}$ satisfies $(F_{h_1}\alpha_{h_1})(P_j^1) = \alpha_{h_1}(P_j^1)$ $(j = 1, 2, \ldots, m_1)$ and solves the discrete system

$$\int_{\Omega_1}\{\nabla(F_{h_1}\alpha_{h_1})\cdot\nabla w + (F_{h_1}\alpha_{h_1})w\}\,\mathrm{d}x = 0, \quad \forall w \in V^0_{h_1} \tag{3.17}$$

where $V_{h_1}^0$ consists of those functions in V_{h_1} which vanish at all interface nodal points P_j^1, $j = 1, 2, \ldots, m_1$.

The next lemma presents some useful properties of F_{h_1}, whose proofs are given in the Appendix.

LEMMA 3.5 For the extension operator F_{h_1} we have

$$\|F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} \eqsim \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)}, \quad \|F_{h_1}\alpha_{h_1}\|_{0,\Omega_1} \lesssim \|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)}, \quad \forall \alpha_{h_1} \in W_{h_1}. \tag{3.18}$$

With the extension operator F_{h_1}, we are ready to propose a modified H^1-norm projection operator $P_h : X \to V_h$, which will play a crucial role in the subsequent error estimates for our mortar element method.

We first construct a transfer operator $G_{h_1} : L^2(\Gamma) \to W_{h_1}$. For any $v_1 \in L^2(\Gamma)$, $G_{h_1}v_1 \in W_{h_1}$ is determined by

$$Q_{h_1}(G_{h_1}v_1) = Q_{h_1}v_1.$$

Noting that m_1 is an odd number, we can easily find that G_{h_1} is well defined, and using the similar deduction for deriving the explicit mortar condition (2.13), we have

$$(G_{h_1}v_1)(M_j) = \frac{1}{|e_j^1|}\int_{e_j^1} v_1(s)\,\mathrm{d}s, \quad j = 1, \ldots, m_1. \tag{3.19}$$

For any $v = (v_1, v_2) \in X$, let $P_{h_i}v_i$ $(i = 1, 2)$ be the H^1-norm projections of v_i as defined in (3.14). Using the following special finite element function in W_{h_1}:

$$\alpha_{h_1} = G_{h_1}(E_{h_2}P_{h_2}v_2 - E_{h_1}P_{h_1}v_1),$$

we define the modified projection operator $P_h v$ as

$$(P_h v)(x) = \begin{cases} (P_{h_1}v_1)(x) + (F_{h_1}\alpha_{h_1})(x) & \text{for } x \in \Omega_1, \\ (P_{h_2}v_2)(x) & \text{for } x \in \Omega_2\,. \end{cases} \tag{3.20}$$

Using the fact that $E_{h_1}F_{h_1}\alpha_{h_1} = \alpha_{h_1}$, it is easy to see that $P_h v \in V_h$. We are now going to establish some error estimates of the operator P_h, for which we need the $H^{1/2}$-stability of G_{h_1}.

LEMMA 3.6 The transfer operator $G_{h_1} : L^2(\Gamma) \to W_{h_1}$ is stable in $H^{1/2}(\Gamma)$, namely

$$\|G_{h_1}v\|_{H^{1/2}(\Gamma)} \lesssim \|v\|_{H^{1/2}(\Gamma)}, \quad \forall v \in H^{1/2}(\Gamma). \tag{3.21}$$

Proof. We first prove for $s = 0, 1$ that

$$\|G_{h_1}v\|_{H^s(\Gamma)} \lesssim \|v\|_{H^s(\Gamma)}, \quad \forall v \in H^s(\Gamma). \tag{3.22}$$

By the standard scaling arguments (Brenner & Scott, 1994; Ciarlet, 1978), (3.19) and the Cauchy-Schwartz inequality we have

$$\begin{aligned}\|G_{h_1}v\|^2_{L^2(\Gamma)} &\lesssim h_1 \sum_{j=1}^{m_1} |(G_{h_1}v)(M_j)|^2 \\ &\lesssim h_1 \sum_{j=1}^{m_1} \left| \frac{1}{|e^1_j|} \int_{e^1_j} v(s)\,\mathrm{d}s \right|^2 \\ &\lesssim \|v\|^2_{L^2(\Gamma)}.\end{aligned}$$

This proves (3.22) with $s = 0$. Similarly, noting $H^1(\Gamma) \subset C^0(\Gamma)$, we have

$$\begin{aligned}|G_{h_1}v|^2_{H^1(\Gamma)} &\lesssim h_1^{-1} \sum_{j=1}^{m_1} \left\{ G_{h_1}v(M_j) - G_{h_1}v(M_{j+1}) \right\}^2 \\ &\lesssim h_1^{-1} \sum_{j=1}^{m_1} \left\{ \frac{1}{|e^1_j|} \int_{e^1_j} v(s)\,\mathrm{d}s - \frac{1}{|e^1_{j+1}|} \int_{e^1_{j+1}} v(s)\,\mathrm{d}s \right\}^2 \\ &\lesssim h_1^{-1} \sum_{j=1}^{m_1} \left\{ \left| \frac{1}{|e^1_j|} \int_{e^1_j} v(s)\,\mathrm{d}s - v(P^1_{j+1}) \right|^2 + \left| \frac{1}{|e^1_{j+1}|} \int_{e^1_{j+1}} v(s)\,\mathrm{d}s - v(P^1_{j+1}) \right|^2 \right\} \\ &\lesssim h_1^{-1} \sum_{j=1}^{m_1} \{ \max_{x \in e^1_j} |v(x) - v(P^1_{j+1})|^2 + \max_{x \in e^1_{j+1}} |v(x) - v(P^1_{j+1})|^2 \} \\ &\lesssim |v|^2_{H^1(\Gamma)}.\end{aligned}$$

(3.21) then follows from (3.22) with $s = 0, 1$ and the interpolation theory of Sobolev spaces (Bergh & Löfstrom, 1976). □

In what follows, for any $1 \leqslant s \leqslant 2$ and $v = (v_1, v_2) \in H^s(\Omega_1) \times H^s(\Omega_2)$, we use the following conventional norms and seminorms:

$$\|v\|_{s,\Omega} = (\|v_1\|^2_{s,\Omega_1} + \|v_2\|^2_{s,\Omega_2})^{1/2}, \quad |v|_{s,\Omega} = (|v_1|^2_{s,\Omega_1} + |v_2|^2_{s,\Omega_2})^{1/2}.$$

LEMMA 3.7 The modified projection operator $P_h : X \to V_h$ defined by (3.20) satisfies the following H^1-norm estimate:

$$\|v - P_h v\|_{1,\Omega} \lesssim h_1 \|v_1\|_{2,\Omega_1} + h_2 \|v_2\|_{2,\Omega_2}, \quad \forall v = (v_1, v_2) \in X. \tag{3.23}$$

Proof. By the definition (3.20), it follows directly from Lemmata 3.4–3.6 that

$$\begin{aligned}\|v - P_h v\|_{1,\Omega} &\lesssim \sum_{i=1}^{2} \|v_i - P_{h_i} v_i\|_{1,\Omega_i} + \|F_{h_1} G_{h_1} (E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1)\|_{1,\Omega_1} \\ &\lesssim \sum_{i=1}^{2} h_i \|v_i\|_{2,\Omega_i} + \|E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1\|_{H^{1/2}(\Gamma)}.\end{aligned} \tag{3.24}$$

But noting that $v_1 = v_2$ on Γ, we have

$$\|E_{h_2}P_{h_2}v_2 - E_{h_1}P_{h_1}v_1\|_{H^{1/2}(\Gamma)} \lesssim \sum_{i=1}^{2}\{\|E_{h_i}(v_i - P_{h_i}v_i)\|_{H^{1/2}(\Gamma)} + \|v_i - E_{h_i}v_i\|_{H^{1/2}(\Gamma)}\}. \tag{3.25}$$

Since $E_{h_i}v_i$ is the continuous and piecewise linear interpolation function of v_i associated with W_{h_i}, we have (Babuska & Aziz, 1972; Brenner & Scott, 1994)

$$\|v_i - E_{h_i}v_i\|_{H^{1/2}(\Gamma)} \lesssim h_i\|v_i\|_{H^{3/2}(\Gamma)} \lesssim h_i\|v_i\|_{2,\Omega_i}. \tag{3.26}$$

Furthermore, by the inverse inequality, Lemma 3.4 and error estimates of the interpolation operator E_{h_i} (Babuska & Aziz, 1972; Brenner & Scott, 1994) we know, for any $\varepsilon \in (0, \frac{1}{2})$, that

$$\begin{aligned}\|E_{h_i}(v_i - P_{h_i}v_i)\|_{H^{1/2}(\Gamma)} &\lesssim h_i^{-1/2}\|E_{h_i}(v_i - P_{h_i}v_i)\|_{L^2(\Gamma)}\\ &\lesssim h_i^{-1/2}\{\|v_i - P_{h_i}v_i\|_{L^2(\Gamma)} + \|(I - E_{h_i})(v_i - P_{h_i}v_i)\|_{L^2(\Gamma)}\}\\ &\lesssim h_i^{-1/2}\{\|v_i - P_{h_i}v_i\|_{L^2(\Gamma)} + h_i^{1/2+\varepsilon}\|v_i - P_{h_i}v_i\|_{H^{1/2+\varepsilon}(\Gamma)}\}\\ &\lesssim h_i^{-1/2}\|v_i - P_{h_i}v_i\|_{L^2(\Gamma)} + h_i^{\varepsilon}\|v_i - P_{h_i}v_i\|_{1+\varepsilon,\Omega_i}\\ &\lesssim h_i^{-1/2}\|v_i - P_{h_i}v_i\|_{L^2(\Gamma)} + h_i\|v_i\|_{2,\Omega_i}.\end{aligned} \tag{3.27}$$

By the Sobolev interpolation theory (Babuska & Aziz, 1972) and Lemma 3.4 we have

$$\begin{aligned}\|v_i - P_{h_i}v_i\|_{L^2(\Gamma)} &\lesssim \|v_i - P_{h_i}v_i\|_{H^{-1/2}(\Gamma)}^{1/2}\|v_i - P_{h_i}v_i\|_{H^{1/2}(\Gamma)}^{1/2}\\ &\lesssim \|v_i - P_{h_i}v_i\|_{H^{-1/2}(\Gamma)}^{1/2}\|v_i - P_{h_i}v_i\|_{1,\Omega_i}^{1/2}\\ &\lesssim h_i^{3/2}\|v_i\|_{2,\Omega_i}.\end{aligned} \tag{3.28}$$

The desired result then follows from (3.24)–(3.28). □

To derive the L^2-norm error estimate of the operator P_h, we need the following result.

LEMMA 3.8 For the L^2 projection operator $Q_{h_1} : L^2(\Gamma) \to \bar{W}_{h_1}$ defined by (2.10), we have the following estimate:

$$\|v - Q_{h_1}v\|_{H^{-1/2}(\Gamma)} \lesssim h_1\|v\|_{H^{1/2}(\Gamma)}, \quad \forall v \in H^{1/2}(\Gamma). \tag{3.29}$$

Proof. By the standard technique as used for the error estimates of L^2 projection operators (Xu, 1989) and the Sobolev interpolation theory, we have for $0 \leqslant s \leqslant 1$,

$$\|v - Q_{h_1}v\|_{L^2(\Gamma)} \lesssim h_1^s\|v\|_{H^s(\Gamma)}, \quad \forall v \in H^s(\Gamma). \tag{3.30}$$

This with the duality argument yields

$$
\begin{aligned}
\|v - Q_{h_1} v\|_{H^{-1/2}(\Gamma)} &= \sup_{w \in H^{1/2}(\Gamma)} \frac{\langle v - Q_{h_1} v, w \rangle}{\|w\|_{H^{1/2}(\Gamma)}} \\
&= \sup_{w \in H^{1/2}(\Gamma)} \frac{\langle v - Q_{h_1} v, w - Q_{h_1} w \rangle}{\|w\|_{H^{1/2}(\Gamma)}} \\
&\lesssim \|v - Q_{h_1} v\|_{L^2(\Gamma)} \sup_{w \in H^1(\Gamma)} \frac{\|w - Q_{h_1} w\|_{L^2(\Gamma)}}{\|w\|_{H^{1/2}(\Gamma)}} \\
&\lesssim h_1 \|v\|_{H^{1/2}(\Gamma)}.
\end{aligned}
$$

□

LEMMA 3.9 The modified projection operator $P_h : X \to V_h$ defined by (3.20) satisfies the following L^2-norm error estimate:

$$
\|v - P_h v\|_{0,\Omega} \lesssim h_1^2 \|v_1\|_{2,\Omega_1} + (h_1 h_2 + h_2^2) \|v_2\|_{2,\Omega_2} \quad \forall v = (v_1, v_2) \in X. \tag{3.31}
$$

Proof. By the definition of (3.20) and Lemmata 3.4–3.5, we have

$$
\begin{aligned}
\|v - P_h v\|_{0,\Omega} &\lesssim \sum_{i=1}^{2} \|v_i - P_{h_i} v_i\|_{0,\Omega_i} + \|F_{h_1} \alpha_{h_1}\|_{0,\Omega_1} \\
&\lesssim \sum_{i=1}^{2} h_i^2 \|v_i\|_{2,\Omega_i} + \|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)}.
\end{aligned} \tag{3.32}
$$

On the other hand, it follows from the identity $Q_{h_1} \alpha_{h_1} = Q_{h_1}(E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1)$ and (3.29) that

$$
\begin{aligned}
\|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} &\leqslant \|\alpha_{h_1} - Q_{h_1} \alpha_{h_1}\|_{H^{-1/2}(\Gamma)} + \|Q_{h_1}(E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1)\|_{H^{-1/2}(\Gamma)} \\
&\lesssim h_1 \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} + \|Q_{h_1}(E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1)\|_{H^{-1/2}(\Gamma)} \\
&\lesssim h_1 \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} + \|(I - Q_{h_1})(E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1)\|_{H^{-1/2}(\Gamma)} \\
&\quad + \|E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1\|_{H^{-1/2}(\Gamma)} \\
&\lesssim h_1 \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} + h_1 \|E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1\|_{H^{1/2}(\Gamma)} \\
&\quad + \|E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1\|_{H^{-1/2}(\Gamma)}.
\end{aligned} \tag{3.33}
$$

By Lemma 3.6 and (3.25) we see

$$
\|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} \lesssim \|E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1\|_{H^{1/2}(\Gamma)} \lesssim h_1 \|v_1\|_{2,\Omega_1} + h_2 \|v_2\|_{2,\Omega_2}, \tag{3.34}
$$

while by the triangle inequality, Lemma 3.4 and the fact that $v_1 = v_2$ on Γ, we obtain

$$
\begin{aligned}
&\|E_{h_2} P_{h_2} v_2 - E_{h_1} P_{h_1} v_1\|_{H^{-1/2}(\Gamma)} \\
&\lesssim \sum_{i=1}^{2} \{\|E_{h_i} P_{h_i} v_i - P_{h_i} v_i\|_{H^{-1/2}(\Gamma)} + \|v_i - P_{h_i} v_i\|_{H^{-1/2}(\Gamma)}\} \\
&\lesssim \sum_{i=1}^{2} \|E_{h_i} P_{h_i} v_i - P_{h_i} v_i\|_{H^{-1/2}(\Gamma)} + \sum_{i=1}^{2} h_i^2 \|v_i\|_{2,\Omega_i}.
\end{aligned} \tag{3.35}
$$

It remains to estimate $\|E_{h_i}P_{h_i}v_i - P_{h_i}v_i\|_{H^{-1/2}(\Gamma)}$. It suffices to give the estimate for the case with $i = 1$. For any $e_j^1 \in \Gamma_{h_1}$, let K_j be the curved element with e_j^1 as one of its edges. Noting that $(P_{h_1}v_1)|_{K_j} \in P_1(K_j)$ (thus the second-order derivatives of $P_{h_1}v_1$ vanish on K_j) and $E_{h_1}P_{h_1}v_1$ is the continuous and piecewise linear interpolation of $P_{h_1}v_1$, and using the inverse inequality we have

$$\begin{aligned}\|E_{h_1}P_{h_1}v_1 - P_{h_1}v_1\|_{L^2(e_j^1)} &\lesssim h_1^2|P_{h_1}v_1|_{H^2(e_j^1)} \lesssim h_1^{5/2}\|P_{h_1}v_1\|_{2,\infty,K_j}\\ &= h_1^{5/2}\|P_{h_1}v_1\|_{1,\infty,K_j} \lesssim h_1^{3/2}\|P_{h_1}v_1\|_{1,K_j}. \qquad (3.36)\end{aligned}$$

Squaring both sides of (3.36) and summing them over all curved elements K_j near the interface, we derive

$$\|E_{h_1}P_{h_1}v_1 - P_{h_1}v_1\|^2_{L^2(\Gamma)} \lesssim h_1^3\|P_{h_1}v_1\|^2_{1,\tilde{\Omega}_1}, \qquad (3.37)$$

where $\tilde{\Omega}_1$ is defined as that introduced in the proof of Lemma 3.3. Similar to the proof of (3.13), we can show that

$$\|v_1\|^2_{1,\tilde{\Omega}_1} \lesssim h_1\|v_1\|^2_{2,\Omega_1}, \quad \forall v_1 \in H^2(\Omega_1),$$

which, together with Lemma 3.4 yields

$$\|P_{h_1}v_1\|^2_{1,\tilde{\Omega}_1} \lesssim \|v_1 - P_{h_1}v_1\|^2_{1,\Omega_1} + \|v_1\|^2_{1,\tilde{\Omega}_1} \lesssim h_1\|v_1\|^2_{2,\Omega_1}. \qquad (3.38)$$

Now it follows from (3.37)–(3.38) that

$$\begin{aligned}\|E_{h_1}P_{h_1}v_1 - P_{h_1}v_1\|_{H^{-1/2}(\Gamma)} &\lesssim \|E_{h_1}P_{h_1}v_1 - P_{h_1}v_1\|_{L^2(\Gamma)}\\ &\lesssim h_1^2\|v_1\|_{2,\Omega_1}. \qquad (3.39)\end{aligned}$$

The desired estimate (3.31) then follows directly from (3.32)–(3.35) and (3.39). □

LEMMA 3.10 For the jumps of the modified projection operator P_h across the interface Γ, we have the following estimate:

$$\|[P_hv]\|_{H^{-1/2}(\Gamma)} \lesssim h_1^2\|v_1\|_{2,\Omega_1} + (h_1h_2 + h_2^2)\|v_2\|_{2,\Omega_2}, \quad \forall v = (v_1, v_2) \in X. \quad (3.40)$$

Proof. The proof will be given in the Appendix as it needs some technique used in the proof of Lemma 3.5. □

4. Error estimates for the mortar finite element method

This section is devoted to the H^1- and L^2-norm error estimates for the mortar finite element method (2.12) with the case $h_1 \geqslant h_2$. The other case with $h_1 < h_2$ (see Remark 2.1) can be dealt with similarly. We assume that $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, and thus the solution $u \in X = H^2(\Omega_1) \cap H^2(\Omega_2) \cap H_0^1(\Omega)$. By the second Strang Lemma (Ciarlet, 1978) we have

$$\|u - u_h\|_{1,\Omega} \lesssim \|u - P_hu\|_{1,\Omega} + \sup_{\xi=(\xi_1,\xi_2)\in V_h} \frac{|a(u,\xi) - (f,\xi) - \langle g, \xi_2\rangle|}{\|\xi\|_{1,\Omega}}. \qquad (4.1)$$

The first term in the right-hand side of (4.1) represents the approximation error, while the second denotes the inconsistency error. From Lemma 3.7 we have

$$\|u - P_h u\|_{1,\Omega} \lesssim h_1 \|u_1\|_{2,\Omega_1} + h_2 \|u_2\|_{2,\Omega_2}. \tag{4.2}$$

Moreover, by integration by parts and using (1.1)–(1.3) we see

$$\begin{aligned} a(u,\xi) &= \sum_{i=1}^{2} \int_{\Omega_i} \beta_i(x) \nabla u_i \cdot \nabla \xi_i \, \mathrm{d}x \\ &= \sum_{i=1}^{2} \int_{\Omega_i} (-\nabla \cdot (\beta_i(x) \nabla u_i) \xi_i \, \mathrm{d}x + \int_\Gamma \beta_1(x) \partial_{\mathbf{n}} u_1 \, \xi_1 \, \mathrm{d}s - \int_\Gamma \beta_2(x) \partial_{\mathbf{n}} u_2 \, \xi_2 \, \mathrm{d}s \\ &= (f,\xi) + \int_\Gamma g \, \xi_1 \, \mathrm{d}s + \int_\Gamma \beta_2(x) \partial_{\mathbf{n}} u_2 \, [\xi] \, \mathrm{d}s. \end{aligned}$$

That implies

$$\begin{aligned} a(u,\xi) - (f,\xi) - \langle g, \xi_2 \rangle &= \int_\Gamma g \, [\xi] \, \mathrm{d}s + \int_\Gamma \beta_2(x) \partial_{\mathbf{n}} u_2 \, [\xi] \, \mathrm{d}s \\ &= \int_\Gamma \beta_1(x) \partial_{\mathbf{n}} u_1 \, [\xi] \, \mathrm{d}s \equiv \mathrm{III}. \end{aligned}$$

We now estimate the term III. We first rewrite it as

$$\begin{aligned} \mathrm{III} &= \int_\Gamma \beta_1(x) \partial_{\mathbf{n}} u_1 \, (E_{h_1}\xi_1 - E_{h_2}\xi_2) \, \mathrm{d}s \\ &\quad + \int_\Gamma \beta_1(x) \partial_{\mathbf{n}} u_1 \, (\xi_1 - E_{h_1}\xi_1) \, \mathrm{d}s - \int_\Gamma \beta_1(x) \partial_{\mathbf{n}} u_1 \, (\xi_2 - E_{h_2}\xi_2) \, \mathrm{d}s \\ &\equiv \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3. \end{aligned} \tag{4.3}$$

For any $e_j^1 \in \Gamma_{h_1}$, noting that $\xi_1|_{K_j} \in P_1(K_j)$ and $E_{h_1}\xi_1$ is the continuous and piecewise linear interpolation of ξ_1 on Γ, and using the inverse inequality we have

$$\|\xi_1 - E_{h_1}\xi_1\|_{L^2(e_j^1)} \lesssim h_1^2 |\xi_1|_{H^2(e_j^1)} \lesssim h_1^{5/2} \|\xi_1\|_{1,\infty,K_j} \lesssim h_1^{3/2} \|\xi_1\|_{1,K_j}. \tag{4.4}$$

Squaring both sides of (4.4) and summing them over all curved elements K_j yield

$$\|\xi_1 - E_{h_1}\xi_1\|^2_{L^2(\Gamma)} \lesssim h_1^3 \|\xi_1\|^2_{1,\Omega_1}.$$

Then by the trace theorem of Sobolev spaces (Grisvard, 1985) we know

$$|\mathrm{III}_2| \lesssim \|\beta_1(x) \partial_{\mathbf{n}} u_1\|_{L^2(\Gamma)} \|\xi_1 - E_{h_1}\xi_1\|_{L^2(\Gamma)} \lesssim h_1^{3/2} \|u_1\|_{2,\Omega_1} \|\xi_1\|_{1,\Omega_1}. \tag{4.5}$$

Similarly, we can derive (noting $h_2 \leqslant h_1$)

$$\begin{aligned} |\mathrm{III}_3| &\lesssim \|\beta_1(x) \partial_{\mathbf{n}} u_1\|_{L^2(\Gamma)} \|\xi_2 - E_{h_2}\xi_2\|_{L^2(\Gamma)} \\ &\lesssim h_2^{3/2} \|u_1\|_{2,\Omega_1} \|\xi_2\|_{2,\Omega_2} \lesssim h_1^{3/2} \|u_1\|_{2,\Omega_1} \|\xi_2\|_{2,\Omega_2}. \end{aligned} \tag{4.6}$$

For III_1, noting the mortar condition $Q_{h_1}(E_{h_1}\xi_1) = Q_{h_1}(E_{h_2}\xi_2)$, we have

$$\begin{aligned}\mathrm{III}_1 &= \int_\Gamma \beta_1(x)\partial_{\mathbf{n}}u_1\ (E_{h_1}\xi_1 - Q_{h_1}(E_{h_1}\xi_1))\,\mathrm{d}s \\ &\quad - \int_\Gamma \beta_1(x)\partial_{\mathbf{n}}u_1\ (E_{h_2}\xi_2 - Q_{h_1}(E_{h_2}\xi_2))\,\mathrm{d}s \\ &= \int_\Gamma \{(I - Q_{h_1})(\beta_1(x)\partial_{\mathbf{n}}u_1)\}\{(I - Q_{h_1})(E_{h_1}\xi_1)\}\,\mathrm{d}s \\ &\quad - \int_\Gamma \{(I - Q_{h_1})(\beta_1(x)\partial_{\mathbf{n}}u_1)\}\{(I - Q_{h_1})(E_{h_2}\xi_2)\}\,\mathrm{d}s.\end{aligned}$$

Hence, by (3.30) we find

$$\begin{aligned}|\mathrm{III}_1| &\lesssim \|(I - Q_{h_1})(\beta_1(x)\partial_{\mathbf{n}}u_1)\|_{L^2(\Gamma)} \sum_{i=1}^{2} \|(I - Q_{h_1})(E_{h_i}\xi_i)\|_{L^2(\Gamma)} \\ &\lesssim h_1\|\beta_1(x)\partial_{\mathbf{n}}u_1\|_{H^{1/2}(\Gamma)} \sum_{i=1}^{2} \|E_{h_i}\xi_i\|_{H^{1/2}(\Gamma)} \\ &\lesssim h_1\|u_1\|_{2,\Omega_1} \sum_{i=1}^{2} \|E_{h_i}\xi_i\|_{H^{1/2}(\Gamma)}.\end{aligned} \tag{4.7}$$

It remains to estimate the term $\|E_{h_i}\xi\|_{H^{1/2}(\Gamma)}$ $(i = 1, 2)$. Let $\bar{Q}_{h_i}$ be the L^2-orthogonal projection operator from $L^2(\Gamma)$ onto W_{h_i}. By the standard argument (Xu, 1989), we have

$$\|v - \bar{Q}_{h_i}v\|_{L^2(\Gamma)} \lesssim h_i\|v\|_{H^1(\Gamma)}, \quad \forall v \in H^1(\Gamma).$$

and for $s = 0, 1$,

$$\|\bar{Q}_{h_i}v\|_{H^s(\Gamma)} \lesssim \|v\|_{H^s(\Gamma)}, \quad \forall\, v \in H^s(\Gamma)\,, \tag{4.8}$$

which implies that (4.8) holds also for $s = 1/2$ by the Sobolev interpolation theory. This, together with the inverse inequality and the trace inequality, yields

$$\begin{aligned}\|E_{h_i}\xi_i\|_{H^{1/2}(\Gamma)} &\leqslant \|\bar{Q}_{h_i}\xi_i\|_{H^{1/2}(\Gamma)} + \|\bar{Q}_{h_i}\xi_i - E_{h_i}\xi_i\|_{H^{1/2}(\Gamma)} \\ &\lesssim \|\xi_i\|_{H^{1/2}(\Gamma)} + h_i^{-1/2}\|\bar{Q}_{h_i}\xi_i - E_{h_i}\xi_i\|_{L^2(\Gamma)} \\ &\lesssim \|\xi_i\|_{H^{1/2}(\Gamma)} + h_i^{-1/2}(\|\xi_i - \bar{Q}_{h_i}\xi_i\|_{L^2(\Gamma)} + \|\xi_i - E_{h_i}\xi_i\|_{L^2(\Gamma)}) \\ &\lesssim \|\xi_i\|_{H^{1/2}(\Gamma)} + \|\xi_i\|_{1,\Omega_i} \lesssim \|\xi_i\|_{1,\Omega_i}.\end{aligned} \tag{4.9}$$

It follows then from (4.3)–(4.9) that

$$|\mathrm{III}| \lesssim (h_1\|u_1\|_{2,\Omega_1} + h_2\|u_2\|_{2,\Omega_2})\|\xi\|_{1,\Omega},$$

which together with (4.1)–(4.2) leads to the following theorem.

THEOREM 4.1 Let u be the solution to the interface problem (2.1) and u_h be the solution to the mortar finite element system (2.12), then we have the following optimal H^1-norm error estimate:

$$\|u - u_h\|_{1,\Omega} \lesssim h_1\|u_1\|_{2,\Omega_1} + h_2\|u_2\|_{2,\Omega_2}. \tag{4.10}$$

REMARK 4.1 Theorem 4.1 still holds when the interface Γ is piecewise C^2-smooth provided the true solution $u \in X$. In this case the nonsmooth points of Γ should be chosen as the nodal points of the triangulations $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$.

We now proceed to give the L^2-norm error estimate for the mortar finite element method (2.12). Consider the auxiliary problem:

$$\begin{aligned} -\nabla\cdot(\beta\nabla\phi) &= P_h u - u_h \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \\ [\phi] = 0, \quad [\beta\partial_{\mathbf{n}}\phi] &= 0 \quad \text{across } \Gamma. \end{aligned} \tag{4.11}$$

Let $\phi_1 = \phi|_{\Omega_1}$, $\phi_2 = \phi|_{\Omega_2}$, then we have the *a priori* estimates by (1.5)

$$\|\phi_1\|_{2,\Omega_1} + \|\phi_2\|_{2,\Omega_2} \lesssim \|P_h u - u_h\|_{0,\Omega}. \tag{4.12}$$

Moreover, let $\phi_h = (\phi_{h_1}, \phi_{h_2}) \in V_h$ be the mortar finite element solution of ϕ through the system (2.12) with $f = P_h u - u_h$ and $g = 0$. From Theorem 4.1 we have

$$\|\phi - \phi_h\|_{1,\Omega} \lesssim h_1\|\phi_1\|_{2,\Omega_1} + h_2\|\phi_2\|_{2,\Omega_2} \lesssim (h_1 + h_2)\|P_h u - u_h\|_{0,\Omega}. \tag{4.13}$$

By the definition of ϕ_h we see

$$\begin{aligned} \|P_h u - u_h\|_{0,\Omega}^2 &= a(\phi_h, P_h u - u_h) \\ &= a(\phi_h - \phi, P_h u - u_h) + a(\phi, P_h u - u) + a(\phi, u - u_h) \\ &\equiv \mathrm{IV}_1 + \mathrm{IV}_2 + \mathrm{IV}_3. \end{aligned} \tag{4.14}$$

It follows from (4.13), Lemma 3.7 and Theorem 4.1 that

$$\begin{aligned} |\mathrm{IV}_1| &\lesssim \|\phi - \phi_h\|_{1,\Omega}\|P_h u - u_h\|_{1,\Omega} \\ &\lesssim (h_1 + h_2)(h_1\|u_1\|_{2,\Omega_1} + h_2\|u_2\|_{2,\Omega_2})\|P_h u - u_h\|_{0,\Omega}. \end{aligned} \tag{4.15}$$

On the other hand, by integration by parts we know

$$\begin{aligned} \mathrm{IV}_2 &= \sum_{i=1}^{2}\int_{\Omega_i}\beta_i(x)\nabla\phi_i\cdot\nabla(P_h u - u)\,|_{\Omega_i}\ \mathrm{d}x \\ &= \int_{\Omega}(P_h u - u_h)(P_h u - u)\,\mathrm{d}x + \int_{\Gamma}[P_h u]\beta_1(x)\partial_{\mathbf{n}}\phi\,\mathrm{d}s. \end{aligned} \tag{4.16}$$

Then by Lemmata 3.9–3.10, the trace theorem and (4.12) we obtain

$$\begin{aligned} |\mathrm{IV}_2| &\lesssim \|P_h u - u_h\|_{0,\Omega}\|u - P_h u\|_{0,\Omega} + \|[P_h u]\|_{H^{-1/2}(\Gamma)}\|\beta_1(x)\partial_{\mathbf{n}}\phi\|_{H^{1/2}(\Gamma)} \\ &\lesssim (h_1^2\|u_1\|_{2,\Omega_1} + (h_1h_2 + h_2^2)\|u_2\|_{2,\Omega_2})\|P_h u - u_h\|_{0,\Omega}. \end{aligned} \tag{4.17}$$

Moreover, by integration by parts (see the deduction of III given above),

$$\begin{aligned} \mathrm{IV}_3 &= a(\phi - P_h\phi, u - u_h) + a(P_h\phi, u - u_h) \\ &= a(\phi - P_h\phi, u - u_h) + a(P_h\phi, u) - (f, P_h\phi) - \langle g, P_h\phi\,|_{\Omega_2}\rangle \\ &= a(\phi - P_h\phi, u - u_h) + \int_{\Gamma}\beta_1(x)\partial_{\mathbf{n}}u_1\,[P_h\phi]\,\mathrm{d}s. \end{aligned}$$

Using Lemma 3.4, Lemma 3.10 and Theorem 4.1 we have (noting $h_1 \geqslant h_2$)

$$
\begin{aligned}
|\mathrm{IV}_3| &\lesssim \|u - u_h\|_{1,\Omega}\|\phi - P_h\phi\|_{1,\Omega} + \|[P_h\phi]\|_{H^{-1/2}(\Gamma)}\|\beta_1(x)\partial_{\mathbf{n}}u_1\|_{H^{1/2}(\Gamma)} \\
&\lesssim \{(h_1 + h_2)(h_1\|u_1\|_{2,\Omega_1} + h_2\|u_2\|_{2,\Omega_2}) + (h_1^2 + h_2^2)\|u_1\|_{2,\Omega_1})\}\|P_h u - u_h\|_{0,\Omega} \\
&\lesssim (h_1^2\|u_1\|_{2,\Omega_1} + (h_1h_2 + h_2^2)\|u_2\|_{2,\Omega_2})\|P_h u - u_h\|_{0,\Omega}. \qquad (4.18)
\end{aligned}
$$

From (4.14)–(4.18) we find

$$\|P_h u - u_h\|_{0,\Omega} \lesssim h_1^2\|u_1\|_{2,\Omega_1} + (h_1h_2 + h_2^2)\|u_2\|_{2,\Omega_2},$$

which together with Lemma 3.9 and the triangle inequality

$$\|u - u_h\|_{0,\Omega} \leqslant \|u - P_h u\|_{0,\Omega} + \|P_h u - u_h\|_{0,\Omega}$$

leads to the following theorem.

THEOREM 4.2 Let u be the solution to the interface problem (2.1) and u_h be the solution to the mortar finite element system (2.12), then we have the following L^2-norm error estimate:

$$\|u - u_h\|_{0,\Omega} \lesssim h_1^2\|u_1\|_{2,\Omega_1} + (h_1h_2 + h_2^2)\|u_2\|_{2,\Omega_2}. \qquad (4.19)$$

REMARK 4.2 The cross term $O(h_1h_2)$ in (4.19) is common to the error estimates for all existing mortar finite element methods for elliptic problems even with smooth coefficients, see, for example, Belgacem (1999); Bernardi *et al.* (1990a).

5. Effect of the numerical integration

So far all our convergence analyses have been carried out under the assumption that the integrals involved in the the mortar finite element method (2.12), namely $a_h(u_h, v_h)$, (f, v) and $\langle g, v_{h_2}\rangle$, were computed exactly. This may cause some technical difficulties in practice for the evaluation of the integrals over those curved elements near the interface Γ. It would make the numerical implementation much easier if we can replace these integrals over the curved elements by the integrals over the corresponding straight elements. This section aims to show that this replacement will not affect the convergence order of the mortar element method (2.12).

To do so, we first replace the original bilinear form $a(u_h, v_h)$ by the following approximate one:

$$a_h(u_h, v_h) = a_{h_1}(u_{h_1}, v_{h_1}) + a_{h_2}(u_{h_2}, v_{h_2}) \qquad (5.1)$$

with

$$a_{h_i}(u_{h_i}, v_{h_i}) = \sum_{K\in\tilde{T}_{h_i}} \mathrm{meas}(K)\beta_i(b_K)\nabla u_{h_i} \cdot \nabla v_{h_i}, \quad i = 1, 2$$

where b_K denotes the barycentre of K, and we have used the conventional quadrature scheme which is exact for polynomials of degree $\leqslant 1$ (Ciarlet, 1978). To treat the interface integral $\langle g, v_{h_2}\rangle$ for $g \in C^0(\Gamma)$, we define $\tilde{g}_{h_2}$ to be the continuous and piecewise linear function defined on the triangulation $\tilde{\Gamma}_{h_2} = \{\tilde{e}_j^2\}_{j=1}^{m_2}$ of Γ such that $g_{h_2}(P_j^2) = g(P_j^2)$, $j = 1, 2, \ldots, m_2$. For $v_{h_2} \in V_{h_2}$, we let $\tilde{v}_{h_2}$ be the linear interpolation of v_{h_2} on the triangulation $\tilde{T}_{h_2}$. Then the mortar finite element method with numerical integration for solving (2.1) is: Find $u_h^* \in V_h$ such that

$$a_h(u_h^*, v_h) = \sum_{i=1}^{2} \int_{\Omega_i} f v_{h_i}\, \mathrm{d}x + \int_{\tilde{\Gamma}_{h_2}} \tilde{g}_{h_2} \tilde{v}_{h_2}\, \mathrm{d}s, \quad \forall v_h = (v_{h_1}, v_{h_2}) \in V_h. \tag{5.2}$$

Recall that $\tilde{T}_{h_i}$ $(i = 1, 2)$ are the triangulations with straight triangular elements (no curved elements included), and $\tilde{\Gamma}_{h_2}$ is the triangulation of Γ with piecewise line segments (no curved segments included). So the major calculations in (5.2) (except for the term involving f) are carried out either on the straight triangular elements or on the line segments. Here, for simplicity, we do not consider the numerical integration of the term involving f in (5.1); this can be done in a same manner as we treat the bilinear form $a(u_h, v_h)$ and the integral $\langle g, v_{h_2} \rangle$.

Let u be the weak solution to the interface problem (2.1) and u_h^* be the finite element solution to (5.2). The rest of this section establishes the H^1-norm and L^2-norm error estimates of $u - u_h^*$.

Consider an element $K \in \mathcal{T}_{h_i}$. If K is a straight triangle, by the standard scaling argument (see Ciarlet (1978)) we have

$$\left| \int_K \beta_i(x)\, \mathrm{d}x - \operatorname{meas}(K)\beta_i(b_K) \right| \lesssim h_i^4 \|\beta_i\|_{2,\infty,K}; \tag{5.3}$$

if K is a curved triangle, let $K' \in \tilde{T}_{h_i}$ be the straight triangle with the same vertices as K, and we have

$$\left| \int_K \beta_i(x)\, \mathrm{d}x - \operatorname{meas}(K')\beta_i(b_{K'}) \right| \lesssim h_i^3 \|\beta_i\|_{2,\infty,K}. \tag{5.4}$$

Using (5.3)–(5.4), we immediately obtain for any $v_h \in V_h$ that

$$|a(v_h, v_h) - a_h(v_h, v_h)| \lesssim (h_1 + h_2) a(v_h, v_h),$$

which implies

$$a_h(v_h, v_h) \eqsim a(v_h, v_h). \tag{5.5}$$

Now we choose $\psi_h = (\psi_{h_1}, \psi_{h_2}) = P_h u - u_h^*$. Then from (5.5) we have

$$
\begin{aligned}
\|\psi_h\|_{1,\Omega}^2 &\lesssim a_h(P_h u - u_h^*, \psi_h) \\
&= a_h(P_h u, \psi_h) - \int_\Omega f \psi_h \, \mathrm{d}x - \int_{\tilde{\Gamma}_{h_2}} \tilde{g}_{h_2} \tilde{\psi}_{h_2} \, \mathrm{d}s \\
&= \{a_h(P_h u, \psi_h) - a(P_h u, \psi_h)\} + a(P_h u - u, \psi_h) \\
&\quad + \left\{ a(u, \psi_h) - \int_\Omega f \psi_h \, \mathrm{d}x - \int_\Gamma g \psi_{h_2} \, \mathrm{d}s \right\} \\
&\quad + \left\{ \int_\Gamma g \psi_{h_2} \, \mathrm{d}s - \int_{\tilde{\Gamma}_{h_2}} \tilde{g}_{h_2} \tilde{\psi}_{h_2} \, \mathrm{d}s \right\} \\
&\equiv \mathrm{V}_1 + \mathrm{V}_2 + \mathrm{V}_3 + \mathrm{V}_4.
\end{aligned} \tag{5.6}
$$

It follows directly from (5.3)–(5.4) and Lemma 3.7 that

$$
\begin{aligned}
|\mathrm{V}_1| &\lesssim (h_1^2 |P_h u|_{1,\Omega_1}^2 + h_2^2 |P_h u|_{2,\Omega_2}^2)^{1/2} |\psi_h|_{1,\Omega} \\
&\lesssim (h_1 \|u_1\|_{2,\Omega_1} + h_2 \|u_2\|_{2,\Omega_2}) |\psi_h|_{1,\Omega}
\end{aligned} \tag{5.7}
$$

and

$$
|\mathrm{V}_2| \lesssim |u - P_h u|_{1,\Omega} |\psi_h|_{1,\Omega} \lesssim (h_1 \|u_1\|_{2,\Omega_1} + h_2 \|u_2\|_{2,\Omega_2}) |\psi_h|_{1,\Omega}. \tag{5.8}
$$

Repeating the same process as for deriving the estimate of III in Section 4 we obtain

$$
|\mathrm{V}_3| \lesssim (h_1 \|u_1\|_{2,\Omega_1} + h_2 \|u_2\|_{2,\Omega_2}) |\psi_h|_{1,\Omega}. \tag{5.9}
$$

Moreover, following the same proof as for deriving Lemma 2.2 in Chen & Zou (1998) we have

$$
|\mathrm{V}_4| \lesssim h_2^{3/2} \|g\|_{H^2(\Gamma)} \|\psi_{h_2}\|_{1,\tilde{\Omega}_2}, \tag{5.10}
$$

where $\tilde{\Omega}_2$ is the union of all curved elements $K \in \mathcal{T}_{h_2}$ with $\bar{K} \cap \Gamma \neq \emptyset$. Now it follows from (5.6)–(5.10) that

$$
\|\psi_h\|_{1,\Omega} \lesssim h_1 \|u_1\|_{2,\Omega_1} + h_2 \|u_2\|_{2,\Omega_2} + h_2^{3/2} \|g\|_{H^2(\Gamma)},
$$

which together with Lemma 3.7 leads to

$$
\begin{aligned}
\|u - u_h^*\|_{1,\Omega} &\lesssim \|u - P_h u\|_{1,\Omega} + \|\psi_h\|_{1,\Omega} \\
&\lesssim h_1 \|u_1\|_{2,\Omega_1} + h_2 \|u_2\|_{2,\Omega_2} + h_2^{3/2} \|g\|_{H^2(\Gamma)},
\end{aligned} \tag{5.11}
$$

thus we have proved the following theorem.

THEOREM 5.1 Let u be the solution to the interface problem (2.1) and u_h^* be the solution to the mortar finite element system (5.2). Then if $g \in H^2(\Gamma)$, the following optimal H^1-norm error estimate holds:

$$
\|u - u_h^*\|_{1,\Omega} \lesssim h_1 \|u_1\|_{2,\Omega_1} + h_2 \|u_2\|_{2,\Omega_2} + h_2^{3/2} \|g\|_{H^2(\Gamma)}.
$$

Next, we use the duality argument to establish the L^2-norm error estimate for the mortar element method (5.2). To do so, we first introduce an auxiliary problem:

$$\begin{aligned} -\nabla\cdot(\beta\nabla\phi) &= \psi_h \quad \text{in } \Omega, \\ \phi &= 0 \quad \text{on } \partial\Omega, \\ [\phi] = 0, \quad [\beta\partial_{\mathbf{n}}\phi] &= 0 \quad \text{across } \Gamma, \end{aligned} \tag{5.12}$$

where $\psi_h = P_h u - u_h^*$. Denote by $\phi_h = (\phi_{h_1}, \phi_{h_2}) \in V_h$ the mortar finite element solution of ϕ through (5.2) with $f = \psi_h$ and $g_{h_2} = 0$. We then have

$$\begin{aligned} \|\psi_h\|_{0,\Omega}^2 &= a_h(\phi_h, P_h u - u_h^*) \\ &= a_h(\phi_h, P_h u) - \int_\Omega f\phi_h\,\mathrm{d}x - \int_{\tilde{\Gamma}_{h_2}} \tilde{g}_{h_2}\tilde{\phi}_{h_2}\,\mathrm{d}s \\ &= \{a_h(\phi_h, P_h u) - a(\phi_h, P_h u)\} \\ &\quad + \{a(\phi_h - \phi, P_h u - u_h)\} + a(\phi, P_h u - u) + a(u - u_h, \phi) \\ &\quad + \left\{\int_\Gamma g\phi_{h_2}\,\mathrm{d}s - \int_{\tilde{\Gamma}_{h_2}} \tilde{g}_{h_2}\tilde{\phi}_{h_2}\,\mathrm{d}s\right\} \\ &\equiv \mathrm{VI}_1 + \mathrm{VI}_2 + \mathrm{VI}_3 + \mathrm{VI}_4 + \mathrm{VI}_5. \end{aligned} \tag{5.13}$$

Using (5.3)–(5.4), Lemma 3.7 and Theorem 5.1, and a careful analysis we can derive

$$\begin{aligned} |\mathrm{VI}_1| &\lesssim \sum_{i=1}^{2}\left\{h_i\int_{\tilde{\Omega}_i}\beta_i(x)|\nabla\phi_h||\nabla P_h u|\,\mathrm{d}x + h_i^2\int_{\Omega_i\setminus\tilde{\Omega}_i}\beta_i(x)|\nabla\phi_h||\nabla P_h u|\,\mathrm{d}x\right\} \\ &\lesssim \sum_{i=1}^{2}\{h_i\|\phi_h\|_{1,\tilde{\Omega}_i}\|P_h u\|_{1,\tilde{\Omega}_i} + h_i^2\|\phi_h\|_{1,\Omega_i}\|P_h u\|_{1,\Omega_i}\} \\ &\lesssim (h_1^2\|u_1\|_{2,\Omega_1} + (h_2^2 + h_1h_2)\|u_2\|_{2,\Omega_2})\|\psi_h\|_{0,\Omega} \end{aligned} \tag{5.14}$$

where $\tilde{\Omega}_i$ $(i = 1, 2)$ is the union of all the curved elements $K \in \mathcal{T}_{h_i}$. We can easily obtain the estimate for VI_2 as follows:

$$\begin{aligned} |\mathrm{VI}_2| &\lesssim \|\phi_h - \phi\|_{1,\Omega}\|P_h u - u_h\|_{1,\Omega} \\ &\lesssim (h_1^2\|u_1\|_{2,\Omega_1} + (h_2^2 + h_1h_2)\|u_2\|_{2,\Omega_2} + h_1h_2^{3/2}\|g\|_{H^2(\Gamma)})\|\psi_h\|_{0,\Omega}. \end{aligned} \tag{5.15}$$

In the same manner as for estimating IV_2 in Section 4 we obtain

$$|\mathrm{VI}_3| \lesssim (h_1^2\|u_1\|_{2,\Omega_1} + (h_2^2 + h_1h_2)\|u_2\|_{2,\Omega_2})\|\psi_h\|_{0,\Omega}, \tag{5.16}$$

while in the same manner as for estimating IV_3 in Section 4 we obtain

$$|\mathrm{VI}_4| \lesssim (h_1^2\|u_1\|_{2,\Omega_1} + (h_2^2 + h_1h_2)\|u_2\|_{2,\Omega_2} + h_1h_2^{3/2}\|g\|_{H^2(\Gamma)})\|\psi_h\|_{0,\Omega}. \tag{5.17}$$

Using (5.10) we know

$$|\mathrm{VI}_5| \lesssim h_2^{3/2}\|g\|_{H^2(\Gamma)}\|\phi_{h_2}\|_{1,\tilde{\Omega}_2},$$

where

$$\|\phi_{h_2}\|_{1,\tilde{\Omega}_2} \leqslant \|\phi - \phi_{h_2}\|_{1,\tilde{\Omega}_2} + \|\phi\|_{1,\tilde{\Omega}_2} \lesssim (h_1 + h_2 + h_2^{1/2})\|\psi_h\|_{0,\Omega},$$

hence

$$|\mathrm{VI}_5| \lesssim (h_1 h_2^{3/2} + h_2^2)\|g\|_{H^2(\Gamma)}\|\psi_h\|_{0,\Omega}. \tag{5.18}$$

Now it follows from (5.13)–(5.18) that

$$\|\psi_h\|_{0,\Omega} \lesssim h_1^2\|u_1\|_{2,\Omega_1} + (h_2^2 + h_1 h_2)\|u_2\|_{2,\Omega_2} + h_1 h_2^{3/2}\|g\|_{H^2(\Gamma)},$$

which together with the triangle inequality

$$\|u - u_h^*\|_{0,\Omega} \lesssim \|u - P_h u\|_{0,\Omega} + \|\psi_h\|_{0,\Omega}$$

and Lemma 3.9 yields the following theorem.

THEOREM 5.2 Let u be the solution to the interface problem (2.1) and u_h^* be the solution to the mortar finite element system (5.2). Then if $g \in H^2(\Gamma)$, the following L^2-norm error estimate holds:

$$\|u - u_h^*\|_{0,\Omega} \lesssim h_1^2\|u_1\|_{2,\Omega_1} + (h_2^2 + h_1 h_2)\|u_2\|_{2,\Omega_2} + h_1 h_2^{3/2}\|g\|_{H^2(\Gamma)}.$$

REMARK 5.1 With a more detailed analysis, the regularity requirement on the interface function g in Theorems 5.1 and 5.2 can be made much weaker (Chen & Zou, 1998).

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Appendix

Proof of Lemma 3.5. The first result of (3.18) is a generalization of the conventional extension theorems for finite element spaces to the current domain with a curved boundary (Xu & Zou, 1998). Note that α_{h_1} can be viewed as the interpolation of $(F_{h_1}\alpha_{h_1})|_\Gamma$ associated with the space W_{h_1}, thus we have

$$\|F_{h_1}\alpha_{h_1} - \alpha_{h_1}\|_{H^{1/2}(\Gamma)} \lesssim h_1^{1/2}|F_{h_1}\alpha_{h_1}|_{H^1(\Gamma)} = h_1^{1/2}\left\{\sum_{j=1}^{m_1}|F_{h_1}\alpha_{h_1}|^2_{H^1(e_j^1)}\right\}^{1/2}. \quad \text{(A.1)}$$

Let $K_j \in \mathcal{T}_{h_1}$ be a curved element with e_j^1 being one of its edges. Then it follows from the inverse inequality that

$$|F_{h_1}\alpha_{h_1}|_{H^1(e_j^1)} \lesssim h_1^{1/2}|F_{h_1}\alpha_{h_1}|_{1,\infty,K_j} \lesssim h_1^{-1/2}|F_{h_1}\alpha_{h_1}|_{1,K_j},$$

which together with (A.1) yields

$$\|F_{h_1}\alpha_{h_1} - \alpha_{h_1}\|_{H^{1/2}(\Gamma)} \lesssim \|F_{h_1}\alpha_{h_1}\|_{1,\Omega_1}.$$

Thus by the trace theorem we immediately have

$$\begin{aligned}\|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} &\lesssim \|F_{h_1}\alpha_{h_1} - \alpha_{h_1}\|_{H^{1/2}(\Gamma)} + \|F_{h_1}\alpha_{h_1}\|_{H^{1/2}(\Gamma)}\\ &\lesssim \|F_{h_1}\alpha_{h_1}\|_{1,\Omega_1}.\end{aligned} \quad \text{(A.2)}$$

On the other hand, $F_{h_1}\alpha_{h_1}$ can be viewed as the finite element approximation of the solution ϕ to the elliptic problem (3.1) with g_1 replaced by α_{h_1}. Note that $\alpha_{h_1} \in W_{h_1}$, and so $\alpha_{h_1} \in H^{1+\epsilon}(\Gamma)$ for any $\epsilon \in (0, 1/2)$ (see Xu (1989)). So the solution ϕ has the regularity $\phi \in H^{3/2+\epsilon}(\Omega_1)$ and meets the estimate (3.3). Following the derivation of (5.5) in Scott (1975), we have

$$\|\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} \lesssim \|\phi - I_{h_1}\phi\|_{1,\Omega_1} + \left(\sup_{v_1\in V_{h_1}^0}\frac{\|v_1\|_{H^{1/2-\epsilon}(\Gamma)}}{\|v_1\|_{1,\Omega_1}}\right)\|\phi\|_{3/2+\epsilon,\Omega_1}, \quad \text{(A.3)}$$

while using Lemma 1 of Scott (1975) with $k = 2$, we have

$$\sup_{v_1\in V_{h_1}^0}\frac{\|v_1\|_{H^{1/2-\epsilon}(\Gamma)}}{\|v_1\|_{1,\Omega_1}} \lesssim h_1^{1+\epsilon},$$

which together with (3.3) and (A.3), Lemma 3.2 and the inverse inequality yields

$$\begin{aligned}\|\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} &\lesssim h_1^{1/2+\epsilon}\|\phi\|_{3/2+\epsilon,\Omega_1} + h_1^{1+\epsilon}\|\phi\|_{3/2+\epsilon,\Omega_1}\\ &\lesssim h_1^{1/2+\epsilon}\|\alpha_{h_1}\|_{H^{1+\epsilon}(\Gamma)} \lesssim \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)}.\end{aligned} \quad \text{(A.4)}$$

Then, by Lemma 3.1 we have

$$\begin{aligned}\|F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} &\lesssim \|\phi\|_{1,\Omega_1} + \|\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1}\\ &\lesssim \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)},\end{aligned} \quad \text{(A.5)}$$

with which, and (A.2), we have proved the first relation in (3.18).

We now use the duality argument to show the second relation in (3.18). For any $\psi \in L^2(\Omega_1)$, introduce an auxiliary problem:

$$-\Delta z + z = \psi \quad \text{in } \Omega_1; \quad z = 0 \quad \text{on } \Gamma. \tag{A.6}$$

By integration by parts and the fact that $F_{h_1}\alpha_{h_1}$ satisfies (3.17) we have

$$\begin{aligned}(\phi - F_{h_1}\alpha_{h_1}, \psi)_{0,\Omega_1} &= (\phi - F_{h_1}\alpha_{h_1}, -\Delta z + z)_{0,\Omega_1} \\ &= (\phi - F_{h_1}\alpha_{h_1}, z - I_{h_1}z)_{1,\Omega_1} + \left\{ \int_\Gamma \partial_{\mathbf{n}}\phi I_{h_1} z \,\mathrm{d}x \right\} \\ &\quad + \left\{ - \int_\Gamma (\phi - F_{h_1}\alpha_{h_1})\partial_{\mathbf{n}} z \,\mathrm{d}s \right\} \\ &\equiv \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned} \tag{A.7}$$

It follows from (A.4) and Lemma 3.2 that

$$|\mathrm{I}_1| \leqslant \|\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} \|z - I_{h_1}z\|_{1,\Omega_1} \lesssim h_1 \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} \|z\|_{2,\Omega_1}, \tag{A.8}$$

and we obtain from Lemma 3.3, the trace theorem and (3.3) that

$$\begin{aligned}|\mathrm{I}_2| &\lesssim \|I_{h_1}z\|_{L^2(\Gamma)} \|\partial_{\mathbf{n}}\phi\|_{L^2(\Gamma)} \\ &\lesssim h_1^2 \|z\|_{2,\Omega_1} \|\phi\|_{3/2,\Omega_1} \lesssim h_1^2 \|z\|_{2,\Omega_1} \|\alpha_{h_1}\|_{H^1(\Gamma)}. \end{aligned} \tag{A.9}$$

For I_3, we have

$$|\mathrm{I}_3| \leqslant \|\phi - F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} \|\partial_{\mathbf{n}} z\|_{H^{1/2}(\Gamma)} \lesssim \|\phi - F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} \|z\|_{2,\Omega_1}. \tag{A.10}$$

By the triangle inequality,

$$\|\phi - F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} \leqslant \|\phi - I_{h_1}\phi\|_{H^{-1/2}(\Gamma)} + \|I_{h_1}\phi - F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)}. \tag{A.11}$$

Note that $I_{h_1}\phi - F_{h_1}\alpha_{h_1} \in V_{h_1}^0$. Again using Lemma 1 of Scott (1975) with $k = 2$ we have

$$\begin{aligned}\|I_{h_1}\phi - F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} &\lesssim \|I_{h_1}\phi - F_{h_1}\alpha_{h_1}\|_{L^2(\Gamma)} \\ &\lesssim h_1^{3/2} \|I_{h_1}\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1}. \end{aligned} \tag{A.12}$$

From (3.3), (A.4) and the inverse inequality we know

$$\begin{aligned}\|I_{h_1}\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} &\leqslant \|\phi - I_{h_1}\phi\|_{1,\Omega_1} + \|\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} \\ &\lesssim h_1^{1/2+\epsilon} \|\phi\|_{3/2+\epsilon,\Omega_1} + \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} \lesssim \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)}. \end{aligned} \tag{A.13}$$

It remains to estimate the term $\|\phi - I_{h_1}\phi\|_{H^{-1/2}(\Gamma)}$ in (A.11). Consider a general curved segment $e_j^1 \in \Gamma_{h_1}$. For any $\xi_j \in H^{1/2}(e_j^1)$, using the local coordinates we have

$$\begin{aligned}\int_{e_j^1}(\phi - I_{h_1}\phi)\xi_j(s)\,\mathrm{d}s &= \int_0^{s_j^{h_1}} \{\phi(x_1^j, \phi_j^1(x_1^j)) - (I_{h_1}\phi)(x_1^j, \phi_j^1(x_1^j))\}\xi_j(\sigma_j^1(x_1^j))\dot{\sigma}_j^1(x_1^j)\,\mathrm{d}x_1^j \\ &= \int_0^{s_j^{h_1}} \{(I_{h_1}\phi)(x_1^j, 0) - (I_{h_1}\phi)(x_1^j, \phi_j^1(x_1^j))\}\xi_j(\sigma_j^1(x_1^j))\dot{\sigma}_j^1(x_1^j)\,\mathrm{d}x_1^j \\ &\quad + \int_0^{s_j^{h_1}} \{\phi(x_1^j, \phi_j^1(x_1^j)) - (I_{h_1}\phi)(x_1^j, 0)\}\xi_j(\sigma_j^1(x_1^j))\dot{\sigma}_j^1(x_1^j)\,\mathrm{d}x_1^j \\ &\equiv \mathrm{II}_1 + \mathrm{II}_2 \end{aligned} \tag{A.14}$$

where $\sigma_j^1(x_1^j) = \int_0^{x_1^j} \sqrt{1 + (\dot{\phi}_j^1(x_1^j))^2}\,\mathrm{d}x_1^j$. By (2.3)–(2.4) and the inverse inequality we obtain

$$\begin{aligned}|\mathrm{II}_1| &\lesssim h_1^2 |I_{h_1}\phi|_{1,\infty,K_j^1} \int_0^{s_j^{h_1}} |\xi_j(\sigma_j^1(x_1^j))|\dot{\sigma}_j^1(x_1^j)\,\mathrm{d}x_1^j \\ &\lesssim h_1^{3/2}|I_{h_1}\phi|_{1,K_j^1}\|\xi_j\|_{0,e_j^1},\end{aligned} \tag{A.15}$$

where K_j^1 is the curved element with e_j^1 as one of its edges.

On the other hand, noting that $\phi|_\Gamma = \alpha_{h_1} \in W_{h_1}$, we see

$$\phi(s)|_{s=\sigma_j^1(s_j^{h_1})/2} = \tfrac{1}{2}(\phi(P_j^1) + \phi(P_{j+1}^1)) = (I_{h_1}\phi)(s_j^{h_1}/2, 0),$$

which yields

$$\mathrm{II}_2 = \int_{e_j^1}\{\phi(s) - \phi(t_j)\}\xi_j(s)\,\mathrm{d}s - \int_0^{s_j^{h_1}} \{(I_{h_1}\phi)(x_1^j, 0) - (I_{h_1}\phi)(s_j^{h_1}/2, 0)\}\xi_j(\sigma_j^1(x_1^j)) \\ \dot{\sigma}_j^1(x_1^j)\,\mathrm{d}x_1^j$$

where $t_j = \sigma_j^1(s_j^{h_1})/2$. Moreover, we easily see that

$$\phi(t_j) = \frac{1}{|e_j^1|}\int_{e_j^1}\phi(s)\,\mathrm{d}s, \quad (I_{h_1}\phi)(s_j^{h_1}/2, 0) = \frac{1}{|\tilde{e}_j^1|}\int_{\tilde{e}_j^1}(I_{h_1}\phi)(x_1^j, 0)\,\mathrm{d}x_1^j,$$

and thus we have the following standard estimates (Brenner & Scott, 1994):

$$\|\phi(s) - \phi(t_j)\|_{H^{-1/2}(e_j^1)} \lesssim h_1|\phi|_{H^{1/2}(e_j^1)}$$

and

$$\|(I_{h_1}\phi)(x_1^j, 0) - (I_{h_1}\phi)(s_j^{h_1}/2, 0)\|_{H^{-1/2}(0,s_j^{h_1})} \lesssim h_1|(I_{h_1}\phi)(x_1^j, 0)|_{H^{1/2}(0,s_j^{h_1})}.$$

With these estimates, we obtain

$$
\begin{aligned}
|\mathrm{II}_2| &\lesssim h_1(|\phi|_{H^{1/2}(e_j^1)} + |(I_{h_1}\phi)(x_1^j, 0)|_{H^{1/2}(0,s_j^{h_1})})\|\xi_j\|_{H^{1/2}(e_j^1)} \\
&\lesssim h_1(|\phi|_{1,K_j^1} + |I_{h_1}\phi|_{1,K_j^1})\|\xi_j\|_{H^{1/2}(e_j^1)}. \qquad \text{(A.16)}
\end{aligned}
$$

Summing both sides of (A.14) over $e_j^1 \in \Gamma_{h_1}$, using (A.15)–(A.16), Lemma 3.2 and the definition of the norm in $H^{-1/2}(\Gamma)$, we have

$$
\begin{aligned}
\|\phi - I_{h_1}\phi\|^2_{H^{-1/2}(\Gamma)} &\lesssim h_1^3 \sum_{j=1}^{m_1} |I_{h_1}\phi|^2_{1,K_j^1} + h_1^2 \sum_{j=1}^{m_1} \{|\phi|^2_{1,K_j^1} + |I_{h_1}\phi|^2_{1,K_j^1}\} \\
&\lesssim h_1^2(|\phi|^2_{1,\Omega_1} + |\phi - I_{h_1}\phi|^2_{1,\Omega_1}) \\
&\lesssim h_1^2(\|\alpha_{h_1}\|^2_{H^{1/2}(\Gamma)} + h_1^{1+2\epsilon}\|\phi\|^2_{H^{3/2+\epsilon}(\Omega_1)}) \qquad \text{(A.17)} \\
&\lesssim \|\alpha_{h_1}\|^2_{H^{-1/2}(\Gamma)}. \qquad \text{(A.18)}
\end{aligned}
$$

Summarizing (A.7)–(A.13) and (A.18) we finally come to

$$
|(\phi - F_{h_1}\phi, \psi)_{0,\Omega_1}| \lesssim (h_1\|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} + h_1^2\|\alpha_{h_1}\|_{H^{1+\epsilon}(\Gamma)} + h_1^{3/2}\|\alpha_{h_1}\|_{H^{1/2}(\Gamma)})\|\psi\|_{0,\Omega_1},
$$

which, together with Lemma 3.1 and the inverse inequalities, implies

$$
\|F_{h_1}\alpha_{h_1}\|_{0,\Omega_1} \leqslant \|\phi - F_{h_1}\alpha_{h_1}\|_{0,\Omega_1} + \|\phi\|_{0,\Omega_1} \lesssim \|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)}.
$$

This completes the proof of the second relation of (3.18). □

Proof of Lemma 3.10. By the definition of (3.20) and the fact that $v_1 = v_2$ on Γ we have

$$
[P_h v] = P_{h_1}v_1 - v_1 + v_2 - P_{h_2}v_2 + F_{h_1}\alpha_{h_1}
$$

where $\alpha_{h_1} = G_{h_1}(E_{h_2}P_{h_2}v_2 - E_{h_1}P_{h_1}v_1)$. From Lemma 3.4 we have

$$
\begin{aligned}
\|[P_h v]\|_{H^{-1/2}(\Gamma)} &\leqslant \sum_{i=1}^{2} \|v_i - P_{h_i}v_i\|_{H^{-1/2}(\Gamma)} + \|F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} \\
&\lesssim \sum_{i=1}^{2} h_i^2\|v_i\|_{2,\Omega_i} + \|F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)}. \qquad \text{(A.19)}
\end{aligned}
$$

Noting that $\alpha_{h_1} = E_{h_1}(F_{h_1}\alpha_{h_1})$, using the techniques employed in deriving (3.36)–(3.37), we obtain

$$
\begin{aligned}
\|F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} &\leqslant \|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} + \|\alpha_{h_1} - F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} \\
&\lesssim \|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} + h_1^{3/2}\|F_{h_1}\alpha_{h_1}\|_{1,\tilde{\Omega}_1}. \qquad \text{(A.20)}
\end{aligned}
$$

Let $\phi \in H^{3/2+\varepsilon}(\Omega_1)$ be the solution of the auxiliary problem (3.1) with g_1 replaced by α_{h_1}. Then using (A.4) and the inverse inequality yields

$$
\|\phi - F_{h_1}\alpha_{h_1}\|_{1,\tilde{\Omega}_1} \leqslant \|\phi - F_{h_1}\alpha_{h_1}\|_{1,\Omega_1} \lesssim \|\alpha_{h_1}\|_{H^{1/2}(\Gamma)} \lesssim h_1^{-1}\|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)}. \qquad \text{(A.21)}
$$

On the other hand, using (3.13) and the interpolation theory of Sobolev spaces (Bergh & Löfstrom, 1976), the regularity estimate (3.3) and the inverse inequality we know

$$\|\phi\|_{1,\tilde{\Omega}_1} \lesssim h_1^{1/4}\|\phi\|_{3/2,\Omega_1} \lesssim h_1^{1/4}\|\alpha_{h_1}\|_{H^1(\Gamma)} \lesssim h_1^{-5/4}\|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)},$$

which together with (A.20)–(A.21) leads to

$$\|F_{h_1}\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} \lesssim \|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)}. \tag{A.22}$$

Following the proof of Lemma 3.9 we have

$$\|\alpha_{h_1}\|_{H^{-1/2}(\Gamma)} \lesssim h_1^2\|v_1\|_{2,\Omega_1} + (h_1 h_2 + h_2^2)\|v_2\|_{2,\Omega_2}$$

which combining with (A.19) gives the desired result. □