L^1 STABILITY FOR THE VLASOV-POISSON-BOLTZMANN SYSTEM AROUND VACUUM

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Based on the global existence theory of the Vlasov-Poisson-Boltzmann system around vacuum in the N-dimensional phase space, in this paper, we prove the uniform L^1 stability of classical solutions for small initial data when $N \geq 4$. In particular, we show that the stability can be established directly for the soft potentials, while for the hard potentials and hard sphere model it is obtained through the construction of some nonlinear functionals. These functionals thus generalize those constructed by Ha for the case without force to capture the effect of the force term on the time evolution of solutions. In addition, the local-in-time L^1 stability is also obtained for the case of $N = 3$.

Keywords: Vlasov-Poisson-Boltzmann system; stability; nonlinear functionals.

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1. Introduction

In this paper, we consider the Cauchy problem of the Vlasov-Poisson-Boltzmann system (VPB in short):

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E_f \cdot \nabla_v f = J(f, f), \\ E_f = \nabla_x \phi_f, \quad \triangle_x \phi_f = \rho_f = \displaystyle\int f dv, \end{cases} \tag{1.1}$$

with initial data

$$f(0, x, v) = f_0(x, v), \tag{1.2}$$

where $f(t,x,v)$ is the distribution function for particles at time $t \geq 0$ with location $x \in \mathbf{R}_x^N$ and velocity $v \in \mathbf{R}_v^N$. The positive integer $N \geq 3$ denotes the dimension of the phase space. The self-consistent electric potential $\phi_f(t,x)$ generating the force field $E_f(t,x)$ is coupled with the distribution function $f(t,x,v)$ through the Poisson equation $\triangle_x \phi_f = \rho_f$. The Boltzmann collision operator $J(f,f)$ describing the binary elastic collision takes the form

$$J(f,f) = Q(f,f) - fR(f), \tag{1.3}$$

with

$$Q(f,f)(t,x,v) = \int_{\mathbf{R}^N \times S^{N-1}} B(\theta, |v-v_*|) f(t,x,v') f(t,x,v'_*)\, dv_* d\omega, \tag{1.4}$$

and

$$fR(f)(t,x,v) = f(t,x,v) \int_{\mathbf{R}^N \times S^{N-1}} B(\theta, |v-v_*|) f(t,x,v_*)\, dv_* d\omega. \tag{1.5}$$

Here $\cos\theta = (v-v_*)\cdot\omega/|v-v_*|$, $\omega \in S^{N-1}$. (v,v_*) and (v',v'_*) are the pre-collision and the post-collision velocity pairs respectively, satisfying

$$v' = v - [(v-v_*)\cdot w]w, \;\; v'_* = v_* + [(v-v_*)\cdot w]w,$$

by the conservation of momentum and energy. $B(\theta, |v-v_*|)$ is the collision kernel characterizing the collision of the charged dilute particles from different physical settings with various interaction potentials.

In the mean field approximation[15], when particles interact only through electromagnetic forces, the density f solves the classical Vlasov-Maxwell-Boltzmann system. For this system, E_f is proportional to the Lorentz force $E + v \times B$ created by the mean electromagnetic field, where E and B are respectively the electric and magnetic fields which satisfy the Maxwell system[10,14]. If magnetic forces are neglected, then one has the VPB system.

In this paper, we shall study the stability of solutions to the VPB system, which is an important subject for some systems in physics[24]. The reason is that not only it shows whether the state is achievable under a small perturbation, but also it provides a possible application in the numerical computation[27]. Furthermore, L^1 norm is natural for the VPB system since the system has the five conservation laws representing the macroscopic conservation of mass, momentum and energy. Therefore, the L^1 stability has been an unsolved interesting problem to this celebrated physical model.

Throughout this paper, we assume that the collision kernel B is nonnegative and continuous in its arguments and satisfies the following physically reasonable assumption:

$$\frac{B(\theta, |v-v_*|)}{|\cos\theta|} \leq C|v-v_*|^{\delta}, \quad -(N-2) < \delta \leq 1. \tag{1.6}$$

Notice that for the hard-sphere model,

$$B(\theta, |v-v_*|) = C|v-v_*|\cos\theta,$$

which satisfies (1.6) with $\delta = 1$. Notice also that both the hard and soft potentials with angular cut-off satisfy the condition (1.6). And for simpler presentation later, we call the cases with $-(N-2) < \delta \leq 0$ and $0 < \delta \leq 1$ as soft and hard potentials respectively.

As usual, we will consider the value $f^{\#}$ of the distribution f along the bi-characteristics. For any fixed $(x, v) \in \mathbf{R}_x^N \times \mathbf{R}_v^N$, the forward bi-characteristics $[X^t(x,v), V^t(x,v)]$ generated by some external force field $E(t,x)$ is defined by

$$\begin{cases} \dfrac{dX^t(x,v)}{dt} = V^t(x,v), \quad \dfrac{dV^t(x,v)}{dt} = E(t, X^t(x,v)), \\ (X^t, V^t)_{t=0} = (x, v). \end{cases} \tag{1.7}$$

Then we denote $f^{\#}$ by

$$f^{\#}(t,x,v) = f(t, X^t(x,v), V^t(x,v)).$$

Furthermore let's introduce some norms for the solutions in consideration[3]. For any $f = f(t,x,v)$ and $f_0 = f_0(x,v)$, define

$$|||f|||_{\alpha,\beta}^E = \sup_{t,x,v} \frac{|f^{\#}(t,x,v)|}{h_\alpha(|x|)m_\beta(|v|)}, \quad \|f_0\|_{\alpha,\beta} = \sup_{x,v} \frac{|f_0(x,v)|}{h_\alpha(|x|)m_\beta(|v|)}, \tag{1.8}$$

where the weight functions h_α and m_β have algebraic decay rates and are in the form of

$$h_\alpha(|x|) = (1+|x|^2)^{-\alpha},\ \alpha > 0 \ \text{ and } \ m_\beta(|v|) = (1+|v|^2)^{-\beta},\ \beta > 0. \tag{1.9}$$

For simplicity, throughout this paper, for any function $f(t,x,v)$, we use notations:

$$\|f(t)\|_1 = \|f(t,\cdot,\cdot)\|_{L^1(\mathbf{R}_x^N \times \mathbf{R}_v^N)}, \quad \|f(t)\|_\infty = \sup_{x,v} |f(t,x,v)|,$$

and

$$\|\nabla_x f(t)\|_p = \sum_{i=1}^3 \|\partial_{x_i} f(t)\|_p, \quad \|\nabla_v f(t)\|_p = \sum_{i=1}^3 \|\partial_{v_i} f(t)\|_p,$$

where $1 \leq p \leq \infty$. Notice that by the measure preservation of the mapping $(x,v) \to (X^t(x,v), V^t(x,v))$ for any $t \geq 0$, we have

$$\|f^{\#}(t,\cdot,\cdot)\|_{L^p(\mathbf{R}_x^N \times \mathbf{R}_v^N)} = \|f(t,\cdot,\cdot)\|_{L^p(\mathbf{R}_x^N \times \mathbf{R}_v^N)}. \tag{1.10}$$

Before stating the stability result in this paper, we first give the following global existence theorem on classical solutions in infinite vacuum to the VPB system with small initial data.

Theorem 1.1. *Suppose that $N \geq 3$, $\alpha > (N+1)/2$, $\beta > N+1$ and the collision kernel B satisfies (1.6) with $-(N-2) < \delta \leq 1$. Then there exist constants δ_0 and C_0 such that the following holds. For any $\delta_1 \in (0, \delta_0]$, if $0 \leq f_0 \in W^{1,\infty}(\mathbf{R}_x^N \times \mathbf{R}_v^N)$ satisfies*

$$\|f_0\|_{\alpha,\beta} + \|\nabla_x f_0\|_{\alpha,\beta} + \|\nabla_v f_0\|_{\alpha,\beta} \leq \delta_1, \tag{1.11}$$

then the Cauchy problem (1.1)-(1.2) has a unique global solution $[f, E_f]$ such that $0 \leq f \in W^{1,\infty}(\mathbf{R}_{\text{loc}}^+ \times \mathbf{R}_x^N \times \mathbf{R}_v^N)$ and $E_f \in C_b(\mathbf{R}_t^+ \times \mathbf{R}_x^N) \cap C_b^1(\mathbf{R}_t^+; \mathbf{R}_x^N)$ satisfy

$$|||f|||_{\alpha,\beta}^{E_f} + |||\nabla_x f|||_{\alpha,\beta}^{E_f} + |||(1+t)^{-1}\nabla_v f|||_{\alpha,\beta}^{E_f} \leq C_0\delta_1 \tag{1.12}$$

and

$$\int_0^\infty (\|E_f(t)\|_\infty + (1+t)\|\nabla_x E_f(t)\|_\infty)\, dt \leq C_0\delta_1. \tag{1.13}$$

Remark 1.1. The inequality (1.13) can be easily improved. In fact by Lemma 4.1 in Section 4 and the Poisson equation $\triangle_x \phi_f = \rho_f$, we obtain the explicit decay rates of $E_f(t,x)$ as follows:

$$\|E_f(t)\|_\infty \leq \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-1}} \quad \text{and} \quad \|\nabla_x E_f(t)\|_\infty \leq \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N(1-\lambda)}}, \tag{1.14}$$

where $\lambda \in (0, 1/(N+1))$ is a constant and $\mathcal{O}(1)$, from now on, denotes the general positive constant independent of δ_1 which may vary for different equations. For the proof of (1.14) when $N = 3$, see Refs. 2 and 12. It is exactly the same for $N > 3$ and thus omitted.

The proof Theorem 1.1 for the realistic physical case, i.e. $N = 3$ can be found in Ref. 12. For $N > 3$, we can use the same method to deal with its proof and hence still omit it for brevity. Instead, we devote ourselves to the proof of the uniform L^1 stability of solutions in sense of Theorem 1.1. In fact we have

Theorem 1.2. *Assume that all conditions in Theorem 1.1 hold and furthermore $N \geq 4$. Let f and g be the classical solutions to the VPB system corresponding to initial data f_0 and g_0 satisfying (1.11). If $\delta_1 \in (0,1)$ is sufficiently small, then it holds that*

$$\|f(t) - g(t)\|_1 \leq \mathcal{O}(1)\|f_0 - g_0\|_1, \quad \forall\ t \geq 0. \tag{1.15}$$

Now we review some previous works on the related topics and then give the main ideas of this paper. Some general knowledge on the VPB system and other related kinetic models can be found in the literature[7,8,14,26]. For the VPB system, the large time asymptotic behavior of weak solutions with some extra regularity was studied by Desvillettes-Dolbeault[10], see also the related topics[6,13]. The global existence of DiPerna-Lions renormalized solutions with arbitrary amplitude to the initial boundary value problem was given by Mischler[22]. For classical solutions in infinite vacuum, the first global existence was obtained by Guo[17] for some soft potentials and Duan-Yang-Zhu[12] for almost general potentials including the hard-sphere model. The global existence of solutions near a global Maxwellian was also studied by Guo[16] and Yang-Yu-Zhao[28] respectively for the space periodic data and the Cauchy problem. Finally we mention some works related to the problems considered in this paper. Arkeryd[1] proved the Lyapunov-type weighted L^1 stability for the space homogeneous Boltzmann equation. Furthermore, Ha completed a series of important works about the uniform L^1 stability for the Boltzmann equation without external forces[18,20], the Vlasov-Poisson system[9] and the Enskog-Boltzmann equation[19], where some new Lyapunov functionals were constructed. For the Boltzmann equation with external forces, Duan-Yang-Zhu[11] recently used the similar method to prove the uniform L^1 stability of small solutions around vacuum.

There are two vital observations for the uniform L^1 stability estimate: One is the decay in time of $f^\#$ in the space $L^1(\mathbf{R}_v^N)$, the other is the decay in time of $J^\#(f,g)$ in the space $L^1(\mathbf{R}_x^N \times \mathbf{R}_v^N)$. To obtain our result, we directly use the Gronwall's inequality to deal with the case of the soft potentials. For the case of the hard potentials, some new nonlinear functionals, which reduce to the same functionals in Ref. 20 when the external force vanishes, are constructed to control the factor $|v-v_*|^\delta$ in the collision kernel $B(\theta,|v-v_*|)$. These functionals can capture the effect of the force term on the time evolution of solutions. Precisely, if $0<\delta\le 1$, we can obtain the following estimates:

- $\displaystyle \frac{d\mathcal{L}(f,g)(t)}{dt} \le \mathcal{O}(1)\Lambda_h(f,g)(t) + \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}}\mathcal{L}(f,g)(t), \quad \forall\, t\ge 0,$
- $\displaystyle \frac{d\mathcal{D}_h(f,g)(t)}{dt} \le -(1-\mathcal{O}(1)\delta_1)\Lambda_h(f,g)(t) + \frac{\mathcal{O}(1)\delta_1^2}{(1+t)^2}\mathcal{L}(f,g)(t), \quad \forall\, t\ge 0.$

Hence by the smallness of $\delta_1>0$, we can choose a proper constant $K>0$ to construct the Glimm-type functional [5,21]:

$$\mathcal{H}_K(f,g)(t) = \mathcal{L}(f,g)(t) + K\mathcal{D}_h(f,g)(t),$$

which will be equivalent with the L^1 distance $\mathcal{L}(f,g)(t)$ of two solutions. Thus the uniform L^1 stability for the case of the hard potentials will follow from the above estimates. See Section 3 for notations and more details.

The rest of this paper is arranged as follows. In Section 2, some preliminary lemmas are given for later use. In Section 3, the L^1 stability estimate is obtained by considering the following two cases: the soft potential and the hard potential. Some known lemmas used in this paper are listed in Section 4.

2. Preliminary

For any fixed $(t,x,v)\in \mathbf{R}_t^+ \times \mathbf{R}_x^N \times \mathbf{R}_v^N$, we also define the backward bi-characteristics $[X(s;t,x,v),\ V(s;t,x,v)]$ generated by some external force field

$E(t,x)$ by solutions to the ODE system

$$\begin{cases} \dfrac{dX(s;t,x,v)}{ds} = V(s;t,x,v), \quad \dfrac{dV(s;t,x,v)}{ds} = E(s, X(s;t,x,v)), \\ (X(s;t,x,v), V(s;t,x,v))_{s=t} = (x,v). \end{cases} \tag{2.1}$$

Then it is easy to see that

$$|f(t,x,v)| \leq |||f|||^{E}_{\alpha,\beta} h_\alpha(|X(0;t,x,v)|) m_\beta(|V(0;t,x,v)|). \tag{2.2}$$

Also notice that (2.1) can be rewritten as the following integral form:

$$\begin{cases} X(s;t,x,v) = x - v(t-s) - \displaystyle\int_s^t \int_\eta^t E(\theta, X(\theta;t,x,v)) d\theta d\eta, \\ V(s;t,x,v) = v - \displaystyle\int_s^t E(\theta, X(\theta;t,x,v)) d\theta. \end{cases} \tag{2.3}$$

First for the backward bi-characteristic, we have

Lemma 2.1. *Suppose that the external force $E(t,x)$ satisfies (1.14). If $N \geq 4$, then we have that for any $(t,x,v) \in \mathbf{R}^+_t \times \mathbf{R}^N_x \times \mathbf{R}^N_v$,*

$$|X(0;t,x,v) - (x - vt)| + |V(0;t,x,v) - v| \leq \mathcal{O}(1)\delta_1 \tag{2.4}$$

and

$$|V^t(x,v) - v| \leq \mathcal{O}(1)\delta_1. \tag{2.5}$$

Proof. From (1.14) and (2.3), we have

$$\begin{aligned} |V(0;t,x,v) - v| &= \left| \int_0^t E(\theta, X(\theta;t,x,v)) d\theta \right| \\ &\leq \int_0^\infty \|E(\theta)\|_\infty d\theta \\ &\leq \mathcal{O}(1)\delta_1 \int_0^\infty \frac{1}{(1+\theta)^{N-1}} d\theta \\ &\leq \mathcal{O}(1)\delta_1, \end{aligned}$$

and

$$\begin{aligned} |X(0;t,x,v) - (x - vt)| &= \left| \int_0^t \int_\eta^t E(\theta, X(\theta;t,x,v)) d\theta d\eta \right| \\ &\leq \int_0^t \int_\eta^\infty \|E(\theta)\|_\infty d\theta d\eta \\ &\leq \mathcal{O}(1)\delta_1 \int_0^t \int_\eta^\infty \frac{1}{(1+\theta)^{N-1}} d\theta d\eta \\ &\leq \mathcal{O}(1)\delta_1, \end{aligned}$$

since $N \geq 4$. Finally (2.5) is proved similarly. This completes the proof of Lemma 2.1. $\square$

Next consider the two integrals I_1 and I_2 defined respectively by

$$I_1(t,y) = \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} \frac{h_\alpha(|X(0;t,x,v)|)m_\beta(|V(0;t,x,v)|)}{|x-y|^{N-1}} dxdv \tag{2.6}$$

and

$$I_2(t,x,v) = \int_{\mathbf{R}_{v_*}^N} |v-v_*|^\delta h_\alpha(|X(0;t,x,v_*)|)m_\beta(|V(0;t,x,v_*)|)dv_*. \tag{2.7}$$

The following lemma shows that both I_1 and I_2 decay with explicit rates in time.

Lemma 2.2. *Suppose that the external force $E(t,x)$ satisfies (1.14). If $N \geq 4$ and $-(N-2) < \delta \leq 0$, then we have that for any $t \geq 0$,*

$$\sup_y I_1(t,y) \leq \frac{\mathcal{O}(1)}{(1+t)^{N-1}} \tag{2.8}$$

and

$$\sup_{x,v} I_2(t,x,v) \leq \frac{\mathcal{O}(1)}{(1+t)^{N+\delta}}. \tag{2.9}$$

Proof. First consider the proof of (2.8). By using Lemmas 2.1 and 4.2, we deduce from (2.6) that for any $t \geq 0$,

$$\begin{aligned} I_1(t,y) &\leq \big(1+\mathcal{O}(1)\delta_1 + (\mathcal{O}(1)\delta_1)^2\big)^{\alpha+\beta} \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} \frac{h_\alpha(|x-vt|)m_\beta(|v|)}{|x-y|^{N-1}} dxdv \\ &\leq \big(1+\mathcal{O}(1)\delta_0 + (\mathcal{O}(1)\delta_0)^2\big)^{\alpha+\beta} \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} \frac{h_\alpha(|x-vt|)m_\beta(|v|)}{|x-y|^{N-1}} dxdv \\ &\leq \mathcal{O}(1) \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} \frac{h_\alpha(|x-vt|)m_\beta(|v|)}{|x-y|^{N-1}} dxdv. \end{aligned} \tag{2.10}$$

Furthermore for any constant $R > 0$, we compute

$$\begin{aligned} &\int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} \frac{h_\alpha(|x-vt|)m_\beta(|v|)}{|x-y|^{N-1}} dxdv \\ &= \left(\int_{\{|x-y|\leq R\}\times \mathbf{R}_v^N} + \int_{\{|x-y|>R\}\times \mathbf{R}_v^N} \right) \frac{h_\alpha(|x-vt|)m_\beta(|v|)}{|x-y|^{N-1}} dxdv \\ &= \int_{\{|x-y|\leq R\}} \frac{1}{|x-y|^{N-1}} \left(\int_{\mathbf{R}_v^N} h_\alpha(|x-vt|)m_\beta(|v|)dv \right) dx \\ &\qquad + \frac{1}{R^{N-1}} \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} h_\alpha(|x-vt|)m_\beta(|v|)dxdv \\ &\leq \int_{\{|x-y|\leq R\}} \frac{1}{|x-y|^{N-1}} \left(\int_{\mathbf{R}_v^N} h_\alpha(|x-vt|)m_\beta(|v|)dv \right) dx + \frac{\mathcal{O}(1)}{R^{N-1}}. \end{aligned} \tag{2.11}$$

Now we claim that

$$\int_{\mathbf{R}_v^N} h_\alpha(|x-vt|)m_\beta(|v|)dv \le \frac{\mathcal{O}(1)}{(1+t)^N}. \tag{2.12}$$

In fact, when $0 \le t \le 1$, it holds that

$$\int_{\mathbf{R}_v^N} h_\alpha(|x-vt|)m_\beta(|v|)dv \le \int_{\mathbf{R}_v^N} m_\beta(|v|) \le \mathcal{O}(1). \tag{2.13}$$

When $t \ge 1$, we can take the change of variable $x - vt = u$ to get

$$\begin{aligned}\int_{\mathbf{R}_v^N} h_\alpha(|x-vt|)m_\beta(|v|)dv &\le \int_{\mathbf{R}_v^N} h_\alpha(|x-vt|)dv\\ &= \frac{1}{t^N}\int_{\mathbf{R}_u^N} h_\alpha(|u|)du\\ &\le \frac{\mathcal{O}(1)}{t^N},\end{aligned} \tag{2.14}$$

which together with (2.13) yields (2.12). Thus putting (2.12) into (2.11), we have

$$\begin{aligned}&\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} \frac{h_\alpha(|x-vt|)m_\beta(|v|)}{|x-y|^{N-1}}dxdv\\ &\le \frac{\mathcal{O}(1)}{(1+t)^N}\int_{\{|x-y|\le R\}} \frac{1}{|x-y|^{N-1}}dx + \frac{\mathcal{O}(1)}{R^{N-1}}\\ &\le \frac{\mathcal{O}(1)R}{(1+t)^N} + \frac{\mathcal{O}(1)}{R^{N-1}}.\end{aligned} \tag{2.15}$$

In particular we take $R = 1 + t$ to obtain

$$\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} \frac{h_\alpha(|x-vt|)m_\beta(|v|)}{|x-y|^{N-1}}dxdv \le \frac{\mathcal{O}(1)}{(1+t)^{N-1}}. \tag{2.16}$$

Hence combining (2.10) and (2.16) yields (2.8).

For (2.9), similar to the proof of (2.10), we have from (2.7) that

$$I_2(t,x,v) \le \mathcal{O}(1)\int_{\mathbf{R}_{v_*}^N} |v-v_*|^\delta h_\alpha(|x-v_*t|)m_\beta(|v_*|)dv_*. \tag{2.17}$$

When $0 \le t \le 1$, it follows from Lemma 4.4 that

$$I_2(t,x,v) \le \mathcal{O}(1)\int_{\mathbf{R}_{v_*}^N} |v-v_*|^\delta m_\beta(|v_*|)dv_* \le \mathcal{O}(1) \tag{2.18}$$

since $-(N-2) < \delta \le 0$. Furthermore when $t \ge 1$, from (2.17), we let $v_*t - x = u$ to obtain

$$\begin{aligned}I_2(t,x,v) &\le \mathcal{O}(1)\int_{\mathbf{R}_{v_*}^N} |v-v_*|^\delta h_\alpha(|x-v_*t|)dv_*\\ &= \frac{\mathcal{O}(1)}{t^{N+\delta}}\int_{\mathbf{R}_u^N} |vt-x-u|^\delta h_\alpha(|u|)du\\ &\le \frac{\mathcal{O}(1)}{t^{N+\delta}},\end{aligned} \tag{2.19}$$

where $-(N-2) < \delta \le 0$ and Lemma 4.4 are used again. Thus both (2.18) and (2.19) lead to

$$I_2(t,x,v) \le \frac{\mathcal{O}(1)}{(1+t)^{N+\delta}}. \tag{2.20}$$

Hence (2.9) holds. The proof of Lemma 2.2 is complete. □

Finally we give a lemma which plays an important role in the proof of the uniform stability estimate (1.15) for the case of the hard potentials.

Lemma 2.3. *Let $N \ge 3$, $\alpha > (N+1)/2$ and $\beta > N+1$. Suppose that the collision kernel B satisfies (1.6) with $-(N-2) < \delta \le 1$ and the external force E satisfies (1.14). Then there exists a positive constant η with $0 < \eta < \beta - N/2$ such that for any $(t,x,v) \in \mathbf{R}_t^+ \times \mathbf{R}_x^N \times \mathbf{R}_v^N$, it holds that*

$$\big|Q^{\#}(f,g)(t,x,v)\big| + \big|f^{\#}R^{\#}(g)(t,x,v)\big| \le \frac{\mathcal{O}(1)|||f|||_{\alpha,\beta}^{E}\,|||g|||_{\alpha,\beta}^{E}}{(1+t)^2} h_{\alpha-1/2}(|x|) m_{\beta-\eta}(|v|). \tag{2.21}$$

For Lemma 2.3, its proof when $N=3$ can be found in Ref. 12. The exactly same method is used to deal with the case of $N \ge 4$ and thus we omit it. The only point we have to mention is that the decay rate $(1+t)^{-2}$ in (2.21) is optimal and independent of the dimension N of the phase space.

3. L^1 stability

In this section, we give the proof of Theorem 1.2, which follows from a series of lemmas. Precisely, we directly use the Gronwall's inequality to deal with the case of the soft potentials. For the case of the hard potentials, some new nonlinear functionals are constructed to balance the singularity effect by the collision kernel $B(\theta, |v-v_*|)$.

To this end, let f and g be two classical solutions to VPB system corresponding to initial data f_0 and g_0 satisfying (1.11) in Theorem 1.1. For use later, let's define the nonnegative bilinear operator $\mathcal{S}$ by

$$\mathcal{S}(f,g)(t,x,v) = [Q(f,g) + fR(g)](t,x,v), \tag{3.1}$$

and the nonlinear functionals $\mathcal{L}$ and Λ by

$$\mathcal{L}(f,g)(t) = \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} |f-g|^{\#}(t,x,v)\,dxdv, \tag{3.2}$$

$$\Lambda(f,g)(t) = \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N \times \mathbf{R}_{v_*}^N} |v-v_*|^{\delta} |f-g|(t,x,v)(f+g)(t,x,v_*)\,dxdvdv_*. \tag{3.3}$$

Notice from (1.10) that for any $t \ge 0$,

$$\mathcal{L}(f,g)(t) = \|(f-g)(t)\|_1,$$

i.e. $\mathcal{L}(f,g)$ is just L^1 distance of two solutions f and g.

First we have the following basic estimate on the evolution of $\mathcal{L}(f,g)(t)$ for the case of the general potentials.

Lemma 3.1. *Suppose that the conditions of Theorem 1.2 hold. If $-(N-2) < \delta \leq 1$, then we have that*

$$\frac{d\mathcal{L}(f,g)(t)}{dt} \leq \mathcal{O}(1)\Lambda(f,g)(t) + \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}}\mathcal{L}(f,g)(t). \tag{3.4}$$

Proof. Since both f and g are solutions to the VPB system, it holds that

$$\begin{aligned} \partial_t f + v\cdot\nabla_x f + E_f\cdot\nabla_v f &= J(f,f),\\ \partial_t g + v\cdot\nabla_x g + E_g\cdot\nabla_v g &= J(g,g). \end{aligned}$$

Taking difference of the above two equations and multiplying it by $\operatorname{sign}(f-g)$ gives

$$\begin{aligned} \partial_t|f-g| &+ v\cdot\nabla_x|f-g| + E_f\cdot\nabla_v|f-g| \\ &\leq \mathcal{S}(|f-g|,f) + \mathcal{S}(g,|f-g|) + |E_g-E_f|\cdot|\nabla_v g|. \end{aligned}$$

Along the forward bi-characteristics generated by the force field $E_f(t,x)$, the above inequality can be rewritten as

$$\partial_t|f-g|^{\#} \leq \mathcal{S}^{\#}(|f-g|,f) + \mathcal{S}^{\#}(g,|f-g|) + |E_g-E_f|^{\#}\cdot|\nabla_v g|^{\#}. \tag{3.5}$$

Integrating it over $\mathbf{R}_x^N\times\mathbf{R}_v^N$ and noting (1.10), we have

$$\frac{d\mathcal{L}(f,g)(t)}{dt} \leq \mathcal{O}(1)\Lambda(f,g)(t) + \int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|E_g-E_f|\cdot|\nabla_v g|dxdv. \tag{3.6}$$

Therefore the rest proof is to show that

$$\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|E_g-E_f|\cdot|\nabla_v g|dxdv \leq \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}}\mathcal{L}(f,g)(t). \tag{3.7}$$

In fact, by the Poisson equation $\triangle\phi = \rho$, we have

$$\begin{aligned}
&\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} |E_g - E_f|\cdot|\nabla_v g|dxdv\\
&\le \mathcal{O}(1)\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N\times\mathbf{R}_y^N} \frac{|(\rho_f-\rho_g)(t,y)|}{|x-y|^{N-1}}|\nabla_v g(t,x,v)|dxdvdy\\
&\le \mathcal{O}(1)\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N\times\mathbf{R}_y^N\times\mathbf{R}_{v_*}^N} \frac{|(f-g)(t,y,v_*)|}{|x-y|^{N-1}}|\nabla_v g(t,x,v)|dxdvdydv_*\\
&\le \mathcal{O}(1)\int_{\mathbf{R}_y^N\times\mathbf{R}_{v_*}^N} |(f-g)(t,y,v_*)|dydv_*\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}\frac{|\nabla_v g(t,x,v)|}{|x-y|^{N-1}}dxdv\\
&\le \mathcal{O}(1)\delta_1(1+t)\int_{\mathbf{R}_y^N\times\mathbf{R}_{v_*}^N} |(f-g)(t,y,v_*)|dydv_*\\
&\qquad\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}\frac{h_\alpha(|X(0;t,x,v)|)m_\beta(|V(0;t,x,v)|)}{|x-y|^{N-1}}dxdv\\
&\le \mathcal{O}(1)\delta_1(1+t)\sup_y I_1(t,y)\int_{\mathbf{R}_y^N\times\mathbf{R}_{v_*}^N} |(f-g)(t,y,v_*)|dydv_*, \qquad (3.8)
\end{aligned}$$

which together with (2.8) yields (3.7). Thus the proof of Lemma 3.1 is complete. □

Next we devote ourselves to use the L^1 distance $\mathcal{L}(f,g)$ of two solutions to control the term $\Lambda(f,g)$ on the right hand of (3.4). For the case of the soft potentials, i.e. $-(N-2)<\delta\le 0$, it can be achieved by the following lemma, which directly leads to the uniform L^1 stability estimate with the help of the Gronwall's inequality.

Lemma 3.2. *Suppose that the conditions of Theorem 1.2 hold. If* $-(N-2)<\delta\le 0$, *then we have*

$$\Lambda(f,g)(t)\le\frac{\mathcal{O}(1)}{(1+t)^{N+\delta}}\mathcal{L}(f,g)(t). \qquad (3.9)$$

Proof. It follows from the representation form (3.3) of $\Lambda(f,g)(t)$ that

$$\begin{aligned}
\Lambda(f,g)(t) &= \int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|f-g|(t,x,v)dxdv\int_{\mathbf{R}_{v_*}^N}|v-v_*|^\delta(f+g)(t,x,v_*)dv_*\\
&\le \mathcal{O}(1)\delta_1\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|f-g|(t,x,v)dxdv\\
&\qquad\int_{\mathbf{R}_{v_*}^N}|v-v_*|^\delta h_\alpha(|X(0;t,x,v_*)|)m_\beta(|V(0;t,x,v_*)|)dv_*\\
&\le \mathcal{O}(1)\delta_1\sup_{x,v}I_2(t,x,v)\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|f-g|(t,x,v)dxdv. \qquad (3.10)
\end{aligned}$$

Thus it follows from (2.9) and (3.10) that (3.9) holds. This ends the proof of Lemma 3.2. □

Remark 3.1. If $-(N-2) < \delta \leq 0$, then both Lemmas 3.1 and 3.2 show that

$$\frac{d\mathcal{L}(f,g)(t)}{dt} \leq \left\{ \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N+\delta}} + \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}} \right\} \mathcal{L}(f,g)(t).$$

Since $-(N-2) < \delta \leq 0$ and $N \geq 4$, the above inequality with the help of the Gronwall's inequality immediately leads to

$$\mathcal{L}(f,g)(t) \leq \mathcal{O}(1)\mathcal{L}(f,g)(0).$$

This gives the uniform L^1 stability estimate (1.15) in Theorem 1.2 for the case of the soft potentials.

Finally we consider the L^1 stability of solutions to the VPB system for the case of the hard potentials, i.e. $0 < \delta \leq 1$. It should be noticed that the estimate similar to (3.9) in Lemma 3.2 fails for this case because of the possible increase at infinity of the term $|v-v_*|^\delta$ in the collision kernel $B(\theta, |v-v_*|)$. To overcome this difficulty, we will construct some new nonlinear functionals motivated by some known works[18,20] on the L^1 stability of solutions to the Boltzmann equation without the external force. For this purpose, let's define

$$v_\infty(x,v) = \int_0^\infty E\left(t, X^t(x,v)\right) dt. \tag{3.11}$$

Then we have

Lemma 3.3. *Suppose that*

$$\int_0^\infty \|E(t)\|_\infty dt < \infty. \tag{3.12}$$

Then for any $(x,v) \in \mathbf{R}^N_x \times \mathbf{R}^N_v$, $v_\infty(x,v)$ *is well-defined with the uniform bound*

$$\sup_{x,v} |v_\infty(x,v)| \leq \int_0^\infty \|E(t)\|_\infty dt. \tag{3.13}$$

Furthermore if there exists some positive constant $\kappa > 0$ *such that*

$$\int_0^\infty (1+t)^\kappa \|E(t)\|_\infty dt < \infty, \tag{3.14}$$

then one has

$$\sup_{x,v} \left| \frac{1}{t}\int_0^t V^s(x,v)ds - v - v_\infty(x,v) \right| \to 0 \quad as \quad t \to \infty. \tag{3.15}$$

Proof. We only prove (3.15). In fact,

$$\begin{aligned}\left|\frac{1}{t}\int_0^t V^s(x,v)ds - v - v_\infty(x,v)\right| &= \left|\frac{1}{t}\int_0^t\int_0^s E(\theta, X^\theta(x,v))d\theta ds - v_\infty(x,v)\right|\\ &= \left|\frac{1}{t}\int_0^t\int_s^\infty E(\theta, X^\theta(x,v))d\theta ds\right|\\ &\le \frac{1}{t}\int_0^t\int_s^\infty \|E(\theta)\|_\infty d\theta ds.\end{aligned}$$

when $t \ge 1$, by the integration part, we have

$$\begin{aligned}&\frac{1}{t}\int_0^t\int_s^\infty \|E(\theta)\|_\infty d\theta ds\\ &= \int_t^\infty \|E(\theta)\|_\infty d\theta + \frac{1}{t}\int_0^t s\|E(s)\|_\infty ds\\ &\le \int_t^\infty \|E(\theta)\|_\infty d\theta + \frac{1}{t}\int_0^1 \|E(s)\|_\infty ds + \frac{1}{t}\int_1^t s\|E(s)\|_\infty ds\\ &\le \int_t^\infty \|E(\theta)\|_\infty d\theta + \frac{1}{t}\int_0^1 \|E(s)\|_\infty ds + \frac{1}{t^{\min\{\kappa,1\}}}\int_1^t s^\kappa\|E(s)\|_\infty ds.\end{aligned}$$

Thus from the assumption (3.14), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\int_s^\infty \|E(\theta)\|_\infty d\theta ds = 0.$$

Hence (3.15) holds. The proof of Lemma 3.3 is complete. □

In order to control the integral $\Lambda(f,g)(t)$ for the case when $0 < \delta \le 1$, as in Ref. 11, let's define nonlinear functionals Λ_h and $\mathcal{D}_h$ as follows:

$$\begin{aligned}\Lambda_h(f,g)(t) = \int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} &|f-g|^{\#}(t,x,v)dxdv\\ &\int_{\mathbf{R}_{v_*}^N} |v_\infty(x,v)+v-v_*|^\delta (f+g)(t, X^t(x,v), v_*)dv_* \end{aligned}\tag{3.16}$$

and

$$\begin{aligned}\mathcal{D}_h(f,g)(t) = \int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} |f-g|^{\#}(t,x,v)dxdv &\int_{\mathbf{R}_{v_*}^N\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta-1}\\ &\times(f+g)(t, X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*), v_*)dv_*d\tau,\end{aligned}\tag{3.17}$$

where $n(z) = z/|z|$ denotes the unit vector along z-direction for any nonzero vector $z \in \mathbf{R}^N$.

Remark 3.2. If the external force $E(t,x)$ satisfies (1.14) and $N \ge 3$, then (3.12) and (3.14) hold. Thus for any self-consistent electric force $E_f(t,x)$ generated by the solution $f(t,x,v)$ to the VPB system in sense of Theorem 1.1, $v_\infty(x,v)$ is always well-defined with the uniform bound:

$$\sup_{x,v} |v_\infty(x,v)| \leq \mathcal{O}(1). \tag{3.18}$$

In addition, if the external force field $E(t,x)$ vanishes, i.e. $E \equiv 0$, then we have

$$v_\infty(x,v) \equiv 0,$$

and

$$X^t(x,v) = x + vt, \quad V^t(x,v) = v.$$

Thus the nonlinear functionals $\Lambda_h(f,g)$ and $\mathcal{D}_h(f,g)$ reduce to

$$\begin{aligned}\Lambda_h(f,g)(t) = \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} &|f-g|^{\#}(t,x,v)dxdv \\ &\int_{\mathbf{R}_{v_*}^N} |v-v_*|^{\delta}(f+g)^{\#}(t,x+t(v-v_*),v_*)dv_*\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_h(f,g)(t) = \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} |f-g|^{\#}(t,x,v)dxdv &\int_{\mathbf{R}_{v_*}^N \times \mathbf{R}_\tau^+} |v-v_*|^{\delta-1} \\ &\times (f+g)^{\#}(t,x+t(v-v_*)+\tau n(v-v_*),v_*)dv_*d\tau,\end{aligned}$$

which are exactly the same as ones in Ref. 20.

Furthermore, we define the integral $I_3(t,x,v)$ by

$$\begin{aligned}I_3(t,x,v) = \int_{\mathbf{R}_{v_*}^N \times \mathbf{R}_\tau^+} &|v_\infty(x,v)+v-v_*|^{\delta-1} \\ &\times h_\alpha(|X(0;t,X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)|) \\ &\times m_\beta(|V(0;t,X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)|)dv_*d\tau. \end{aligned} \tag{3.19}$$

We first claim that $\mathcal{D}_h(f,g)$ can be bounded by $\mathcal{L}(f,g)$, which comes from the following lemma.

Lemma 3.4. *Suppose that the external force $E(t,x)$ satisfies (1.14). If $N \geq 4$ and $0 < \delta \leq 1$, then we have that*

$$\sup_{t,x,v} I_3(t,x,v) \leq \mathcal{O}(1). \tag{3.20}$$

Proof. Fix any $(t,x,v) \in \mathbf{R}_t^+ \times \mathbf{R}_x^N \times \mathbf{R}_v^N$. It follows from Lemma 2.1 and Lemmas 4.2-4.4 that

$$\begin{aligned} I_3(t,x,v) &\leq \mathcal{O}(1) \int_{\mathbf{R}_{v_*}^N \times \mathbf{R}_\tau^+} |v_\infty(x,v) + v - v_*|^{\delta-1} m_\beta(|v_*|) \\ &\qquad\qquad h_\alpha \left(\left| X^t(x,v) - v_* t + \tau n(v_\infty(x,v) + v - v_*) \right| \right) dv_* d\tau \\ &\leq \mathcal{O}(1) \int_{\mathbf{R}_{v_*}^N} |v_\infty(x,v) + v - v_*|^{\delta-1} m_\beta(|v_*|) dv_* \\ &\qquad\qquad \int_{\mathbf{R}_\tau^+} h_\alpha \left(\left| X^t(x,v) - v_* t + \tau n(v_\infty(x,v) + v - v_*) \right| \right) d\tau \\ &\leq \mathcal{O}(1) \int_{\mathbf{R}_{v_*}^N} |v_\infty(x,v) + v - v_*|^{\delta-1} m_\beta(|v_*|) dv_* \\ &\leq \mathcal{O}(1). \end{aligned}$$

Thus the proof of Lemma 3.4 is complete. □

From Lemma 3.4 above, it is easy to see that for any $t \geq 0$,

$$\begin{aligned} \mathcal{D}_h(f,g)(t) &\leq \mathcal{O}(1)\delta_1 \sup_{t,x,v} I_3(t,x,v) \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} |f-g|^{\#}(t,x,v) dx dv \\ &\leq \mathcal{O}(1)\delta_1 \mathcal{L}(f,g)(t). \end{aligned} \tag{3.21}$$

Hence we shall construct the Glimm-type nonlinear functional[5,21]

$$\mathcal{H}_K(f,g)(t) = \mathcal{L}(f,g)(t) + K\mathcal{D}_h(f,g)(t), \tag{3.22}$$

where $K > 0$ is a positive constant. In view of (3.21), we see that for any $K > 0$, $\mathcal{H}_K(f,g)$ is equivalent with the L^1 distance $\mathcal{L}(f,g)$ of two solutions, i.e.

$$\mathcal{L}(f,g)(t) \leq \mathcal{H}_K(f,g)(t) \leq \mathcal{O}(1)\mathcal{L}(f,g)(t), \quad \forall\, t \geq 0. \tag{3.23}$$

Thus in order to obtain the uniform L^1 stability estimate, it suffices to choose the proper constant $K > 0$ such that

$$\mathcal{H}_K(f,g)(t) \leq \mathcal{H}_K(f,g)(0), \quad \forall\, t \geq 0. \tag{3.24}$$

This can be achieved by the following vital lemma.

Lemma 3.5. *Suppose that the conditions of Theorem 1.2 hold. If* $0 < \delta \leq 1$, *then we have that*

$$\frac{d\mathcal{D}_h(f,g)(t)}{dt} \leq -(1 - \mathcal{O}(1)\delta_1)\Lambda_h(f,g)(t) + \frac{\mathcal{O}(1)\delta_1^2}{(1+t)^2}\mathcal{L}(f,g)(t). \tag{3.25}$$

Proof. First, notice that

$$\begin{aligned}
\partial_t\,[f(t,\ &X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)]\\
&=(\partial_t f)(t,X^t(x,v)+\tau n,v_*)+V^t(x,v)\cdot\nabla_x f(t,X^t(x,v)+\tau n,v_*)\\
&=J(f,f)(t,X^t(x,v)+\tau n,v_*)-v_*\cdot\nabla_x f(t,X^t(x,v)+\tau n,v_*)\\
&\quad -E_f(t,X^t(x,v)+\tau n)\cdot\nabla_v f(t,X^t(x,v)+\tau n,v_*)\\
&\quad +V^t(x,v)\cdot\nabla_x f(t,X^t(x,v)+\tau n,v_*)\\
&=J(f,f)(t,X^t(x,v)+\tau n,v_*)\\
&\quad -E_f(t,X^t(x,v)+\tau n)\cdot\nabla_v f(t,X^t(x,v)+\tau n,v_*)\\
&\quad +(V^t(x,v)-v-v_\infty(x,v))\cdot\nabla_x f(t,X^t(x,v)+\tau n,v_*)\\
&\quad +\partial_\tau(|v_\infty(x,v)+v-v_*|f(t,X^t(x,v)+\tau n,v_*)),
\end{aligned}\tag{3.26}$$

where for simplicity we have used n to denote the unit vector $n(v_\infty(x,v)+v-v_*)$. By the dominated convergence theorem, the following integral

$$\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N\times\mathbf{R}_{v_*}^N}\begin{aligned}&|v_\infty(x,v)+v-v_*|^\delta h_\alpha(|x|)m_\beta(|v|)\\ &\times h_\alpha(|X^t(x,v)+\tau n-v_*t|)m_\beta(|v_*|)dxdvdv_*\end{aligned}\tag{3.27}$$

tends to zero as τ goes to infinity. Hence (3.26) together with (3.5) yields

$$\begin{aligned}
&\partial_t\,\left[|f-g|^\#(t,x,v)(f+g)(t,X^t(x,v)+\tau n,v_*)\right]\\
&=\partial_t|f-g|^\#(t,x,v)(f+g)(t,X^t(x,v)+\tau n,v_*)\\
&\quad +|f-g|^\#(t,x,v)\partial_t\left[(f+g)(t,X^t(x,v)+\tau n,v_*)\right]\\
&\le(\mathcal{S}^\#(|f-g|,f)+\mathcal{S}^\#(g,|f-g|))(t,x,v)(f+g)(t,X^t(x,v)+\tau n,v_*)\\
&\quad +|E_g-E_f|^\#(t,x,v)|\nabla_v g|^\#(t,x,v)(f+g)(t,X^t(x,v)+\tau n,v_*)\\
&\quad +|f-g|^\#(t,x,v)(\mathcal{S}(f,f)+\mathcal{S}(g,g))(t,X^t(x,v)+\tau n,v_*)\\
&\quad +|f-g|^\#(t,x,v)(\|E_f(t)\|_\infty|\nabla_v f|+\|E_g(t)\|_\infty|\nabla_v g|)(t,X^t(x,v)+\tau n,v_*)\\
&\quad +|f-g|^\#(t,x,v)|V^t(x,v)-v-v_\infty(x,v)|(|\nabla_x f|+|\nabla_x g|)(t,X^t(x,v)+\tau n,v_*)\\
&\quad +\partial_\tau\left[|v_\infty(x,v)+v-v_*||f-g|^\#(t,x,v)(f+g)(t,X^t(x,v)+\tau n,v_*)\right].
\end{aligned}\tag{3.28}$$

Multiplying the inequality (3.28) by $|v_\infty(x,v)+v-v_*|^{\delta-1}$ and integrating it over the domain $D=\mathbf{R}_x^N\times\mathbf{R}_v^N\times\mathbf{R}_{v_*}^N\times\mathbf{R}_\tau^+$ leads to

$$\frac{d\mathcal{D}_h(f,g)(t)}{dt}\le-\Lambda_h(f,g)(t)+\sum_{i=1}^5 J_i(t),\tag{3.29}$$

where $J_i(t),\ i = 1, 2, 3, 4$ are defined as follows:

$$
\begin{aligned}
J_1(t) &= \int_D |v_\infty(x,v) + v - v_*|^{\delta-1}(\mathcal{S}^{\#}(|f-g|, f) + \mathcal{S}^{\#}(g, |f-g|))(t,x,v) \\
&\qquad (f+g)(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau, \\
J_2(t) &= \int_D |v_\infty(x,v) + v - v_*|^{\delta-1}|E_g - E_f|^{\#}(t,x,v)|\nabla_v g|^{\#}(t,x,v) \\
&\qquad (f+g)(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau, \\
J_3(t) &= \int_D |v_\infty(x,v) + v - v_*|^{\delta-1}|f-g|^{\#}(t,x,v) \\
&\qquad (\mathcal{S}(f,f) + \mathcal{S}(g,g))(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau, \\
J_4(t) &= \int_D |v_\infty(x,v) + v - v_*|^{\delta-1}\|E(t)\|_\infty |f-g|^{\#}(t,x,v) \\
&\qquad (|\nabla_v f| + |\nabla_v g|)(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau, \\
J_5(t) &= \int_D |v_\infty(x,v) + v - v_*|^{\delta-1}|V^t(x,v) - v - v_\infty(x,v)| \cdot |f-g|^{\#}(t,x,v) \\
&\qquad (|\nabla_x f| + |\nabla_x g|)(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau.
\end{aligned}
$$

For $J_1(t)$, it follows from Lemmas 2.1, 3.4 and 4.2 that

$$
\begin{aligned}
J_1(t) &= \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} (\mathcal{S}^{\#}(|f-g|, f) + \mathcal{S}^{\#}(g, |f-g|))(t,x,v)dxdv \\
&\qquad \int_{\mathbf{R}_{v_*}^N \times \mathbf{R}_\tau^+} |v_\infty(x,v) + v - v_*|^{\delta-1}(f+g)(t, X^t(x,v) + \tau n, v_*)dv_*d\tau \\
&\le \mathcal{O}(1)\delta_1 \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} (\mathcal{S}^{\#}(|f-g|, f) + \mathcal{S}^{\#}(g, |f-g|))(t,x,v)dxdv \\
&\qquad \int_{\mathbf{R}_{v_*}^N \times \mathbf{R}_\tau^+} |v_\infty(x,v) + v - v_*|^{\delta-1} h_\alpha(|X(0;t, X^t(x,v) + \tau n, v_*)|) \\
&\qquad\qquad m_\beta(|V(0;t, X^t(x,v) + \tau n, v_*)|)dv_*d\tau \\
&\le \mathcal{O}(1)\delta_1 \sup_{t,x,v} I_3(t,x,v) \int_{\mathbf{R}_x^N \times \mathbf{R}_v^N} (\mathcal{S}^{\#}(|f-g|, f) + \mathcal{S}^{\#}(g, |f-g|))(t,x,v)dxdv \\
&\le \mathcal{O}(1)\delta_1 \Lambda(f,g)(t).
\end{aligned}
\tag{3.30}
$$

Furthermore we have from Lemmas 2.1, 3.2, 3.3 and 4.2 that

$$
\begin{aligned}
&\Lambda(f,g)(t)\\
&=\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|f-g|(t,x,v)dxdv\int_{\mathbf{R}_{v_*}^N}|v-v_*|^\delta(f+g)(t,x,v_*)dv_*\\
&=\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|f-g|^\#(t,x,v)dxdv\\
&\qquad\int_{\mathbf{R}_{v_*}^N}|V^t(x,v)-v_*|^\delta(f+g)(t,X^t(x,v),v_*)dv_*\\
&\le\mathcal{O}(1)\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|f-g|^\#(t,x,v)dxdv\int_{\mathbf{R}_{v_*}^N}(f+g)(t,X^t(x,v),v_*)\\
&\qquad\qquad\left(|v_\infty(x,v)|^\delta+|V^t(x,v)-v|^\delta+|v_\infty(x,v)+v-v_*|^\delta\right)dv_*\\
&\le\mathcal{O}(1)\Lambda_h(f,g)(t)+\mathcal{O}(1)\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N\times\mathbf{R}_{v_*}^N}|f-g|(t,x,v)(f+g)(t,x,v_*)dxdvdv_*\\
&\le\mathcal{O}(1)\Lambda_h(f,g)(t)+\frac{\mathcal{O}(1)\delta_1}{(1+t)^N}\mathcal{L}(f,g)(t).
\end{aligned}
\tag{3.31}
$$

Putting (3.31) into (3.30), we have

$$
J_1(t)\le\mathcal{O}(1)\delta_1\Lambda_h(f,g)(t)+\frac{\mathcal{O}(1)\delta_1^2}{(1+t)^N}\mathcal{L}(f,g)(t). \tag{3.32}
$$

For $J_2(t)$, we have from Lemma 3.4 and (3.7) that

$$
\begin{aligned}
J_2(t)&=\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|E_g-E_f|^\#(t,x,v)|\nabla_vg|^\#(t,x,v)dxdv\\
&\qquad\int_{\mathbf{R}_{v_*}^N\times\mathbf{R}_\tau^+}|v_\infty(x,v)+v-v_*|^{\delta-1}(f+g)(t,X^t(x,v)+\tau n,v_*)dv_*d\tau\\
&\le\mathcal{O}(1)\delta_1\sup_{t,x,v}I_3(t,x,v)\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|E_g-E_f|^\#(t,x,v)|\nabla_vg|^\#(t,x,v)dxdv\\
&=\mathcal{O}(1)\delta_1\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N}|E_g-E_f|(t,x,v)|\nabla_vg|(t,x,v)dxdv\\
&\le\frac{\mathcal{O}(1)\delta_1^2}{(1+t)^{N-2}}\mathcal{L}(f,g)(t).
\end{aligned}
\tag{3.33}
$$

From Lemmas 2.3 and 3.4, $J_3(t)$ is estimated as follows:

$$
\begin{aligned}
J_3(t) &= \int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} |f-g|(t,x,v)dxdv \int_{\mathbf{R}_{v_*}^N\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta-1} \\
&\qquad (\mathcal{S}(f,f)+\mathcal{S}(g,g))(t,X^t(x,v)+\tau n,v_*)dv_*d\tau \\
&\le \frac{\mathcal{O}(1)\delta_1^2}{(1+t)^2}\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} |f-g|(t,x,v)dxdv \int_{\mathbf{R}_{v_*}^N\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta-1} \\
&\qquad h_{\alpha-1/2}\left(\left|X(0;t,X^t(x,v)+\tau n,v_*)\right|\right) \\
&\qquad m_{\beta-\eta}\left(\left|V(0;t,X^t(x,v)+\tau n,v_*)\right|\right)dv_*d\tau \\
&\le \frac{\mathcal{O}(1)\delta_1^2}{(1+t)^2}\mathcal{L}(f,g)(t),
\end{aligned}
\tag{3.34}
$$

where the last inequality comes from the same proof of Lemma 3.4 and we have used $0<\delta\le 1$, $\alpha>(N+1)/2$ and $0<\eta<\beta-N/2$. Similarly, for $J_4(t)$, we have from Lemma 3.4 and (1.14) that

$$
\begin{aligned}
J_4(t) &= \max\{\|E_f(t)\|_\infty,\|E_g(t)\|_\infty\}\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} |f-g|^\#(t,x,v)dxdv \\
&\quad \int_{\mathbf{R}_{v_*}^N\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta-1}(|\nabla_v f|+|\nabla_v g|)(t,X^t(x,v)+\tau n,v_*)dv_*d\tau \\
&\le \mathcal{O}(1)\delta_1(1+t)\max\{\|E_f(t)\|_\infty,\|E_g(t)\|_\infty\}\int_{\mathbf{R}_x^N\times\mathbf{R}_v^N} |f-g|^\#(t,x,v)dxdv \\
&\quad \int_{\mathbf{R}_{v_*}^N\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta-1} \\
&\qquad h_\alpha\left(\left|X(0;t,X^t(x,v)+\tau n,v_*)\right|\right) m_\beta\left(\left|V(0;t,X^t(x,v)+\tau n,v_*)\right|\right)dv_*d\tau \\
&\le \mathcal{O}(1)\delta_1(1+t)\max\{\|E_f(t)\|_\infty,\|E_g(t)\|_\infty\}\mathcal{L}(f,g)(t) \\
&\le \frac{\mathcal{O}(1)\delta_1^2}{(1+t)^{N-2}}\mathcal{L}(f,g)(t).
\end{aligned}
\tag{3.35}
$$

Finally, to estimate $J_5(t)$, by noticing from (1.14) that

$$
\begin{aligned}
|V^t(x,v)-v-v_\infty(x,v)| &= \left|\int_0^t E(\theta,X^\theta(x,v))d\theta - \int_0^\infty E(\theta,X^\theta(x,v))d\theta\right| \\
&= \left|\int_t^\infty E(\theta,X^\theta(x,v))d\theta\right| \\
&\le \int_t^\infty \|E(\theta)\|_\infty d\theta \\
&\le \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}},
\end{aligned}
\tag{3.36}
$$

similarly we have

$$J_5(t) \le \frac{\mathcal{O}(1)\delta_1^2}{(1+t)^{N-2}}\mathcal{L}(f,g)(t). \tag{3.37}$$

Since $N \ge 4$, combining (3.29) and (3.32)-(3.37) gives (3.25). Thus the proof of Lemma 3.5 is complete. □

Corollary 3.1. *Suppose that the conditions of Theorem 1.2 hold. For the case of the hard potentials, i.e. $0 < \delta \le 1$, if $\delta_1 > 0$ is sufficiently small, then we have the L^1 stability estimate (1.15).*

Proof. First if $0 < \delta \le 1$, then it follows from Lemma 3.1 and the inequality (3.31) that

$$\begin{aligned}
\frac{d\mathcal{L}(f,g)(t)}{dt} &\le \mathcal{O}(1)\Lambda(f,g)(t) + \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}}\mathcal{L}(f,g)(t)\\
&\le \left\{\mathcal{O}(1)\Lambda_h(f,g)(t) + \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N}}\mathcal{L}(f,g)(t)\right\} + \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}}\mathcal{L}(f,g)(t)\\
&\le \mathcal{O}(1)\Lambda_h(f,g)(t) + \frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}}\mathcal{L}(f,g)(t).
\end{aligned} \tag{3.38}$$

Recall (3.22). Multiplying (3.25) by the constant $K > 0$ and adding it to (3.38), we have that

$$\begin{aligned}
\frac{d\mathcal{H}_K(f,g)(t)}{dt} &\le \left(\mathcal{O}(1) - K(1-\mathcal{O}(1)\delta_1)\right)\Lambda_h(f,g)(t)\\
&\quad + \left\{\frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}} + \frac{\mathcal{O}(1)K\delta_1^2}{(1+t)^2}\right\}\mathcal{L}(f,g)(t)\\
&\le \left(\mathcal{O}(1) - K(1-\mathcal{O}(1)\delta_1)\right)\Lambda_h(f,g)(t)\\
&\quad + \left\{\frac{\mathcal{O}(1)\delta_1}{(1+t)^{N-2}} + \frac{\mathcal{O}(1)K\delta_1^2}{(1+t)^2}\right\}\mathcal{H}_K(f,g)(t).
\end{aligned} \tag{3.39}$$

Now let $\delta_1 \in (0,1)$ sufficiently small such that $1 - \mathcal{O}(1)\delta_1 > 1/2$ and furthermore $K > 1$ sufficiently large independent of δ_1 such that $\mathcal{O}(1) - K/2 < -1$. Then since $N \ge 4$, we have from (3.39) that

$$\frac{d\mathcal{H}_K(f,g)(t)}{dt} + \Lambda_h(f,g)(t) \le \frac{\mathcal{O}(1)\delta_1}{(1+t)^2}\mathcal{H}_K(f,g)(t), \tag{3.40}$$

which together with the Gronwall's inequality yields

$$\mathcal{H}_K(f,g)(t) + \int_0^t \Lambda_h(f,g)(s)ds \le \mathcal{O}(1)\mathcal{H}_K(f,g)(0). \tag{3.41}$$

Immediately we have from (3.23) that

$$\mathcal{L}(f,g)(t) \le \mathcal{H}_K(f,g)(t) \le \mathcal{O}(1)\mathcal{H}_K(f,g)(0) \le \mathcal{O}(1)\mathcal{L}(f,g)(0), \tag{3.42}$$

i.e.

$$\|f(t) - g(t)\|_1 \leq \mathcal{O}(1)\|f_0 - g_0\|_1. \tag{3.43}$$

This ends the proof of Corollary 3.1. □

Finally combining Remark 3.1 and Corollary 3.1 leads to the proof of Theorem 1.2. Thus we are done.

Remark 3.3. If we consider the more realistic physical case, i.e. the VPB system in the phase space $\mathbf{R}_x^N \times \mathbf{R}_v^N$ with the dimension $N = 3$, then the uniform L^1 stability estimate (1.15) can not be obtained by using the same method in the proof of Theorem 1.2. In fact for $N = 3$, we can only obtain the local-in-time L^1 stability estimate under the same assumptions of Theorem 1.2 as follows:

$$\|(f - g)(t)\|_1 \leq (1 + t)^{\mathcal{O}(1)\delta_1}\|f_0 - g_0\|_1.$$

Even though we have solved the uniform L^1 stability only for $N \geq 4$, the analysis could be useful for the case $N = 3$ and it will shed some light on the stability analysis for more complicated system such as the Vlasov-Maxwell-Boltzmann system. We will pursue the proof of the uniform L^1 stability for these physically important models in the future.

4. Some Known Lemmas

The following lemmas are known and hence their proofs are omitted. Interested readers may refer to References[2,3,4,11,19,21,22] for details of proofs.

Lemma 4.1. *Let $\rho(x) \in L^1(\mathbf{R}^N) \cap W^{1,\infty}(\mathbf{R}^N)$ and $\phi = 1/|x|^{N-2} * \rho$. Then we have*

$$\|\phi\|_\infty \leq \mathcal{O}(1)\|\rho\|_1^{2/N}\,\|\rho\|_\infty^{(N-2)/N},$$

$$\|\nabla\phi_x\|_\infty \leq \mathcal{O}(1)\|\rho\|_1^{1/N}\,\|\rho\|_\infty^{(N-1)/N},$$

$$\|\nabla_x^2\phi\|_\infty \leq \mathcal{O}(1)\|\nabla_x\rho\|_\infty^{N\lambda}\,\|\rho\|_1^{\lambda}\,\|\rho\|_\infty^{1-(N+1)\lambda},$$

where $\lambda \in (0, 1/(1 + N))$ is any constant.

Lemma 4.2. *For any $\alpha > 0$ and $(x, y) \in \mathbf{R}^N \times \mathbf{R}^N$, we have that*

$$(1 + |y| + |y|^2)^{-\alpha} \leq \frac{h_\alpha(|x|)}{h_\alpha(|x + y|)} \leq (1 + |y| + |y|^2)^{\alpha}.$$

Lemma 4.3. *For any $\alpha > 1/2$ and $u \in \mathbf{R}^N$ with $u \neq 0$, we have that*

$$\sup_x \int_0^\infty h_\alpha(|x + su|)\,ds \leq \frac{\mathcal{O}(1)}{|u|}.$$

Lemma 4.4. *For any $0 \le \gamma < N$ and $\beta > N/2$, we have that*

$$\sup_v \int_{\mathbf{R}^N} \frac{1}{|v-u|^\gamma} m_\beta(|u|)du \le \mathcal{O}(1).$$

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