Recent Developments in Fractals and Related Fields

Julien Barral
Stéphane Seuret
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Editors
Julien Barral
LAGA (UMR CNRS 7539)
Département de Mathématiques
Institut Galilée, Université Paris 13
93430 Villetaneuse, France
barral@math.univ-paris13.fr

Stéphane Seuret
Laboratoire d'Analyse et de Mathématiques
Appliquées (UMR CNRS 8050)
Université Paris-Est
Bâtiment P3 4ème étage Bureau 441
94 010 CRETEIL Cedex, France
seuret@univ-paris12.fr

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Cantor Boundary Behavior of Analytic Functions

Xin-Han Dong[1*] and Ka-Sing Lau[2]

[1] Department of Mathematics, Hunan Normal University, Chang Sha, China
xhdong@hunnu.edu.cn
[2] Department of Mathematics, The Chinese University of Hong Kong, Hong Kong
kslau@math.cuhk.edu.hk

Summary. Let $\mathbb{D}$ be the open unit disc and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For $f(z)$ analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, it follows from the open mapping theorem that $\partial f(\mathbb{D}) \subset f(\partial\mathbb{D})$. These two sets have very rich and intrigue geometric properties. When $f(z)$ is univalent, then they are equal and there is a large literature to study their boundary behaviors. Our interest is on the class of analytic functions $f(z)$ for which the image curves $f(\partial\mathbb{D})$ form infinitely many loops everywhere, they are not univalent of course. We formulate this as the *Cantor boundary behavior*. We give sufficient conditions for such property, making use of the distribution of the zeros of f' and the mean growth rate of f'. Examples includes the complex Weierstrass functions, and the Cauchy transform of the canonical Hausdorff measure on the Sierpiski gasket.

1 Introduction

Let $\mathbb{D}$ be the open unit disk and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For f analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, it follows from the open mapping theorem that $\partial f(\mathbb{D}) \subset f(\partial\mathbb{D})$. These two sets have very rich and intriguing geometric properties. In fact, when f is conformal, then they are equal and there is a large literature on the study of their boundary behaviors; the reader can refer to Pommerenke [19] and Duren [8] for the classical developments, and to Lawler [13] for the more recent development in connection with the Brownian motion. Also, the well-known conjecture that the Mandelbrot set M is locally connected can be treated as a problem of boundary behavior of conformal maps, because the complement of M in $\mathbb{C}_\infty(= \mathbb{C} \cup \{\infty\})$ is the image of a conformal map f on $\mathbb{D}$ [2,7]. Hence, the problem is equivalent to whether the f can be extended continuously to the boundary of $\mathbb{D}$ [19].

Our interest is in the class of analytic functions f for which the image curve $f(\partial\mathbb{D})$ forms infinitely many loops everywhere; they are not univalent of course. Intuitively, for any open arc I on $\partial\mathbb{D}$, $f(I)$ contains at least one loop (which is inside $f(\mathbb{D})$). If we let $C = f^{-1}(\partial f(\mathbb{D}))$, then $C = \partial\mathbb{D} \setminus \bigcup_{i=1}^\infty I_i$,

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where I_i are open arcs of $\partial\mathbb{D}$, $f(I_i) \subset f(\mathbb{D})$, and $\overline{\bigcup_{i=1}^{\infty} I_i} = \partial\mathbb{D}$. The condition of loops everywhere implies that C is a nowhere dense close set (a Cantor-type set) and the image stretches out to be $f(C) = \partial f(\mathbb{D})$. This boundary behavior was first observed by Strichartz et al. [14] through some computer graphics of the Cauchy transform on the Sierpinski gasket (see Fig. 3).

We formulate this property as the *Cantor boundary behavior* on $\mathbb{D}$ and carry out an investigation via (a) the distribution of zeros of f' and (b) the fast mean growth rate of $|f'|$ for z near the boundary (faster than the well-known rate for univalent functions [18]). Our theorems allow us to use the infinite Blaschke product to construct examples with the Cantor boundary behavior. We show that the complex Weierstrass functions will have this property (see Fig. 1). In the fractal case, we show that the Cauchy transform of the canonical Hausdorff measure on the Sierpinski gasket also possesses this property, which answers the Cantor set conjecture in [14].

The detail of proofs will appear elsewhere.

2 The Basic Setup

The geometry of the curve $f(\partial\mathbb{D})$ can be very complicated, and there are difficulties in obtaining a precise meaning of "infinitely many loops" from the intuitive idea. Our approach is to use a weaker topological concept of the connected components determined by $f(\partial\mathbb{D})$.

By a component of a set E in a topological space, we mean a maximal connected subset of E. Let $K \subset \mathbb{C}$ be a compact subset, then $\mathbb{C}_\infty \setminus K$ has at most countably many components, they are simply connected if E is connected. Furthermore, if K is locally connected, then each component will have a locally connected boundary [21].

For Ω a bounded domain in $\mathbb{C}$, we will consider the components of $\mathbb{C}_\infty \setminus f(\partial\Omega)$ and $\Omega \setminus f^{-1}\big(f(\partial\Omega)\big)$. The former is used as a rigorous setup for the loose concept of loops of $f(\partial\Omega)$, and the second one divides Ω into connected subregions that map onto components of $\mathbb{C}_\infty \setminus f(\partial\Omega)$. These two classes of components play a key role in our consideration. In view of the facts stated in the last paragraph, we have the following.

Proposition 1. *Let Ω be a bounded simply connected domain. Let f be a non-constant analytic function in Ω and continuous on $\overline{\Omega}$. Suppose $\mathbb{C}_\infty \setminus f(\partial\Omega) = \bigcup_{j\geq 0} \mathcal{W}_j$ is the unique decomposition into components. Then*

(i) *Each $\mathcal{W}_j$ is a simply connected domain.*
(ii) *$f^{-1}(f(\partial\Omega))$ is connected and each component of $\Omega \setminus f^{-1}(f(\partial\Omega))$ is a simply connected domain.*

Let $n_f(w; K)$ denote the number of roots $z \in K$ for the equation $f(z) = w$, counting according to multiplicity. The more precise relationship of the components is as follows.

Proposition 2. *With the above assumption, suppose that* $\mathcal{W}_j \cap f(\Omega) \neq \emptyset$. *Let* $f^{-1}(\mathcal{W}_j) = \bigcup_{k=1}^{q_j} O_j^k$ *be the decomposition of the open set* $f^{-1}(\mathcal{W}_j)$ *into components. Then* $1 \leq q_j < +\infty$; *each* O_j^k *is a simply connected component of* $\Omega \setminus f^{-1}(f(\partial\Omega))$ *and*

$$f(O_j^k) = \mathcal{W}_j, \qquad f(\partial O_j^k) = \partial\mathcal{W}_j. \tag{1}$$

Moreover, for each $w \in \mathcal{W}_j$, $n_f(w; O_j^k) \equiv n_{j,k}$ *and* $\sum_{k=1}^{q_j} n_{j,k} \equiv n_f(w, \Omega)$.

If, in addition, $\partial\Omega$ *is locally connected, then all the* $\partial\mathcal{W}_j$ *and* ∂O_j^k *are locally connected.*

The above O_j^k has a close relationship with the zeros of f'.

Proposition 3. *With the above assumption and notation,* f' *has* $n_{j,k}-1$ *zeros in* O_j^k.

The proof depends on the following lemma and the Riemann mapping theorem.

Lemma 1. *Let* f *be analytic in* $\mathbb{D}$ *with* $f(\mathbb{D}) = \mathbb{D}$. *Suppose* $n_f(w; \mathbb{D}) \equiv k$ *for all* $w \in \mathbb{D}$; *then* f *is a finite Blaschke product of degree* k, *and* $f'(z)$ *has* $k-1$ *zeros in* $\mathbb{D}$.

We need a special result on the finite Blaschke product f, which provides a way to cut up the domain $\mathbb{D}$ into simply connected subregions so that f is univalent in each of the subregions. It will be applied to f from O_j^k onto $\mathcal{W}_j$ (Lemma 2). For clarity, we use $\mathbb{D}_z$ and $\mathbb{D}_w$ to denote the unit disk $\mathbb{D}$ as domain and range.

Proposition 4. *Let* f *be a Blaschke product of degree* k *and let* $\mathcal{Z}$ *be the set of zeros of* f' *in* $\mathbb{D}_z$. *Suppose* $f(\mathcal{Z}) \subset L$ *where* L *is a Jordan curve in* $\mathbb{D}_w$ *except for an end point* $\xi_0 \in \partial\mathbb{D}_w$. *Let* $G = \mathbb{D}_w \setminus L$ *(it is simply connected), and let* $f^{-1}(G) = \bigcup_{j=1}^{d} O_j$ *be the connected component decomposition as in Proposition 2. Then* $d = k$, *and* f *is univalent in* O_j *with* $f(O_j) = G$.

3 The Cantor Boundary Behavior

With the preceding notation, we can define the Cantor boundary behavior for f.

Definition 1. *Let* f *be analytic in* $\mathbb{D}$ *and continuous on* $\overline{\mathbb{D}}$. *We say that* f *has the* Cantor boundary behavior *if* $f^{-1}(\partial f(\mathbb{D}))$ *and* $\partial O \cap \partial\mathbb{D}$ *are Cantor type sets in* $\partial\mathbb{D}$ *(whenever it is non-empty) where* O *is any simply connected component of* $\mathbb{D} \setminus f^{-1}(f(\partial\mathbb{D}))$ *(as in Proposition 1).*

The geometric meaning of the definition is as follows: for $C := f^{-1}(\partial f(\mathbb{D})) \subset \partial\mathbb{D}$ to be a Cantor type set, $C = \partial\mathbb{D} \setminus \bigcup_{k=1}^{\infty} I_k$ where I_k are disjoint open arcs of $\partial\mathbb{D}$, with $\overline{\bigcup_k I_k} = \partial\mathbb{D}$ and $f(I_k) \subset f(\mathbb{D})$. Intuitively, the curve $f(I_k)$ forms a loop (closed curves) inside the image $f(\mathbb{D})$, and the outer boundary of the image $f(\mathbb{D})$ comes from the nowhere dense closed set C in $\partial\mathbb{D}$. The same explanation applies for $O_j^k \cap \partial\mathbb{D}$ with its image in the boundary of $f(O_j^k) = \mathcal{W}_j$ (as in Proposition 2). Putting these together, we can perceive that for each loop $f(I_k)$, there is another family of loops inside $f(I_k)$ with the Cantor boundary behavior, and inductively we can see that for $f(\partial\mathbb{D})$ there is an infinite family of loops inside loops.

Also, it is clear that the definition implies the following: for any subarc $I \subset \partial\mathbb{D}$, $f(I) \not\subseteq \partial\mathcal{W}$ for any component $\mathcal{W}$ of $\mathbb{C}_\infty \setminus f(\partial\mathbb{D})$.

Our main lemma is as follows.

Lemma 2. *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If there is a non-degenerated arc $J \subset \partial\mathbb{D}$ such that $f(J) \subset \partial f(\mathbb{D})$, then there exists a non-degenerated subarc $I \subset J$ and a bounded simply connected domain $D \subset \mathbb{D}$ such that $I \subset \partial D$, ∂D is locally connected, and f is univalent in D.*

Sketch of proof. Let $J = \{e^{i\theta} : 0 \le \theta_1 \le \theta \le \theta_2 < 2\pi\}$. We choose a Jordan curve γ such that $\gamma^o \subset \mathbb{D}$ and has two end points $e^{i\theta_1}, e^{i\theta_2}$. Let Ω be the closed region enclosed by the simple closed curve $J \cup \gamma$ and let $\widetilde{f} = f|\overline{\Omega}$. Then, by assumption, we have $\widetilde{f}(J) \subset \partial\widetilde{f}(\Omega)$. Let $\Gamma = \widetilde{f}(J \cup \gamma)$; then, by applying Propositions 1 and 2, we have the decompositions

$$C_\infty \setminus \Gamma = \bigcup_{j \ge 1} \mathcal{W}_j \quad \text{and} \quad \widetilde{f}^{-1}(\mathcal{W}_j) = \bigcup_{k=1}^{q_j} O_j^k.$$

As $\widetilde{f}(J) \subset \partial\widetilde{f}(\Omega)$, we can show that one of the O_j^k will contain a subarc $\ell \subset J$. We denote this simply connected domain by O^* and the corresponding $\mathcal{W}_j$ by $\mathcal{W}^*$.

Now consider $f : O^* \to \mathcal{W}^*$. By Proposition 2, $f(O^*) = \mathcal{W}^*$, $f(\partial O^*) = \partial\mathcal{W}^*$, and each $w \in \mathcal{W}^*$ has multiplicity, say, q. Let $\mathcal{Z}$ denote the $q-1$ zeros of f' in O^* and let L be a Jordan curve in $\mathcal{W}^*$ with one end point at $\partial\mathcal{W}^*$. We can apply Proposition 4 (through the Riemann mapping theorem) to divide O^* into simply connected regions $D_i, i = 1, \ldots, q$ and f is univalent on each of the regions. We select the one D_i such that $\ell \cap D_i$ is a non-degenerated arc of $\partial\mathbb{D}$. we denote this D_i by D, and the arc $\ell \cap D_i$ by I. □

We also need a similar lemma on the components.

Lemma 3. *Lemma 2 still holds if we replace the assumption $f(J) \subset \partial f(\mathbb{D})$ by $f(J) \subset \partial\mathcal{W}$ for some component $\mathcal{W}$ of $f(\mathbb{D}) \setminus f(\partial\mathbb{D})$.*

Now we can state our first theorem for the Cantor boundary behavior.

Theorem 1. *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose the set of limit points of $\mathcal{Z} = \{z \in \mathbb{D} : \ f'(z) = 0\}$ is $\partial\mathbb{D}$. Then f has the Cantor boundary behavior.*

The proof is simple by the two lemmas. We show that $C = f^{-1}(\partial f(\mathbb{D}))$ does not contain any subarc of $\partial\mathbb{D}$; this will imply that C is a Cantor-type set. Suppose otherwise, then there exists a circular arc $J = \{e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\} \subset f^{-1}(\partial f(\mathbb{D}))$. It follows that $f(J) \subset \partial f(\mathbb{D})$. By Lemma 2, there exists a simply connected domain $D \subset \mathbb{D}$ and a non-degenerated subarc $I \subset J$ such that $I \subset \partial D$ and f is univalent in D. Hence, $f'(z) \neq 0$ in D, i.e., $\mathcal{Z} \cap D = \emptyset$ and $\mathcal{Z}$ does not have a limit point in I^o. This is a contradiction, and therefore C is a Cantor set. The case for the components $\mathcal{W}_j$ follows from the same proof.

We can construct analytic functions with the Cantor boundary behavior explicitly using the theorem and the infinite Blaschke product. For example, we let $\theta_{k,m} = m/k$, $m = 1, 2, \ldots, k-1$, $k = 2, 3, \ldots$, and let $z_{k,m} = (1 - k^{-s})e^{i2\pi\theta_{k,m}}$. Since $\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}(1 - |z_{k,m}|) = \sum_{k=2}^{\infty}(k-1)k^{-s} < \infty$ if $s > 2$, then the Blaschke product

$$p_s(z) = \prod_{k=2}^{\infty}\prod_{m=1}^{k-1} \frac{|z_{k,m}|}{z_{k,m}} \frac{z_{k,m} - z}{1 - \overline{z}_{k,m} z}$$

converges uniformly for $|z| \leq r < 1$ and $|p_s(z)| \leq 1$ for $z \in \mathbb{D}$. For $s > 2$, we define $f(z) = \int_0^z p_s(\xi)d\xi$. Then f satisfies the assumptions in Theorem 1 and hence has the desired property.

In general, the zeros of f' are not easy to locate. We will give another sufficient condition of different nature for f to have the Cantor boundary behavior. It is related to the growth rate of the integral mean of $|f'|$.

Let $\mathcal{S}$ denote the class of all analytic functions f with $f(0) = 0, f'(0) = 1$, that are univalent in $\mathbb{D}$. For $\lambda > 0$, we define

$$\beta(\lambda) = \sup_{f \in \mathcal{S}} \left(\limsup_{r \to 1^-} \frac{\log\left(\int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta\right)}{-\log(1-r)} \right) \tag{1}$$

and call it the *integral mean spectrum* of $\mathcal{S}$ [18, 19]. A nice survey of this and related topics can be found in [3]. It follows easily that for any $f \in \mathcal{S}$ and for any fixed $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) > 0$ such that

$$\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta \leq \frac{C}{(1-r)^{\beta(\lambda)+\varepsilon}}, \quad \frac{1}{2} < r < 1. \tag{2}$$

The estimate of $\beta(\lambda)$ is a difficult problem. Up to now, the best upper bound estimate was given by Pommerenke:

$$\beta(\lambda) \leq \lambda - \frac{1}{2} + \left(4\lambda^2 - \lambda + \frac{1}{4}\right)^{1/2} < 3\lambda^2 + 7\lambda^3, \qquad \lambda > 0. \tag{3}$$

The lower bound was considered by Makarov [15], and a sharper estimate was given by Kayumov [12] more recently: $\beta(\lambda) \geq \frac{1}{5}\lambda^2$ for $0 < \lambda \leq \frac{2}{5}$.

Theorem 2. *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose, for any non-degenerated interval $I \subset [0, 2\pi]$, there exist $\kappa > 0, C > 0$, and $0 < r_0 < 1$ such that, for sufficiently small $\lambda > 0$,*

$$\int_I |f'(re^{i\theta})|^\lambda d\theta \geq \frac{C}{(1-r)^{\lambda\kappa}}, \qquad r_0 < r < 1. \tag{4}$$

Then f has the Cantor boundary behavior.

Note that, by assumption, when $\lambda > 0$ is small, the mean growth rate of $|f'|$ is greater than the rate for all the univalent functions in $\mathcal{S}$ (i.e., $\lambda\kappa > 3\lambda^2 + 7\lambda^3$ in (3)). This, together with Lemmas 2 and 3 and a contrapositive argument (using the Riemann mapping theorem on D), yields the theorem.

4 The Complex Weierstrass Functions

In the following we consider the class of complex Weierstrass functions:

$$f(z) := f_{q,\beta}(z) = \sum_{n=1}^{\infty} q^{-\beta n} z^{q^n}, \ z \in \overline{\mathbb{D}},$$

where $0 < \beta < 1$ and $q \geq 2$ is an integer. It is well known that f is a Lipschitz function of order β and the Hausdorff dimension of $f(\partial\mathbb{D})$ is $1 < 1/\beta < 2$ for $\beta > \frac{1}{2}$ [9].

For $0 \leq \theta < 2\pi$, $0 < \alpha < \pi/2$, $\tau > 0$, we let

$$S_\alpha(\theta, \tau) = \{z : |z - e^{i\theta}| \leq \tau, \ |\arg(1 - e^{-i\theta} z)| \leq \alpha\}$$

to denote the *Stolz angle* at $e^{i\theta}$. By some rather delicate estimations, we show that the class of $f_{q,\beta}$ satisfies the following lemma.

Lemma 4. *For $\theta_{k,m} := 2\pi m q^{-k}$ with $m = 0, \ldots, q^k - 1$, $k = 1, 2, \ldots$, there exist $C > 0$, $0 < \alpha < 1$, and $0 < \tau_k < \delta q^{-k}$ such that*

$$\mathrm{Re}\big(e^{i\theta_{k,m}} f'(z)\big) \geq \frac{C}{(1-|z|)^{1-\beta}}, \qquad z \in S_\alpha(\theta_{k,m}, \tau_j) \setminus \{e^{i\theta_{k,m}}\}.$$

In order to apply Theorem 2, it is more convenient to modify the integral mean growth condition to be a discretized growth condition of $|f'|$.

Lemma 5. *For $\theta_{k,m} := 2\pi m q^{-k}$ with $m = 0, \ldots, q^k - 1$, $j = 1, 2, \ldots$, suppose there exist $\kappa > 0$, $\delta > 0$, and $\eta \in (0, \pi/2)$ such that*

$$|f'(z)| \geq c(1 - |z|)^{-\kappa} \tag{1}$$

for $z \in S_\eta(\theta_{k,m}, \delta/2^k)$ and $\delta/2^{k+1} \leq 1 - |z| < \delta/2^k$. Then the integral mean condition in (4) of Theorem 2 is satisfied.

By using the two lemmas and Theorem 2, we prove the following.

Theorem 3. *For $0<\beta<1$, $q\geq 2$ an integer, the complex Weierstrass function $f_{q,\beta}$ has the Cantor boundary behavior.*

In Fig. 1, we display some graphics of the complex Weierstrass functions $f(z) = \sum_{n=1}^{\infty} q^{-\beta n} z^{q^n}$ for different values of q and β. It is seen that the

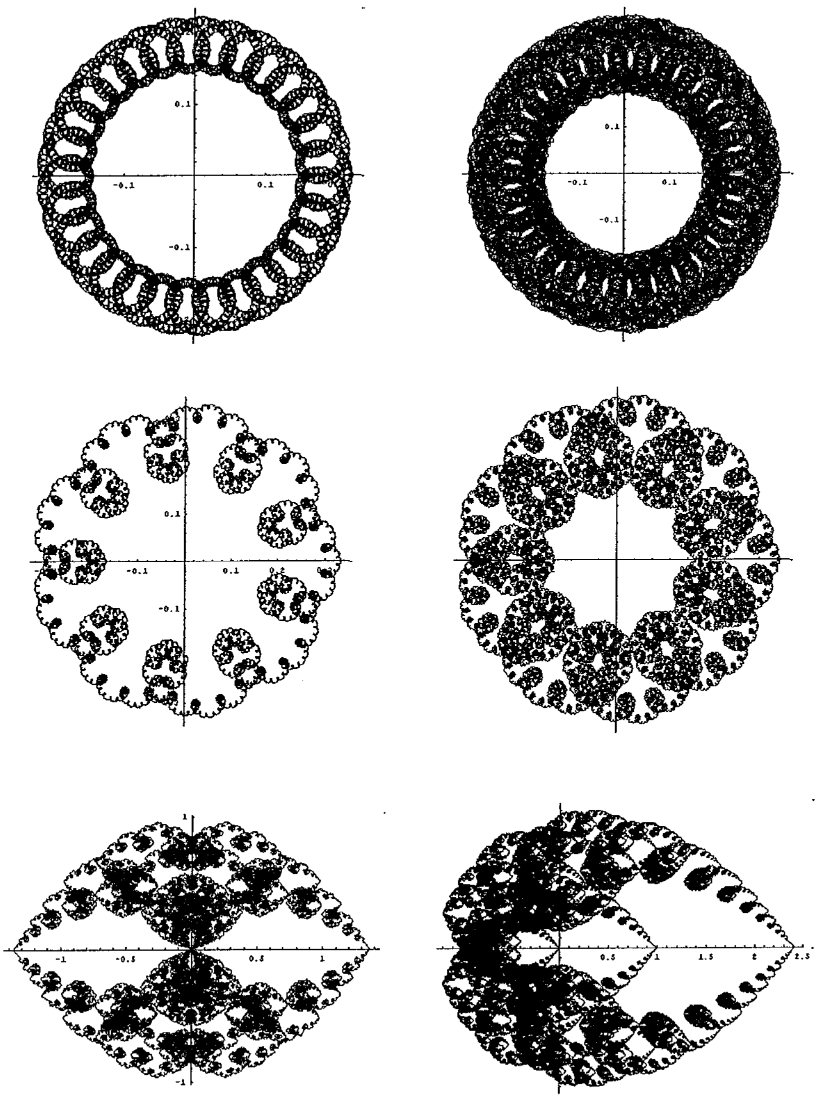

Fig. 1. The images $f_{q,\beta}(\partial\mathbb{D})$. The first two are $q=30, \beta=0.5, 0.4$; the second two are $q=10, \beta=0.6,\ 0.5$; the last two are $q=3,2,\ \beta=0.5$

number of pedals depends on q. Indeed, there are $q-1$ symmetric pedals due to $q^n \equiv 1 \pmod{q-1}$. The curve $f(\partial\mathbb{D})$ can be space-filling on some regions inside $f(\mathbb{D})$, as was first observed by Salem and Zygmund. In [1], Barański proved this further for $q \geq 2$ and for β sufficiently close to 0. It is seen from the picture that as β is closer to 0, the mean growth rate of $|f'|$ is larger and the curve $f(\mathbb{D})$ loops more violently.

In our theorem, we make use of the fact that the gap ratio q of the series is an integer and the coefficients are a geometric progression. We do not know whether the more general lacunary series still have the Cantor boundary behavior. Also, it is well known that for Ref and Imf, the box dimension of the graph is known to be $2-\beta$ [9]; however, the question for the Hausdorff dimension is still open (see [10,11,17]). It is seen that $f(\partial\mathbb{D})$ is a fractal curve, and it will be interesting to find the dimension in connection with the results in [20] and [1], and in particular for the dimension or Hausdorff measure of the Cantor set C and the outside boundary of the image $f(C)(=\partial f(\mathbb{D}))$.

5 Cauchy Transform on Sierpinski Gasket

Let $S_k z = \varepsilon_k + (z-\varepsilon_k)/2,\ k = 0,1,2$, where $\varepsilon_k = e^{2k\pi i/3}$. The attractor of this iterated function system $\{S_k\}_{k=0}^2$ is the Sierpinski gasket K (see Fig. 2). Recall that the α-Hausdorff measures satisfies $\mathcal{H}^\alpha(2E) = 2^\alpha \mathcal{H}^\alpha(E)$. For $\mu = \mathcal{H}^\alpha|_K$, where μ is a self-similar measure and normalized to 1, it satisfies $\mu = 3^{-1}\sum_{j=0}^2 \mu \circ S_i^{-1}$ [11,16]. The Cauchy transform of $\mu = \mathcal{H}^\alpha|_K$ is defined by

$$F(z) = \int_K \frac{d\mathcal{H}^\alpha(w)}{z-w}.$$

It is clear that F is analytic away from K and $F(\infty) = 0$. In [14], Strichartz et al. showed that F has a unique extension to be a Hölder continuous function over K of order $\log 3/\log 2 - 1$ (see also [4,5]). Let Δ_0 be the unbounded connected component of $\mathbb{C} \setminus K$, then $F(\Delta_0)$ is a bounded domain. In [14] they

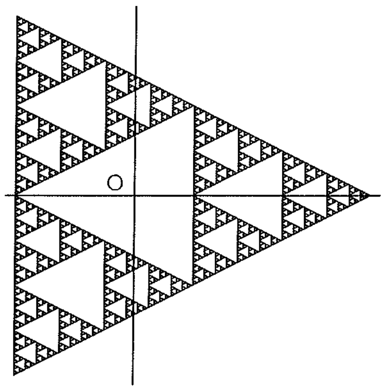

Fig. 2. The Sierpinski gasket K with vertices 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$

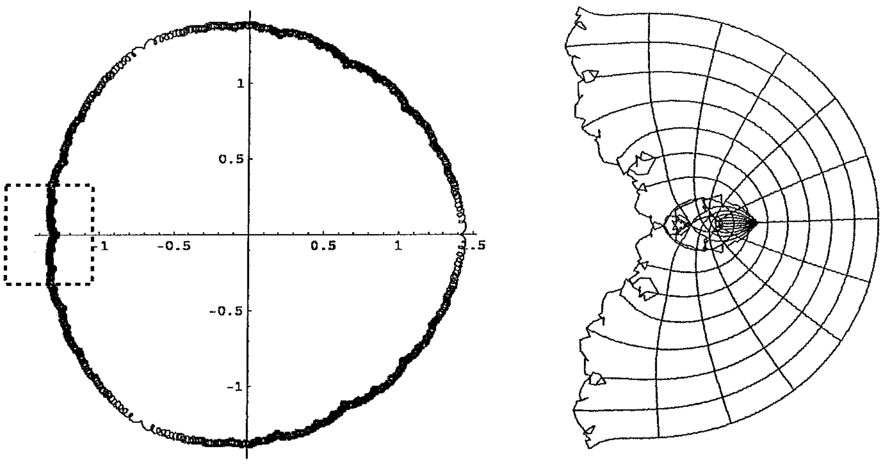

Fig. 3. The image of the outer triangle of K under $F(z)$; the picture on the right is the magnification around $F(-\frac{1}{2})$

also observed from computer graphics that the image $F(\partial\Delta_0)$ is a curve consisting of infinitely many fractal-looking loops (see Fig. 3), and they proposed the *Cantor set conjecture: there exists a Cantor-type set $C \subset \partial\Delta_0$ such that* $F(C) = \partial F(\Delta_0)$. This is actually the motivation of our investigation of the Cantor boundary behavior.

By symmetry we only consider the vertical line segment $\partial\Delta_0$; the dyadic points $z_{k,m}$ (not including the two end points) are of the form: for $1 \le m \le 2^k - 1$ and $k \ge 1$,

$$z_{k,m} = \frac{m}{2^k}\varepsilon_1 + \left(1 - \frac{m}{2^k}\right)\varepsilon_2 = -\frac{1}{2} + \frac{m - 2^{k-1}}{2^k}\sqrt{3}i. \tag{1}$$

For $\theta \in (0, \pi/2]$ and $r > 0$, we use the notation

$$\Omega(\theta) = \{z : \ |\arg z - \pi| < \theta\} \quad \text{and} \quad \Omega(\theta; r) = \{z \in \Omega(\theta) : |z| < r\}$$

to replace the Stolz angle on $\mathbb{D}$.

Theorem 4. *There exists a function $\mathcal{G}$ such that, for any $z_{m,k}$,*

$$F(z + z_{k,m}) = F(z_{k,m}) + \mathcal{G}(z)z^{\alpha-1} + zp_{k,m}(z), \quad 0 < \arg z < 2\pi,$$

where

(i) *$\mathcal{G}$ is continuous on $\mathbb{C}\backslash\{0\}$, analytic in $\Omega(\pi/2)$, and $\mathcal{G}(2z) = \mathcal{G}(z)$ in $0 \le \arg z < 2\pi$.*

(ii) *$p_{k,m}(z)$ is bounded continuous on $\mathbb{C}$, and analytic in $\Omega(\pi/2) \cup \{z : \ |z| < 3/2^{k+1}\}$.*

From this we can draw the following conclusion on the growth rate of F near $\partial\Delta_0$.

Proposition 5. *There exists $C > 0$ such that*

$$\max_{dist\ (z,K)\geq t} |F'(z)| \leq Ct^{\alpha-2};$$

and the order is attained at the dyadic points of $\partial\Delta_0$, in the sense that there exists $0 < \eta < \pi/2, \delta > 0$ and $c > 0$ such that for any $z \in \Omega(\eta; 2^{-k}\delta)$,

$$|F'(z + z_{k,m})| \geq c|z|^{\alpha-2}.$$

Let φ be the Riemann mapping that transforms the closed unit disk $\overline{\mathbb{D}}$ onto $\overline{\Delta_0} \cup \{\infty\}$ conformally. We can use Proposition 5 to show that Lemma 5 (with a slight modification on the $\theta_{k,m}$) is satisfied. Hence, $f(z) = F(\varphi(z))$ satisfies the growth rate condition in Theorem 2. Therefore, we have the following theorem which answers the Cantor set conjecture proposed by Strichartz et al. in [14].

Theorem 5. *The Cauchy transform F has the Cantor boundary behavior.*

The main idea in the proof of Theorem 4 and Proposition 5 is to make use of the following auxiliary functions:

$$g_k(z) = \int_{A_k} \frac{d\mathcal{H}^\alpha(w)}{w(z-w)}, \qquad H_k(z) = \int_{A_k} \frac{d\mathcal{H}^\alpha(w)}{(z-w)^2}$$

with $0 \leq k \leq 5$, where $A_k = e^{k\pi i/3}A_0$ and A_0 is the "Sierpinski cone" generated by the relocated gasket $T = 1 - K$ with vertex at 0 (see Fig. 4).

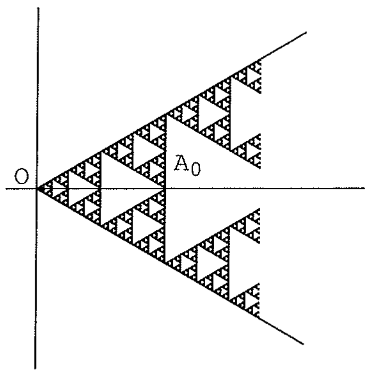

Fig. 4. The Sierpinski cones

These functions have the multiplicative periodic property (period 2). Formally $(zg_k(z))' = -H_k(z)$. The bounded function $\mathcal{G}$ in Theorem 4 is given by

$$\mathcal{G}(z) = z^{2-\alpha}\big(g_1(z) + g_5(z)\big).$$

The H_k's are used in the derivative F' in Proposition 5:

$$F'(z + z_{k,m}) = -(H_1(z) + H_5(z)) + O(1)$$

as $z \in \Omega(\pi/2)$ and $z \to 0$.

From the self-similar property of F, we see that there are "loops inside loops" in the image $F(\partial\Delta_0)$ (Fig. 2). The image points in these loops have multiplicity (from Δ_0) at least 2 and can be any large number. It is natural to ask whether the area of the Riemann region $F(\Delta_0)$ (counting according to multiplicity) is finite. We prove the following.

Theorem 6. *The area of the Riemann region $F(\Delta_0)$ is finite, but it is infinite for $F(\mathbb{C} \setminus K)$.*

The Cantor boundary behavior suggests that $F(\partial\Delta_0)$ is a fractal curve. Indeed, observe that $F(z)$ is Hölder continuous of order $\alpha - 1$ on K. We have immediately (by [9, p. 29]) the following proposition.

Proposition 6. $\dim_{\mathcal{H}} F(\partial\Delta_0) \leq (\alpha - 1)^{-1} (\approx 1.70951)$.

On the other hand, by using Theorem 4,

$$F(z + z_{m,k}) = F(z_{m,k}) + \mathcal{G}(z) z^{\alpha-1} + O(z), \tag{2}$$

we see that the order $\alpha - 1$ is attained on a dense subset of $\partial\Delta_0$. It is natural to make the following conjecture:

The box dimension and the Hausdorff dimension of $F(\partial\Delta_0)$ are $(\alpha-1)^{-1}$.

Let $\mathrm{Gr}(f; I) = \{(t, f(t)) : t \in I\}$ denote the graph of f on an interval I. It is known that if f is Hölder continuous of order $0 < s \leq 1$, then the upper box dimension $\overline{\dim}_B \mathrm{Gr}(f; I) \leq 2 - s$. It is easy to show ([9, p. 146]) that if there exists $c > 0$ such that for any dyadic subinterval $I_{k,m} \subset I$, $m = 0, \dots, 2^k - 1$, $k > 0$,

$$\mathrm{Osc}_f(I_{m,k}) \geq c 2^{-sk},$$

then $\underline{\dim}_B \mathrm{Gr}(f; I) \geq 2 - s$. Based on this and the estimation on the oscillation of $\mathrm{Re}F(z)$ and $\mathrm{Im}F(z)$, we have the following.

Proposition 7. $\dim_B \mathrm{Gr}(\mathrm{Re}F; \partial\Delta_0)$ *and* $\dim_B \mathrm{Gr}(\mathrm{Im}F; \partial\Delta_0)$ *are* $3 - \alpha$.

We do not know if the Hausdorff dimension of the graphs of ReF and ImF is $3 - \alpha$. This question may be difficult, as the approximating function $\mathcal{G}$ in (2) has a series expression $\sum_{n \in \mathbb{Z}} 2^{(\alpha-2)} \phi(2^{-n} z)$ [6]. It is analogous to the well-known Weierstrass function, and as already mentioned, the Hausdorff dimension of its graph is still unsolved.

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