Differentiability of pressure functions for products of non-negative matrices*†

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Abstract

For nonnegative matrices $M_1, \cdots, M_m$, we define the pressure function $P(q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|J|=n} \|M_J\|^q$. We prove that if $\sum_{i=1}^m M_i$ is irreducible, then $P(q)$ is differentiable. The result is important when we consider the multifractal formalism for the iterated function systems with overlaps. The proof is a simplification of an earlier version of the authors [FL] on the more general setting of continuous matrix-valued functions on the subshift of finite type.

Contents

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1 Introduction

Given a family of non-negative $d \times d$ real matrices $\{M_1, \cdots, M_m\}$, we define the *pressure function* $P(q)$ by

$$P(q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|J|=n} \|M_J\|^q, \qquad q > 0 \tag{1.1}$$

where $M_J = M_{j_1} \ldots M_{j_n}$ for $J = j_1 \ldots j_n \in \{1, \ldots m\}^n$, $\|\cdot\|$ denotes the matrix norm defined by $\|A\| := \mathbf{1}^t A \mathbf{1}$, $\mathbf{1}^t = (1, \cdots, 1)$. The existence of the limit in the definition follows from a subadditive argument.

The pressure function of the scalar case (in terms of the potential functions) was studied in great detail in statistical mechanics [R] and dynamical systems ([B], [P]) in conjunction with the entropy and variational principle; it has also been used to study the multifractal structure of the self-similar (or self-conformal) measures generated by iterated function systems (IFS) with no overlap (the *open set condition*)([MU], [FL]). In those cases, the pressure functions under consideration are differentiable (actually real analytic). This property is essential to investigate the phase transition in thermodynamics, and for the validity of the multifractal formalism in the dimension theory of fractals.

In the recent investigation of the self-similar measures generated by iterated function systems with overlaps, it is known that in many interesting cases, the measure can be put into a vector form with a new non-overlapping IFS and with matrix weights ([LN 1,2], [LNR], [Fe1,2], [FeO]). In this way the related multifractal formalism depends on the differentiability of the above $P(q)$ ([LN2], [LW], [Fe1,2], [FeLW]). In another direction, the expression of the matrix product in (1.1) also appears in the study of the scaling functions in wavelet theory (the matrices are allowed to be non-positive) in the form of the L^q-joint spectral radius and the L^q-Lipschitz exponent ([DL], [LM]) and the problem of differentiability of the $P(q)$ also appears there. For a survey of these, the reader can refer to [L].

A systematic study of the differentiability of the pressure functions of product of non-negative matrices was carried out in [FL] (see also [Fe3]). The setup is in the more general subshift of finite type (Σ_A, σ) where $\Sigma = \{1, 2, \cdots, m\}^{\mathbb{N}}, m \geq 2$, σ is the shift map on Σ, A is an $m \times m$ matrix with entries 0 and 1. For a Hölder continuous matrix-valued function $M(x)$ on Σ_A, the pressure function is defined by

$$P(q) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{J\in\Sigma_{A,n}} \sup_{x\in[J]} \|M(x)M(\sigma x)\cdots M(\sigma^{n-1}x)\|^q$$

where $[J]$ is the cylinder set determined by J. The reason of this generality is to include the consideration in dynamical system. The proof is however more complicated due to some technicalities. In view of the many unsolved problems and the major applications, it is worthwhile to reduce to the special case (1.1) and give a simpler and more transparent proof. The main theorem is

Theorem 1.1. *Let $M_1, \ldots, M_m$ be non-negative $d \times d$ matrices such that $H = \sum_{i=1}^m M_i$ is irreducible. Then the pressure function $P(q)$ is differentiable for $q > 0$.*

The proof is to apply a technique of Brown, Michon and Peyriere [BMP] and Carleson [C] to construct, from the given matrices, a certain ergodic measure that possesses the "Gibbs property"; the differentiability of $P(q)$ is obtained via such measure by extending an idea of Heurteaux [H].

We note that if H is not irreducible, then Theorem 1.1 is not true. Some examples and remarks will be given in the last section. As an application we let

$$E(\alpha) := \left\{ J = (j_i) \in \Sigma : \lim_{n\to\infty} \frac{\log \|M_{j_1} \cdots M_{j_n}\|}{n} = \alpha \right\}$$

where $\Sigma = \{1, \cdots m\}^{\mathbb{N}}$. We prove the following dimension formula which is in essence derived from the variational principle.

Theorem 1.2. *Under the same assumption of Theorem 1.1, we have for any $\alpha = P'(q)$, $q > 0$,*

$$\dim_H E(\alpha) = \frac{1}{\log m}(-\alpha q + P(q)).$$

To relate this to the classical random product of matrices, we let $\{Y_n\}$ be the i.i.d. random variables that take values $M_1, \ldots, M_m$ with uniform distribution, then under certain assumption on the M_j's,

$$\lim_{n\to\infty} \frac{1}{n} \log \|Y_n \ldots Y_1\| = \lambda \quad \text{a.s.}$$

and λ is called the upper Lyapunov exponent ([FK], [BL, Chapter 1]). In comparison with the above theorem, it corresponds to the case for $q = 0$ and $\lambda = P'(0)$ (it exists under the additional assumptions on the M_j ([BL, Theorem 4.3]) and $\dim_H E(\lambda) = P(0)/\log m = 1$.

2 Basic setup.

Let $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ be the symbolic space endowed with the standard metric $d(I, J) = m^{-n}$ where $I = (i_k)$, $J = (j_k)$ and n is the smallest of the k such that $i_k \neq j_k$. Let σ be the shift operator on Σ defined by $\sigma(i_1, i_2, \ldots) = (i_2, i_3, \ldots)$. For each positive integer n, we use Σ_n to denote the set of all the indices of length n over $1, \ldots, m$. Let $\Sigma^* = \bigcup_{n \geq 1} \Sigma_n$. For $I, J \in \Sigma^*$, we write $|I|$ as its length and IJ as the concatenation. For $J = j_1 \ldots j_n \in \Sigma^*$, we let $[J] = \{(i_k) \in \Sigma : \ i_k = j_k, \ 1 \leq k \leq n\}$.

For any two families of positive numbers $\{a_i\}_{i \in \mathcal{I}}$, $\{b_i\}_{i \in \mathcal{I}}$, we write, for brevity, $a_i \approx b_i$ to mean the existence of a constant $C > 0$ such that $C^{-1} a_i \leq b_i \leq C a_i$ for all $i \in \mathcal{I}$.

Proposition 2.1. *There exists a σ-invariant measure μ on Σ such that*

$$\mu([i_1 \ldots i_n]) \approx \rho^{-n} \|M_{i_1} \ldots M_{i_n}\| \qquad \forall\, n > 0, \ i_1 \ldots i_n \in \Sigma_n \tag{2.1}$$

where ρ is the largest eigenvalue of $H := \sum_{i=1}^{m} M_i$.

Proof. Since H is an irreducible non-negative matrix, by the Perron-Frobenius theorem, there exist two strictly positive column vectors $\mathbf{u}$, $\mathbf{v}$ such that

$$\mathbf{u}^t H = \rho \mathbf{u}^t \qquad \text{and} \qquad H\mathbf{v} = \rho \mathbf{v}$$

with $\mathbf{u}^t \mathbf{v} = 1$. By the Kolmogorov consistence theorem, there is an invariant measure μ on Σ such that

$$\mu([i_1 \ldots i_n]) = \rho^{-n} \mathbf{u}^t M_{i_1} \ldots M_{i_n} \mathbf{v} \qquad \forall\, n > 0, \ i_1 \ldots i_n \in \Sigma_n. \tag{2.2}$$

It is easy to see that $\mathbf{u}^t M_{i_1} \ldots M_{i_n} \mathbf{v} \approx \|M_{i_1} \ldots M_{i_n}\|$ and the desire result follows. $\square$

The measure μ will play a significant role in the study of the pressure function $P(q)$. It is actually uniquely determined by the property (2.1), the proof will be given in the last section of remarks, making use of some facts in Section 3. As an immediate consequence of (2.2), we have

Corollary 2.2. *There is a positive constant C such that*

$$\mu([IJ]) \leq C\mu([I])\mu([J]) \qquad \forall\, I, J \in \Sigma^*. \tag{2.3}$$

For $q \in \mathbb{R}$, let $\tau(q) := \tau_\mu(q)$ be the L^q-spectrum of μ, i.e.,

$$\tau(q) = \liminf_{n\to\infty} \frac{\log \sum_I \mu([I])^q}{\log m^{-n}}$$

where the summation is taken over all $I \in \Sigma_n$ with $\mu([I]) > 0$. By Proposition 2.1, we have

$$\tau(q) = \frac{1}{\log m}(q \log \rho - P(q)) \qquad \forall \quad q > 0.$$

Therefore to prove that $P(q)$ is differentiable for $q > 0$, it suffices to prove that for $\tau(q)$. The proof depends on the following result of Heurteaux [H].

Proposition 2.3. *Let ν be a Borel probability measure on Σ. Assume that there exists a constant $C > 0$ such that*

$$\nu([IJ]) \leq C\nu([I])\nu([J]) \qquad \forall\, I,\ J \in \Sigma^*. \tag{2.4}$$

Then $\tau_\nu'(1)$ exists if ν is a Young measure (i.e., $\lim_{n\to\infty} \frac{\log \nu(I_n(x))}{\log m^{-n}} =$ constant *for ν almost all $x = (j_i) \in \Sigma$ and $I_n(x) = [j_i \cdots j_n]$).*

The main idea is to make use of the estimation of the dimensions:

$$\dim_*(\nu) := \sup\{\alpha > 0 : \dim(E) < \alpha \Rightarrow \nu(E) = 0\}$$

and

$$\operatorname{Dim}^*(\nu) := \inf\{\operatorname{Dim}(E) : \nu(\mathbb{R}^d \setminus E) = 0\}$$

where $\dim(E)$ and $\operatorname{Dim}(E)$ denote the Hausdorff dimension and the packing dimension of E respectively. If we let

$$\underline{D}(\nu, x) = \liminf_{n\to\infty} \frac{\log \nu(I_n(x))}{\log m^{-n}}$$

and let $\overline{D}(\nu, x)$ be defined similarly, then $\dim_*(\nu) = \text{ess inf}\{\underline{D}(\nu, x)\}$ and $\operatorname{Dim}^*(\nu) = \text{ess sup}\{\overline{\mathrm{D}}(\nu, \mathrm{x})\}$ ([F], [FLR]). It was proved in [H, Theorem 1.3] (also [N]) that

$$\tau_+'(1) \leq \dim_*(\nu) \leq \operatorname{Dim}^*(\nu) \leq \tau_-'(1).$$

Under the assumption (2.4), we have $\tau'_+(1) = \dim_*(\nu)$ and $\mathrm{Dim}^*(\nu) \leq \tau'_-(1)$ [H, Theorem 2.1]. Furthermore if ν is a Young measure, then the two dimensions are equal and hence $\tau'(1)$ exists (it is the entropy of ν).

For more detail of the relationship of the dimensions, the reader can refer to [F], [FLR]. Proposition 2.3 is a kind of inverse of a result of Ngai [N]. In fact, Ngai proved that the existence of $\tau'_\nu(1)$ implies that ν is a Young measure, without the assumption (2.4). By using Proposition 2.3, Heurteaux proved that $\tau_\nu(q)$ is differentiable on $\mathbb{R}$ if ν has the so-called "quasi-Bernoulli" property: there exists $C > 0$ such that

$$C^{-1}\nu([I])\nu([J]) \leq \nu([IJ]) \leq C\nu([I])\nu([J]) \qquad \forall\, I,\ J \in \Sigma^*.$$

We point out that under our setting, the measure μ may not be quasi-Bernoulli unless all the entries of the matrices M_i are positive (see the examples and remarks in Section 4). However, in the following lemma, we can still obtain a weaker form of the left inequality and apply an idea of [H] to retrieve the differentiability of $\tau(q)$ for $q > 0$.

It is easy to show that the given assumption on the irreducibility of H is equivalent to the existence of an $r \in \mathbb{N}$ such that $\sum_{k=1}^r H^k > 0$.

Lemma 2.4. *For $q > 0$, there exists a constant $C' > 0$ (depends on q and r) such that*

$$\sum_{k=1}^{r} \sum_{K\in\Sigma_k} \mu([IKJ])^q \geq C'\mu([I])^q\mu([J])^q \qquad \forall\, I,\ J \in \Sigma^*. \tag{2.5}$$

Proof. Let $I \in \Sigma_\ell,\ J \in \Sigma_n$. By (2.2), we have

$$\begin{aligned}\sum_{k=1}^{r} \sum_{K\in\Sigma_k} \mu([IKJ]) &= \sum_{k=1}^{r} \sum_{K\in\Sigma_k} \rho^{-\ell-k-n}\mathbf{u}^t M_I M_K M_J \mathbf{v} \\ &= \rho^{-\ell-n}\mathbf{u}^t M_I\Big(\sum_{k=1}^{r} \rho^{-k}H^k\Big)M_J\mathbf{v}.\end{aligned}$$

Since $\sum_{k=1}^r H^k$ is positive, so is $\sum_{k=1}^r \rho^{-k}H^k$; there is a constant $C_1 > 0$ such that $\sum_{k=1}^r \rho^{-k}H^k \geq C_1\mathbf{v}\mathbf{u}^t$, which implies

$$\sum_{k=1}^{r} \sum_{K\in\Sigma_k} \mu([IKJ]) \geq \rho^{-n-\ell}\mathbf{u}^t M_I(C_1\mathbf{v}\mathbf{u}^t)M_J\mathbf{v} = C_1\mu([I])\mu([J]).$$

Let $K_0 \in \bigcup_{k=1}^r \Sigma_k$ such that $\mu(IK_0J)$ maximizes the $\mu(IKJ)$'s, then $rm^r\mu([IK_0J]) \geq C_1\mu([I])\mu([J])$. It follows that

$$\sum_{k=1}^{r}\sum_{K\in\Sigma_k} \mu([IKJ])^q \geq \mu(IK_0J)^q \geq C_1^q r^{-q} m^{-qr}\mu([I])^q\mu([J])^q,$$

which concludes the proof by letting $C' = C_1^q r^{-q} m^{-qr}$. □

3 Proof of the theorem

For $q > 0$, $n \in \mathbb{N}$, we define $s_n(q) = \sum_{I\in\Sigma_n}\mu([I])^q$, then $\tau(q) = \liminf_{n\to\infty} \log s_n(q)/\log m^{-n}$. It follows from Lemma 2.4 that

Lemma 3.1. *For $q > 0$, there exists a constant $C' > 0$ (depends on q and r) such that for $n \geq r$ and $I \in \Sigma^*$,*

$$\sum_{J\in\Sigma_n}\mu([IJ])^q \geq C's_n(q)\mu([I])^q \qquad \textit{and} \qquad \sum_{J\in\Sigma_n}\mu([JI])^q \geq C's_n(q)\mu([I])^q.$$

Proof. Observe that for any $1 \leq k \leq r$ and $I \in \Sigma^*$,

$$\sum_{J\in\Sigma_n}\mu([IJ])^q = \sum_{K\in\Sigma_k}\sum_{J'\in\Sigma_{n-k}}\mu([IKJ'])^q \geq m^{-r}\sum_{K\in\Sigma_k}\sum_{J''\in\Sigma_n}\mu([IKJ''])^q$$

(the last inequality follows from $\mu([L]) \geq \mu([LK_1])$ for any $K_1 \in \Sigma_k$). Therefore by Lemma 2.4 we have

$$\begin{aligned}\sum_{J\in\Sigma_n}\mu([IJ])^q &\geq r^{-1}m^{-r}\sum_{k=1}^{r}\sum_{K\in\Sigma_k}\sum_{J''\in\Sigma_n}\mu([IKJ''])^q \\ &\geq r^{-1}m^{-r}C_1\mu([I])^q\sum_{J''\in\Sigma_n}\mu([J''])^q = C's_n(q)\mu([I])^q.\end{aligned}$$

The proof of the second part is similar, we need only adjust the inequality in the parenthesis by $\mu([L]) = \mu(\sigma^{-k}([L])) \geq \mu([K_1L]), K_1 \in \Sigma_k$, using the σ-invariant property of μ. □

Lemma 3.2. *For $q > 0$ and for any $\ell, n \in \mathbb{N}$, we have* (i) $s_{\ell+n}(q) \approx s_\ell(q)s_n(q)$; (ii) $s_n(q) \approx m^{-n\tau(q)}$.

Proof. From Lemma 3.1 and Corollary 2.2, there exist $C, C' > 0$ such that

$$C' s_\ell(q) s_n(q) \le s_{\ell+n}(q) \le C s_\ell(q) s_n(q),$$

which proves (i). To prove (ii), we can write $C s_{\ell+n}(q) \le (C s_\ell(q))\,(C s_n(q))$ (by (i)). Hence the subadditivity property implies

$$(-\log m)\tau(q) = \lim_{n\to\infty} \frac{\log C s_n(q)}{n} = \inf_n \frac{\log C s_n(q)}{n},$$

so that $C^{-1} m^{-n\tau(q)} \le s_n(q)$. The reverse inequality in (ii) follows from a similar argument. □

For each integer $n > 0$, let $\mathcal{B}_n$ be the σ-algebra generated by the cylinders $[I]$, $I \in \Sigma_n$. We define a sequence of probability measures $\{\nu_{n,q}\}$ on $\mathcal{B}_n$ by

$$\nu_{n,q}([I]) = \frac{\mu([I])^q}{s_n(q)} \qquad \forall\, I \in \Sigma_n. \tag{3.1}$$

Then there is a subsequence $\{\nu_{n_k,q}\}_{k\ge 1}$ converges in the weak-star topology to a probability measure ν_q. The following assertion shows that ν_q has the "Gibbs property".

Lemma 3.3. *$\nu_q([I]) \approx \mu([I])^q m^{n\tau(q)}$ for all $n > 0$, $I \in \Sigma_n$.*

Proof. Take any $I \in \Sigma_n$ and $\ell > n + r$, we have

$$\begin{aligned} \nu_{n,q} &= \sum_{J\in\Sigma_{\ell-n}} \nu_{\ell,q}([IJ]) = \sum_{J\in\Sigma_{\ell-n}} \frac{\mu([IJ])^q}{s_\ell(q)} \\ &\approx \sum_{J\in\Sigma_{\ell-n}} \frac{\mu([I])^q \mu([J])^q}{s_\ell(q)} \qquad \text{(by (2.3) and Lemma 3.1)} \\ &= \frac{\mu([I])^q s_{\ell-n}(q)}{s_\ell(q)} \approx \mu([I])^q m^{n\tau(q)} \qquad \text{(by Lemma 3.2).} \end{aligned}$$

Letting $\ell = n_k \nearrow \infty$, we obtain the desired result. □

We can strengthen the above measure to be an ergodic measure.

Proposition 3.4. *For each $q > 0$, there exists an ergodic measure η_q on Σ such that*

$$\eta_q([I]) \approx \mu([I])^q m^{n\tau(q)} \qquad \forall\, n > 0,\ I \in \Sigma_n. \tag{3.2}$$

Proof. Let η_q be a limit point of a subsequence of $\{\frac{1}{n}(\nu_q+\nu_q\circ\sigma^{-1}+\ldots+\nu_q\circ\sigma^{-(n-1)}\}$ in the weak-star topology. Then η_q is a σ-invariant measure on Σ. We have for each $I\in\Sigma_n$ and $\ell\in\mathbb{N}$,

$$\begin{aligned}\nu_q\circ\sigma^{-\ell}([I]) &= \sum_{J\in\Sigma_\ell}\nu_q([JI]) \\ &\approx \sum_{J\in\Sigma_\ell}\mu([JI])^q m^{(\ell+n)\tau(q)} && (\text{ by Lemma 3.3}) \\ &\approx s_\ell(q)\mu([I])^q m^{(\ell+n)\tau(q)} && (\text{by Corollary 2.2, Lemma 3.1}) \\ &\approx \mu([I])^q m^{n\tau(q)} && (\text{by Lemma 3.2}). \qquad (3.3)\end{aligned}$$

This proves the desired $\approx$ for η_q. In what follows we prove that η_q is ergodic. First we show that there is a constant $C>0$ such that for each $I\in\Sigma_n$, $J\in\Sigma_\ell$,

$$\lim_{p\to\infty}\sum_{i=0}^{p-1}\eta_q\Big([I]\cap\sigma^{-i}([J])\Big)\geq C\eta_q([I])\eta_q([J]). \tag{3.4}$$

To see this, we note that when $i>n$,

$$\begin{aligned}&\sum_{k=1}^{r}\eta_q\Big([I]\cap\sigma^{-i-k}([J])\Big) \\ &= \sum_{k=1}^{r}\sum_{K_1\in\Sigma_k}\sum_{K_2\in\Sigma_{i-n}}\eta_q([IK_1K_2J]) \\ &\geq C_1\sum_{k=1}^{r}\sum_{K_1\in\Sigma_k}\sum_{K_2\in\Sigma_{i-n}}\mu([IK_1K_2J])^q m^{(i+\ell+k)\tau(q)} && (\text{by (3.2)}) \\ &\geq C_1 m^{(i+\ell)\tau(q)} m^{-r|\tau(q)|}\sum_{k=1}^{r}\sum_{K_1\in\Sigma_k}\sum_{K_2\in\Sigma_{i-n}}\mu([IK_1K_2J])^q.\end{aligned}$$

By Lemmas 3.1, 3.2, 3.3, we have

$$\begin{aligned}\sum_{k=1}^{r}\sum_{K_1\in\Sigma_k}\sum_{K_2\in\Sigma_{i-n}}\mu([IK_1K_2J])^q &\geq C_2\mu([I])^q\sum_{K_2\in\Sigma_{i-n}}\mu([K_2J])^q \\ &\geq C_3 s_{i-n}(q)\mu([I])^q\mu([J])^q \\ &\geq C_4 m^{(n-i)\tau(q)}\mu([I])^q\mu([J])^q.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{k=1}^{r}\eta_q\Big([I]\cap\sigma^{-i-k}([J])\Big) &\geq C_5 m^{-r|\tau(q)|}m^{(n+\ell)\tau(q)}\mu([I])^q\mu([J])^q\\ &\geq C_6 m^{-r|\tau(q)|}\eta_q([I])\eta_q([J])\\ &:= C\eta_q([I])\eta_q([J]),\end{aligned}$$

from which (3.4) follows. Since the collection $\{[I] : I \in \Sigma_n,\ n \in \mathbb{N}\}$ is a semi-algebra that generates the Borel σ-algebra on Σ, a standard argument (e.g., see the proof of [W, Theorem 1.17]) shows that for any Borel sets $A, B \subset \Sigma$,

$$\lim_{p\to\infty}\sum_{i=0}^{p-1}\eta_q\Big(A\cap\sigma^{-i}(B)\Big) \geq C\eta_q(A)\eta_q(B).$$

This implies that for any Borel sets $A, B \subset \Sigma$ with $\eta_q(A) > 0$, $\eta_q(B) > 0$, there exists $n > 0$ with $\eta_q(A \cap \sigma^{-n}B) > 0$. By [W, Theorem 1.5], η_q is ergodic. □

Proof of Theorem 1.1. It suffices to prove that $\tau(q)$ is differentiable for $q > 0$. By (3.2) and a direct computation, we have

$$\tau_{\eta_q}(t) = \tau(qt) - t\tau(q). \tag{3.5}$$

where $\tau_{\eta_q}(t)$ is the L^t-spectrum of η_q. Since η_q is ergodic on Σ, it is a Young measure by the Shannon-McMillan-Brieman theorem (i.e, $\lim_{n\to\infty}\dfrac{-\log\eta_q(I_n(x))}{n}$ equals the topological entropy of η_q for η_q almost all $x = (j_i) \in \Sigma$ and $I_n(x) = [j_i \cdots j_n]$). Hence by Proposition 2.3, $\tau_{\eta_q}(t)$ is differentiable at $t = 1$. For any fixed q and for any $\epsilon > 0$, let $\epsilon' = \epsilon/q$, then by (3.5), we have

$$\frac{\tau(q+\epsilon)-\tau(q)}{\epsilon} = \frac{\tau(q(1+\epsilon'))-\tau(q)}{q\epsilon'} = \frac{\tau_{\eta_q}(1+\epsilon')-\tau_{\eta_q}(1)}{q\epsilon'} + \frac{\tau(q)}{q}.$$

This implies that $\tau(q)$ is differentiable for $q > 0$. □

Proof of Theorem 1.2. Let μ be the σ-invariant measure constructed in the proof of Proposition 2.1. We see from the definition of $\tau_\mu(q)$ that

$$\tau_\mu(q) = \frac{q\log\rho - P(q)}{\log m} \qquad \forall\, q > 0$$

and

$$E(\alpha) = \left\{ x \in \Sigma : \lim_{n\to\infty} \frac{\log \mu([x_1 \cdots x_n])}{\log m^{-n}} = \frac{\log \rho - \alpha}{\log m} \right\}$$

By [BMP, Theorem 1] or [LN2, Theorem 4.1], we have

$$\dim_H E(\alpha) \le \frac{\log \rho - \alpha}{\log m} q - \tau_\mu(q) = \frac{1}{\log m}(-\alpha q + P(q)) \tag{3.6}$$

for any $q > 0$.

Now assume that $\alpha = P'(q)$ for some $q > 0$. Let η_q be the corresponding ergodic "Gibbs measure" we constructed. From the proof of Theorem 1.1, we see that $\tau'_{\eta_q}(1)$ exists and

$$\tau'_{\eta_q}(1) = q\tau'_\mu(q) - \tau_\mu(q) = \frac{-qP'(q) + P(q)}{\log m}.$$

By [N], we have

$$\lim_{n\to\infty} \frac{\log \eta_q([i_1, \dots, i_n]}{\log m^{-n}} = \frac{-qP'(q) + P(q)}{\log m}, \qquad \eta_q \ a.a.\ I \in \Sigma,$$

which implies

$$\lim_{n\to\infty} \frac{\log \|M_{i_1} \cdots M_{i_n}\|}{n} = P'(q), \qquad \eta_q \ a.a.\ I = (i_j) \in \Sigma.$$

Therefore we have

$$\dim_H E(\alpha) \ge \dim_H \eta_q = \frac{-qP'(q) + P(q)}{\log m},$$

which together with (3.6) yield the theorem. □

4 Remarks

We first give an example that the irreducibility is needed for Theorem 1.1.

Example 4.1. Let $M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Then $H = M_1 + M_2$ is reducible; $M_J = \begin{pmatrix} 2^n & 0 \\ 0 & 3^k \end{pmatrix}$ where $|J| = n$ and k is the number of M_2's in M_J. Hence $\sum_{|J|=n} \|M_J\|^q = \sum_{k=0}^{n} nk(2^n + 3^k)^q$. Note that

$$\sum_{k=0}^{n} \binom{n}{k} (2^n+3^k)^q \geq \max\{\sum_{k=0}^{n} \binom{n}{k} 2^{nq}, \sum_{k=0}^{n} \binom{n}{k} 3^{kq}\} = \max\{2^{n(q+1)}, (1+3^q)^n\}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (2^n + 3^k)^q \leq \sum_{k=0}^{n} \binom{n}{k} 2^q(2^{nq} + 3^{kq}) = 2^q(2^{n(q+1)} + (1 + 3^q)^n).$$

We hence have $P(q) = \max\{(q + 1) \log 2, \log(1 + 3^q)\}$, which is not differentiable at $q = 1$

In Propositon 2.1, the measure μ is actually *unique* to satisfy

$$\mu([I]) \approx \rho^{-n}\|M_J\| \qquad \text{for} \quad J \in \Sigma_n. \tag{4.1}$$

Indeed for $q = 1$, $s_n(1)$ in (3.1) converges to 1 and $\tau(1) = 0$. This means that the ν_1 in Lemma 3.3 and the η_1 in Proposition 3.4 equal μ. It is ergodic by Proposition 3.4. Hence if there are μ and μ' satisfy (4.1), they are absolutely continuous to each other, and being ergodic, they must be equal.

In regard to the quasi-Bernoulli property of the μ in Proposition 2.1, we have

Proposition 4.2. *If all the entries of M_j, $1 \leq j \leq m$ are positive, then the μ in (1.1) will have the quasi-Bernoulli property. Moreover, $P(q)$ is differentiable over $\mathbb{R}$.*

Proof. The lower bound in the quasi-Bernoulli property can be obtained by observing that for each j, $M_j \geq \alpha E M_j$ (co-ordinatewise) where E is the $d \times d$ matrices with 1 on all the coordinates and $\alpha = (\min_k \min_{i,j} M_k(i,j))/(d \max_k \max_{i,j} M_k(i,j))$. Hence by using the notations in the proof of Proposition 2.1, we have $E \geq c\mathbf{v}\mathbf{u}^t$ and

$$\mathbf{u}^t M_I M_J \mathbf{v} \geq c\alpha(\mathbf{u}^t M_I \mathbf{v})(\mathbf{u}^t M_J \mathbf{v}).$$

and the lower bound for μ follows.

Since μ is quasi-Bernoulli, by [H, Theorem 3.3], $\tau_\mu(q)$ is differentiable over $\mathbb{R}$. Note that

$$\tau_\mu(q) = \frac{q \log \rho - P(q)}{\log m} \qquad \forall\, q \in \mathbb{R}.$$

It implies the differentability of $P(q)$. $\square$

The differentiablity of $P(q)$ can be proved in a more general subshift setting. To see this, let A be a primitive $m \times m$ matrix with entries 0 and 1, and Σ_A the subshift space over $\{1, \ldots, m\}$ (cf. [B]). Denote by $\Sigma_{A,n}$ the collection of all admissible words of length n, i.e.,

$$\Sigma_{A,n} := \{i_1 \ldots i_n : \ A(i_j, i_{j+1}) = 1 \text{ for all } 1 \leq j \leq n-1\}.$$

If $\{M_j\}_{j=1}^m$ is a family of positive $d \times d$ matrices, then $p(q)$ can be defined as

$$P(q) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{J \in \Sigma_{A,n}} \|M_J\|^q.$$

In [FeL], using a similar idea we proved that $P(q)$ is differentiable over $\mathbb{R} \backslash \{0\}$ (see [FeL, Theorem 1.2]) (there is a trouble in handling the case $q = 0$ by that method). Later Feng proved the differentability of $P(q)$ over $\mathbb{R}$ in a different way ([Fe3, Corrolary 4.2]). Actually, he generalized Walters' variational principle from the scalar function setting to the non-negative matrix-valued function setting and gave an analogue of Ruelle's formula for the derivative of pressure functions (see [Fe3, Theorem 1.2]); when M_j ($j = 1, \ldots, m$) are all positive, there is a unique equilibrium state for each $q \in \mathbb{R}$, which implies the differentaibilty of $P(q)$.

Without the assumption of positivity, the measure μ in Proposition 2.1 is not necessary quasi-Bernoulli, so that Lemma 2.4 is essential.

Example 4.3. Let $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and M_2 an arbitrarily positive matrix, then $H = M_1 + M_2$ is an irreducible positive matrix, let μ be the measure constructed in Proposition 2.1. Let $J = 1 \ldots 1$ (n-times), then $\|M_J\| = n + 2$ and hence

$$\|M_J\| \, \|M_J\| \ \geq \ \frac{n}{2} \|M_{JJ}\| \ .$$

This implies that there does not exist $C' > 0$ such that $C'\mu([I])\mu([J]) \leq \mu([IJ])$ for all $I, J \in \Sigma^*$.

Besides the above simple example, there are natural examples that do not satisfy the quasi-Bernoulli property. For example, let $\rho = (\sqrt{5}-1)/2$, the reciprocal of the golden ratio, let ν be the Bernoulli convolution corresponding to ρ, i.e., the unique self-similar measure associated with the IFS $\{\rho x, \rho x + (1-\rho)\}$ with weights $\{\frac{1}{2}, \frac{1}{2}\}$. This IFS has overlaps, however one can reduce the IFS to three maps with no overlap

$$R_1(x) = \rho^2 x, \quad R_2(x) = \rho^3 x + \rho^2, \quad \text{and} \quad R_3(x) = \rho^2 x + \rho,$$

and the measure ν satisfies

$$\nu([i_1 \dots i_n]) \approx \frac{1}{4^n} \|M_{i_1} \dots M_{i_n}\|$$

where $[i_1 \dots i_n] = R_{i_1} \dots R_{i_n}([0,1])$ and

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

([LN1], [Fel], [FeO]). Then ν does not have the quasi-Bernoulli property by the same reason as in Example 4.3., but it will satisfy the assumption in Theorem 1.1. We point out that there is no simple relation between the pressure function $P(q)$ generated by M_1, M_2, M_3 and the L^q-spectrum $\tau(q)$ of the ν, since R_i $(i = 1, 2, 3)$ are not uniformly contractive. Actually more can be said about the L^q-spectrum $\tau(q)$ of the ν: an explicit formula was given in [LN1] for $q > 0$ (in terms the product of M_1 and M_3) and was extended to $q < 0$ in [Fel]. By using the formula, it was proved that $\tau(q)$ is differentiable (actually real analytic) on $\mathbb{R}$ except one point in $\mathbb{R}^-$.

Another instructive example similar to the above is the 3-fold convolution of the Cantor measure. It can easily be reduced to vector form and the L^q-spectrum is in terms the product of matrices [LW]. The L^q-spectrum behaves like the above example, but the dimension spectrum has an usually behavior [HL]. The problem has been dealt in [FeLW] (see also [S]).

Note that the behavior for $q \le 0$ is also important in the multifractal analysis. For this we have to modify the the pressure function $P(q)$ slightly:

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{J \in \mathcal{N}_n} \|M_J\|^q \tag{4.2}$$

where $\mathcal{N}_n$ consists of all the $J \in \Sigma_n$ such that $M_J \neq \mathbf{0}$. It is clear that if $M_J \neq \mathbf{0}$ for all $J \in \Sigma_n$, then the super-additivity of the sum in (4.2) implies that the limit exists. In our case we have

Proposition 4.4. *Suppose $M_1, \cdots M_m$ are non-negative matrices and $H = \sum_{i=1}^m M_i$ is irreducible, then the limit in (4.2) exists for each $q \le 0$*

Proof. By the irreducibility, there exists integer r with $\sum_{k=1}^r H^k > 0$. From the proof of Lemma 2.4, there is a constant $C > 0$ such that for each $I, J \in \Sigma^*$, there exists $K_0 \in \bigcup_{k=1}^r \Sigma_k$ and $i \in \Sigma_1$ satisfying

$$0 < \|M_{IK_0J}\| \le C\|M_I\|\|M_J\|, \qquad 0 < \|M_{Ii}\| \le C\|M_I\|. \tag{4.3}$$

Denote by $s_n = \sum_{J\in\mathcal{N}_n} \|M_J\|^q$. Then (4.3) implies $s_n s_\ell \le C^{-q}\sum_{k=1}^r s_{n+\ell+k}$. From (4.3) we also deduce that for any $I \in \Sigma*$, there exists $i \in \Sigma_1$ such that $M_{Ii} \ne 0$. Since $\|M_{Ii}\| \le D\|M_I\|$ for some constant$C_1 > 0$, we have $s_n \le C_1^{-q} s_{n+1}$ for any integers n, ℓ. It follows that $s_n s_\ell \le C' s_{n+\ell+r}$ for some constant $C' > 0$ (depending on q), which implies that $C' s_{n+r}$ is super-multiplicative. This yields the existence of the limit. □.

The differentiability of such $P(q)$ for $q \le 0$ is still unknown. We see from the above Bernoulli convolution of the golden ratio that the above $P(q)$ can be non-differentiable at a point of $q < 0$. On the other hand, it is known that by imposing some stronger conditions on the matrices (but can have negative entries), the pressure function $P(q)$ is analytic near $q = 0$ (see e.g., [BL, Theorem 4.3]).

The above example of Bernoulli convolution also gives rise to another interesting question. Note that it is a special case of the overlapping IFS that can be reduced to new sets of IFS with no overlap and the calculation of the $\tau(q)$ can be converted into the product of matrices as in (1.1). Such IFS is an important class of those that satisfy the *weak separation condition* ([LN2], [LNR]). Under this condition it was proved that the multifractal formalism is valid provided that $\tau(q)$ is differentiable [LN2]. In view of this it will be useful to find the differentiability of the $\tau(q)$ for the self-similar measures generated by the IFS with the weak separation condition. Indeed in [Fe2], Feng considered the IFS that are defined on $\mathbb{R}$ and satisfy a stronger *finite type condition*, he established the representing matrices for the corresponding self-similar measures and applied Theorem 1.1 to yield the differentiability of $\tau(q)$. The case on $\mathbb{R}^d$ is still unsolved.

Finally we remark that we do not know whether Theorem 1.1 can be extended to *non-positive* matrices. An important theorem concerning this is in [BL, Theorem 4.3] for the analyticity of $P(q)$ near zero. More close to our development is the result of Daubechies and Lagarias [DL] on the multifractal formalism of the well known scaling function D_4. They showed the differentiability of the corresponding $\tau(q)$ (which is modified to the L^q-Lipschitz exponent of the function), but the consideration depends on the two involved 2×2 matrices to have a common eigenvector. There are some extensions in [LM].

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