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On the connectedness and classification of self-affine tiles

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1. Introduction

We call an integer matrix $A \in M_n(\mathbb{Z})$ *expanding* if all the eigenvalues of A have moduli > 1. Let $|\det(A)| = q$, a subset $\mathcal{D} = \{d_1, \cdots, d_q\} \subseteq \mathbb{Z}^n$ of q distinct vectors is called a *q-digit set.* The affine maps w_j defined by

$$w_j(x) = A^{-1}(x + d_j), \quad 1 \leq j \leq q,$$

are all contractions under a suitable norm in $\mathbb{R}^n$ (see [LW2, pp. 29-30]). The family $\{w_j\}_{j=1}^q$ is called an *iterated function system* (IFS) and there is a unique nonempty compact set satisfying $T = \bigcup_{j=1}^q w_j(T)$ [F], T is called the *attractor* of the system and is explicitly given by

$$T := T(A, \mathcal{D}) = \Big\{ \sum_{i=1}^{\infty} A^{-i} d_{j_i} : d_{j_i} \in \mathcal{D} \Big\}. \tag{1.1}$$

Let $\mu(T)$ denote the Lebesgue measure of T, we call T an *integral self-affine tile* if $\mu(T) > 0$. The study of self-affine tiles was originally motivated by the work of Thurston [T] and Kenyon [K] on quasi-periodic self-similar tilings, and that of Gröchenig and Madych [GM], Gröchenig and Haas [GH] on compactly supported Haar-type orthonormal wavelet basis. In a series of paper [LW 1-5], Lagarias and Wang have developed the basic theory of the self-affine tiles. In this note, we will consider two problems concerning such tiles: the connectedness of the tiles and the classification of the expanding integer matrix A.

There is very little known about the connectedness of the self-affine tiles. In [O], Odlyzko gave a characterization of such tiles T in $\mathbb{R}$ to be the finite union of intervals by using the strict product form of the digits. In $\mathbb{R}^2$, Bandt and Gelbrich [BG] investigated the disc like tiles for $|\det A| = 2$ or 3. In $\mathbb{R}^n$, Hacon *et al* [H] proved all 2-digit tiles T are pathwise connected (actually the tile can be filled up by a space filling curve).

It is almost trivial to see that in $\mathbb{R}$, for $A = [q]$ and $D = \{0, v, ..., (q-1)v\}$ with $q, v \in \mathbb{Z}^+$, T is an interval. Hence it is natural to investigate this simplest class of tiles in $\mathbb{R}^n$. For $v \in \mathbb{R}^n \backslash \{0\}$, we will call a digit set $\mathcal{D} = \{d_1 v, d_2 v, ..., d_{q-1} v\}$ a *collinear digit set.* We are particularly interested in those "consecutive" collinear

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digit set $\{0, v, ..., (q-1)v\}$. By setting up a general criterion for connectedness, we show that for a tile defined by the consecutive collinear digit sets, this topological property can be reduced into an algebraic property of the characteristic polynomial (c.p.) of A. We say that a monic, integer polynomial $f(x) \in \mathbb{Z}[x]$ has *property* $(*)$ if there exists $g(x) \in \mathbb{Z}[x]$ such that

$$g(x)f(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x \pm q \tag{1.2}$$

with $|a_i| \leq q-1$, $i = 1, \cdots, k-1$.

THEOREM 1.1. *Let* $A \in M_n(\mathbb{Z})$ *be expanding with* $|\det A| = q$ *and let*

$$\mathcal{D} = \{0, v, 2v, \ldots, (q-1)v\}$$

be a collinear digit set in $\mathbb{Z}^n \backslash \{0\}$. *Then* $T = T(A, \mathcal{D})$ *is connected if the characteristic polynomial* $f(x)$ *of* A *has property* $(*)$.

We call a polynomial expanding if all its roots have moduli > 1.

THEOREM 1.2. *All the monic, expanding integer polynomials of degree* ≤ 3 *have property* $(*)$. *Consequently if* T *is a self-affine tile in* $\mathbb{R}^n, n \leq 3$, *generated by an expanding integer matrix* A *with* $|\det A| = q$ *and digit set*

$$\mathcal{D} = \{0, v, 2v, \ldots, (q-1)v\}, \quad v \in \mathbb{Z}^n,\ n \leq 3,$$

then T *is connected.*

The case for $n = 3$ is proved by Rao in [KLR]. Property $(*)$ has its own interest from the algebraic point of view. We are not able to prove this property for all degree n, monic, expanding integer polynomials. However we can prove a statement close to it. We will discuss this in Section 2.

Our next consideration is on the $\mathbb{Z}$-similarity of the expanding integer matrices. We say that $A, B \in M_n(\mathbb{Z})$ are *$\mathbb{Z}$-similar*, denoted by $A \sim B$, if there exists a unimodular matrix $P \in M_n(\mathbb{Z})$ (i.e., P is invertiable and $P^{-1} \in M_n(\mathbb{Z})$) such that $PAP^{-1} = B$. $\mathbb{Z}$-similarity is an equivalent relationship, its equivalence classes are called *$\mathbb{Z}$-similar classes*. The $\mathbb{Z}$-similar classification is useful in studying the tiles. For example it is known that the measure $\mu(T)$ has integer value [LW3, Theorem 1.1], but in general it is difficulty to determine if it is positive, in particular, if $\mu(T) = 1$ ([B], [LW 1-5]). The measure $\mu(T) > 0$ and $\mu(T) = 1$ are invariant under $\mathbb{Z}$-similarity, it is therefore possible to reduce the measure problem to a few classes of matrices for consideration. Other properties of the tiles such as the connectedness ([BG], [GH], [HSV], [KL1]), the dimension of the boundary ([KLSW], [SW]) and the tiling problem ([LW2]) are all invariant under the $\mathbb{Z}$-similarity.

The $\mathbb{Z}$-similar classification of expanding integer matrices was first studied by Lagarias and Wang [LW1] using the characteristic polynomials. They showed that each integer expanding polynomial $f(x) = x^2 + ax + q$, $|q| = 2$ corresponds to exactly one $\mathbb{Z}$-similar class of expanding matrices $A \in M_2(\mathbb{Z})$. Since there are six such polynomials, there are only six $\mathbb{Z}$-similar classes of 2×2 integer matrices with $|\det(A)| = 2$. The situation is considerably more complicate for $|\det(A)| > 2$. In Section 3, we will give the complete classification for $A \in M_2(\mathbb{Z})$ with $|\det A| = 3, 4, 5$. By using this classification we can sort out those that are

$\mathbb{Z}$-similar to the self-similar matrices, i.e., those that are constant multiples of orthonormal matrices. They are the most important ones in the geometry of fractals. We also make use of the classification to consider some cases of $\mu(T(A,\mathcal{D}) > 0$ that is not covered by the general theorem.

The detail of theorems will appear in [KL1], [KL2], [KLR].

2. Connectedness

Let $A \in M_n[\mathbb{Z}]$ be expanding and $|\det A| = q$ and $\mathcal{D} \subset \mathbb{Z}^n$ be a q-digit set. Let $L = Z[A, D]$ be the lattice generated by D, AD,..., $A^{n-1}D$, then $\mathcal{D}$ is called a *complete set of coset representatives* in $L/A(L)$ (in short, $\mathcal{D}$ is complete) if

$$L = \bigcup_{i=1}^{q}(d_i + A(L)) \quad \text{and} \quad (d_i + A(L)) \cap (d_j + A(L)) = \emptyset \ \text{ for } \ i \neq j.$$

It is well known that $\mathcal{D}$ is complete implies $\mu(T(A, D)) > 0$ [B], but the converse is not true and in general it is difficult to determine $\mathcal{D}$ to be complete (see Section 3). But for collinear digit set, the problem is much simpler.

THEOREM 2.1. *Suppose $A \in M_n(\mathbb{Z})$ is an expanding matrix with $|\det\ A| = q$, $q \geq 2$ is a prime. Let $\mathcal{D} = \{d_1 v, ..., d_q v\}$ with $v \in \mathbb{R}^n \setminus \{0\}$, $d_i \in \mathbb{Z}$. Then T is a self-affine tile (i.e., $\mu(T) > 0$) if and only if $\{v, Av, ..., A^{n-1}v\}$ is a linearly independent set and $\{d_1, ..., d_q\} = q^l\{d'_1, ..., d'_q\}$ where $\{d'_1, ..., d'_q\}$ is complete in $\mathbb{Z}_q$.*

In particular if $v \in \mathbb{Z}^n \setminus \{0\}$, then $\{v, Av, ..., A^{n-1}v\}$ is automatically a linearly independent set. Hence T is a self-affine tile if and only if the above $\{d'_1, ..., d'_q\}$ is complete in $\mathbb{Z}_q$.

For the case $\det A = q$ is not a prime, if we assume further that the eigenvalues of A has n independent eigenvectors and if we take suitable $v \in \mathbb{Z}^n \setminus \{0\}$, then the same conclusion of the theorem holds. However we do not have a complete answer without the additional assumption on the eigenvalues of A.

In the following, we give a general criterion of connectedness by using a “graph” argument on $\mathcal{D}$. Let $(A, \mathcal{D})$ be given, we define

$$\mathcal{E} = \{(d_i, d_j) : \ (T + d_i) \cap (T + d_j) \neq \emptyset, \ d_i, d_j \in \mathcal{D}\}$$

to be the set of ‘edges’ for the set $\mathcal{D}$; we say that d_i and d_j are *$\mathcal{E}$-connected* if there exists a finite sequence $\{d_{j_1}, ..., d_{j_k}\}R \subseteq \mathcal{D}$ such that $d_{j_1} = d_i$, $d_{j_k} = d_j$ and $(d_{j_l}, d_{j_{l+1}}) \in \mathcal{E}$, $1 \leq l \leq k-1$. It is not difficult to prove

PROPOSITION 2.2. *Let $A \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det\ A| = q$ and let $\mathcal{D} = \{d_1, ..., d_q\} \subseteq \mathbb{R}^n$ be a q-digit set. Then T is connected if and only if for any two $d_i, d_j \in \mathcal{D}$, d_i and d_j are $\mathcal{E}$-connected.*

For the consecutive collinear digit set, we can reduce the above $\mathcal{E}$-connectedness to a simple algebraic condition. First note that for such T, it is connected if and only if $\{0, v\} \in \mathcal{E}$, which is equivalent to $T\bigcap(T+v) \neq \emptyset$. Let u be in the intersection, then

$$u = \sum_{j=1}^{\infty} \alpha_j A^{-j} v = v + \sum_{j=1}^{\infty} \alpha'_j A^{-j} v$$

where $\alpha_j, \alpha'_j = 0, 1, \cdots, (q-1)$. It follows that for $\mathcal{D} = \{0, \cdots, (q-1)\}$, T is connected if and only if

$$(2.1) \qquad v = \sum_{j=1}^{\infty} \beta_j A^{-j} v \qquad \text{for some} \quad \beta_j = 0, \pm 1, \cdots, \pm(q-1).$$

Next we will reduce this infinite series to a finite sum. Let $s_m := \sum_{j=1}^{m} \beta_j A^{m-j} v$, $s'_k = \sum_{j=1}^{k} \beta'_j A^{k-j} v$ where $\beta_j, \beta'_j = 0, \pm 1, ..., \pm(q-1)$. We define a *shift* p_k by

$$(2.2) \qquad p_k(s_m) = A^k s_m + s'_k.$$

Lemma 2.3. *Let $A \in M_n(\mathbb{Z})$ be expanding and let $\mathcal{D} = \{0, v, ..., (q-1)v\}$ with $v \in \mathbb{R}^n \setminus \{0\}$. Then $T(A, \mathcal{D})$ is connected if there exists a monic polynomial $\sum_{j=1}^{m} \beta_j x^{m-j}$ with $s_m = \sum_{j=1}^{m} \beta_j A^{m-j} v$ and a shift p_k such that $p_k(s_m) = s_m$.*

Proof. In view of (2.1), it suffices to construct a sequence $\{\beta_j\}_{j=0}^{\infty}$ with $\beta_0 = 1$ and $\beta_j = 0, \ \pm 1, ..., \pm(q-1)$ for $j \geq 1$ such that $\sum_{j=0}^{\infty} \beta_j A^{-j} v = 0$. By the hypothesis, we have

$$\begin{aligned} p_k^l(s_m) &= \underbrace{p_k \circ p_k \circ \cdots \circ p_k}_{l \text{ times}}(s_m) \\ &= A^{kl} s_m + \sum_{j=0}^{l-1} A^{kj} s'_k \\ &= s_m. \end{aligned}$$

Let $t_l := A^{-kl}(p_k^l(s_m)) = s_m + \sum_{j=1}^{l} A^{-kj} s'_k$ and $t := \lim_{l \to \infty} t_l$. Then $t = 0$ since $t_l = A^{-kl} s_m \to 0$. Write $0 = A^{-m+1} t = \sum_{j=0}^{\infty} \beta_j A^{-j} v$ by inserting the full expressions for s_m and s'_k. This series converges since starting from $j = m$, $\{\beta_j\}_{j=0}^{\infty}$ is periodic (repeated coefficients of s'_k). Also, $\sum_{j=0}^{\infty} \beta_j A^{-j} v = 0$. We have $\beta_0 = 1$ since s_m is monic. Therefore $\{\beta_j\}_{j=0}^{\infty}$ is the required sequence. □

The above discussion leads to the following algebraic method to check the connectedness.

Theorem 2.4. *Let $A \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det A| = q$ and characteristic polynomial $f(x)$. Let $\mathcal{D} = \{0, v, ..., (q-1)v\}$ with $v \in \mathbb{R}^n \setminus \{0\}$. Suppose $f(x)$ has property $(*)$ as defined in Section 1, then $T(A, \mathcal{D})$ is connected.*

Proof. Let $s_1 = v$ and let $h(x) = g(x)f(x)$ be defined as in the definition of property $(*)$. We have two cases. (i) $h(0) = -q$. We take

$$p_k(s_1) = A^k v + (a_{k-1}A^{k-1} + a_{k-2}A^{k-2} + ... + a_1 A - (q-1)I)v.$$

Then $p_k(s_1) = h(A)v + v = s_1$ since $h(A) = 0$. By Lemma 2.3, $T(A, \mathcal{D})$ is connected. (ii) $h(0) = q$. We take

$$q_k(s_1) = -A^k v + (-a_{k-1}A^{k-1} - a_{k-2}A^{k-2} - ... - a_1 A - (q-1)I)v$$

so that $q_k(s_1) = -h(A)v + v = s_1$. Note that in the above expression of $q_k(s_1)$, the leading coefficient is $-A^k$ instead of A^k in (2.2). Lemma 2.3 still apply to this case by a slight modification of the argument. □

In view of the theorem we will consider property (∗) in the rest of this section.

PROPOSITION 2.5. [B] *Let $f(x) = x^2 + ax \pm q$, where $a \in \mathbb{Z}$, $1 \neq q \in \mathbb{N}$. Then $f(x)$ is expanding if and only if $|a| \leq q$ for $f(0) = q$, and $|a| \leq q - 2$ for $f(0) = -q$.*

Note that all the degree 2 expanding polynomial, except for $f(x) = x^2 \pm qx + q$, are already in the form of property (∗). If we multiply $g(x) = x \mp 1$ by $f(x)$, then

$$h(x) = g(x)f(x) = x^3 \pm (q-1)x^2 \mp q$$

and hence $f(x)$ also has property (∗). By Theorem 2.4 we have

COROLLARY 2.6. *All the self-affine integral tiles in $\mathbb{R}^2$ generated by the consecutive collinear digit sets are connected.*

To prove the connectedness of such tiles in $\mathbb{R}^3$, we need to establish a characterization of the polynomials similar to Proposition 2.5. Let $f(x)$ and $\tilde{f}(x)$ denote the characteristic polynomial of A and $-A \in M_n(\mathbb{Z})$ respectively. It is easy to see that $\tilde{f}(x) = (-1)^n f(-x)$. This enables us to consider the degree 3 characteristic polynomials with positive constant terms, i.e. $\det A < 0$. Furthermore we can assume that

$$f(x) = x^3 \pm ax^2 \pm bx + q \tag{2.3}$$

where $a, b \geq 0, ab \neq 0$ and $q \geq 2$ (the case $ab = 0$ is trivial). We want to determine the coefficients a, b so that $f(x)$ is an expanding polynomial. It is elementary to prove

LEMMA 2.7. *Suppose $f(x)$ is of the form (2.3) and is expanding, then*

$$f(1),\ f(-1),\ f(q) > 0 \quad \text{and} \quad f(-q) < 0.$$

Conversely suppose the $f(x)$ in (2.3) has a real root in $(-q, -1)$ or $(1, q)$ and has no other real root in $[-1, 1]$. Then $f(x)$ is expanding.

By repeatedly using the lemma we can prove

PROPOSITION 2.8. *Consider the $f(x)$ in (2.3).*
(i) If $f(x) = x^3 - ax^2 + bx + q$, then $f(x)$ is expanding if and only if $q \geq a + b + 2$;
(ii) If $f(x) = x^3 - ax^2 - bx + q$, then $f(x)$ is expanding if and only if $q \geq a + b$;
(iii) Let $f(x) = x^3 + ax^2 - bx + q$, then $f(x)$ is expanding if and only if
(a) $a = q - 1, \quad b \leq q - 2$ *or*
(b) $a < q - 1, \quad b \leq q + a$;
(iv) Let $f(x) = x^3 + ax^2 + bx + q$, then $f(x)$ is expanding if and only if
(a) $a = q + 1, \quad q + 2 \leq b \leq 2q - 1$; *or*
(b) $a = q, \quad 2 \leq b \leq 2q - 2$; *or*
(c) $a \leq q - 1, \quad b \leq q + a - 2$.

THEOREM 2.9. *Let $f(x) \in \mathbb{Z}[x]$ be degree 3, monic and expanding, then $f(x)$ has property $(*)$.*

It follows that all the self-affine tiles in $\mathbb{R}^3$ generated by the expanding integer matrix A and $\mathcal{D} = \{0, v, \cdots, (q-1)v\} \subseteq \mathbb{Z}^3$ are connected.

Proof. For the $f(x)$ in Proposition 2.8 (iii)(b) and (iv)(a,b,c) that are not already in the form of property $(*)$, we will need to find a $g(x) \in \mathbb{Z}[x]$ to reduce its coefficients to have moduli $\leq q-1$. We can also assume that $q \geq 4$ (because of case II in the following). For the case $q = 2$, it is easy to check using Proposition 2.8. For $q = 3$, the theorem is true by Table 1.

(I) In Proposition 2.8(iii)(b), $a < q-1$ and we divide the $b \leq q+a$ into two cases. If $b = q+a$, we let $g(x) = (x^2+1)(x+1)$, then

$$g(x)f(x) = x^6 + (a+1)x^5 + (1-q)x^4 + x^3 - ax + q.$$

If $q \leq b < q+a$, we let $g(x) = x+1$, then

$$g(x)f(x) = x^4 + (a+1)x^3 + (a-b)x^2 + (q-b)x + q$$

(II) In Proposition 2.8(iv)(a), $a = q+1$ and we divide the b into three cases. If $b = (q+2)$, we let $g(x) = (x-1)^2$, then

$$g(x)f(x) = x^5 + (q-1)x^4 - (q-1)x^3 - 3x^2 - (q-2)x + q.$$

If $b = 2q-1$, we let $g(x) = (x^2+1)(x-1)^2$, then

$$g(x)f(x) = x^7 + (q-1)x^6 - x^5 - (q-2)x^4 - 3x^3 - (q-3)x^2 - x + q.$$

If $q+2 < b < 2q-1$, we take $g(x) = (x-1)^2(x^2+1)(x^3+1)$, then

$$g(x)f(x) = x^{10} + (q-1)x^9 + (b-2q)x^8 + (3q-2b+1)x^7 + (2b-3q-2)x^6$$
$$+(q-b+1)x^5 + (q-b)x^4 + (2b-3q-1)x^3 + (3q-2b+1)x^2 + (b-2q)x + q,$$

and property $(*)$ is satisfied.

(III) For Proposition 2.8(iv)(b) and (iv)(c), we take $g(x) = x-1$, then

$$g(x)f(x) = x^4 + (a-1)x^3 + (b-a)x^2 + (q-b)x + q.$$

By (I), (II), (III), all the cases are taken care, and the theorem is proved. □

To illustrate the proof of Theorem 2.9, we include in the following the degree 3 expanding integer polynomials for $q = 3, 4, 5$ that need to multiply the factors $g(x)$. Note that the table $q = 5$ includes all the cases in the theorem and the factors $g(x)$ we used in the proof.

There are 25 expanding characteristic polynomials $f(x)$ with $f(0) = \det A = 3$; 17 of them are already in the form of property $(*)$. The rest of them can be reduced to the form of property $(*)$ by multiplying a polynomial $g(x)$ as is exhibited in Table 1.

There are 51 such expanding polynomials with $f(0) = \det A = 4$; 35 of the $f(x)$ are already in the form of property $(*)$. The rest can be shown to be in such form by multiplying suitable $g(x)$, as is in Table 2.

There are 85 such expanding polynomial with $f(0) = \det A = 5$; 26 of them need to multiply a factor $g(x)$. They are listed in Table 3. Note that from the proof

c.p.= $p(x)$	$g(x)$	$h(x) = g(x)p(x)$
$x^3 - 3x + 3$	$(x^2+1)(x+1)$	$x^6 + x^5 - 2x^4 + x^3 + 3$
$x^3 + x^2 - 4x + 3$	$(x^2+1)(x+1)$	$x^6 + 2x^5 - 2x^4 + x^3 - x + 3$
$x^3 + x^2 - 3x + 3$	$x+1$	$x^4 + 2x^3 - 2x^2 + 3$
$x^3 + 2x^2 + 3x + 3$	$x-1$	$x^4 + x^3 + x^2 - 3$
$x^3 + 3x^2 + 2x + 3$	$x-1$	$x^4 + 2x^3 - x^2 + x - 3$
$x^3 + 3x^2 + 3x + 3$	$x-1$	$x^4 + 2x^3 - 3$
$x^3 + 3x^2 + 4x + 3$	$x-1$	$x^4 + 2x^3 + x^2 - x - 3$
$x^3 + 4x^2 + 5x + 3$	$(x^2-x+1)(x-1)$	$x^6 + 2x^5 - x^4 + x - 3$

TABLE 1. $g(x)$ for the case det $A = -3$

c.p.= $p(x)$	$g(x)$	$h(x) = g(x)p(x)$
$x^3 - 4x + 4$	$(x^2+1)(x+1)$	$x^6 + x^5 - 3x^4 + x^3 + 4$
$x^3 + x^2 - 4x + 4$	$x+1$	$x^4 + 2x^3 - 3x^2 + 4$
$x^3 + x^2 - 5x + 4$	$(x^2+1)(x+1)$	$x^6 + 2x^5 - 3x^4 + x^3 - x + 4$
$x^3 + 2x^2 - 6x + 4$	$(x^2+1)(x+1)$	$x^6 + 3x^5 - 3x^4 + x^3 - 2x + 4$
$x^3 + 2x^2 - 5x + 4$	$x+1$	$x^4 + 3x^3 - 3x^2 - x + 4$
$x^3 + 2x^2 - 4x + 4$	$x+1$	$x^4 + 3x^3 - 2x^2 + 4$
$x^3 + 2x^2 + 4x + 4$	$x-1$	$x^4 + x^3 + 2x^2 - 4$
$x^3 + 3x^2 + 4x + 4$	$x-1$	$x^4 + 2x^3 + x^2 - 4$
$x^3 + 3x^2 + 5x + 4$	$x-1$	$x^4 + 2x^3 + 2x^2 - x - 4$
$x^3 + 4x^2 + 2x + 4$	$x-1$	$x^4 + 3x^3 - 2x^2 + 2x - 4$
$x^3 + 4x^2 + 3x + 4$	$x-1$	$x^4 + 3x^3 - x^2 + x - 4$
$x^3 + 4x^2 + 4x + 4$	$x-1$	$x^4 + 3x^3 - 4$
$x^3 + 4x^2 + 5x + 4$	$x-1$	$x^4 + 3x^3 + x^2 - x - 4$
$x^3 + 4x^2 + 6x + 4$	$x-1$	$x^4 + 3x^3 + 2x^2 - 2x - 4$
$x^3 + 5x^2 + 6x + 4$	$(x-1)^2$	$x^5 + 3x^4 - 3x^3 - 3x^2 - 2x + 4$
$x^3 + 5x^2 + 7x + 4$	$(x^2+1)(x-1)^2$	$x^7 + 3x^6 - x^5 - 2x^4 - 3x^3 - x^2 - x + 4$

TABLE 2. $g(x)$ for the case det $A = -4$

we see that these are all the $g(x)$ needed to use for all the expanding polynomial of $f(0) = q$.

For the higher degree expanding integer polynomials, Garsia [G, Lemma 1.6] proved a weaker statement than property $(*)$: there exists $g(x) \in \mathbb{Z}[x]$ such that

$$g(x)f(x) = a_k x^k + \cdots + a_1 x \pm q \quad \text{where } |a_j| \leq q \ \ \forall\, j = 1, \ldots, k.$$

His proof makes use of the pigeon hole principle. By using a geometric argument of the tiling set, we can improve Garsia's result as follows.

THEOREM 2.10. *Let $f(x) \in \mathbb{Z}[x]$ be an irreducible expanding, monic polynomial. Then there exists $g(x) \in \mathbb{Z}[x]$ such that*

$$h(x) = f(x)g(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x \pm q$$

where $|a_j| \leq q - 1$ for $j = 1, \cdots, k$.

c.p.= $p(x)$	$g(x)$	$h(x) = g(x)p(x)$
$x^3 - 5x + 5$	$(x^2+1)(x+1)$	$x^6 + x^5 - 4x^4 + x^3 + 5$
$x^3 + x^2 - 5x + 5$	$x+1$	$x^4 + 2x^3 - 4x^2 + 5$
$x^3 + x^2 - 6x + 5$	$(x^2+1)(x+1)$	$x^6 + 2x^5 - 4x^4 + x^3 - x + 5$
$x^3 + 2x^2 + 5x + 5$	$x-1$	$x^4 + x^3 + 3x^2 - 5$
$x^3 + 2x^2 - 5x + 5$	$x+1$	$x^4 + 3x^3 - 3x^2 + 5$
$x^3 + 2x^2 - 6x + 5$	$x+1$	$x^4 + 3x^3 - 4x^2 - x + 5$
$x^3 + 2x^2 - 7x + 5$	$(x^2+1)(x+1)$	$x^6 + 3x^5 - 4x^4 + x^3 - 2x + 5$
$x^3 + 3x^2 + 6x + 5$	$x-1$	$x^4 + 2x^3 + 3x^2 - x - 5$
$x^3 + 3x^2 + 5x + 5$	$x-1$	$x^4 + 2x^3 + 2x^2 - 5$
$x^3 + 3x^2 - 5x + 5$	$x+1$	$x^4 + 4x^3 - 2x^2 + 5$
$x^3 + 3x^2 - 6x + 5$	$x+1$	$x^4 + 4x^3 - 3x^2 - x + 5$
$x^3 + 3x^2 - 7x + 5$	$x+1$	$x^4 + 4x^3 - 4x^2 - 2x + 5$
$x^3 + 3x^2 - 8x + 5$	$(x^2+1)(x+1)$	$x^6 + 4x^5 - 4x^4 + x^3 - 3x + 5$
$x^3 + 4x^2 + 7x + 5$	$x-1$	$x^4 + 3x^3 + 3x^2 - 2x - 5$
$x^3 + 4x^2 + 6x + 5$	$x-1$	$x^4 + 3x^3 + 2x^2 - x - 5$
$x^3 + 4x^2 + 5x + 5$	$x-1$	$x^4 + 3x^3 + x^2 - 5$
$x^3 + 5x^2 + 8x + 5$	$x-1$	$x^4 + 4x^3 + 3x^2 - 3x - 5$
$x^3 + 5x^2 + 7x + 5$	$x-1$	$x^4 + 4x^3 + 2x^2 - 2x - 5$
$x^3 + 5x^2 + 6x + 5$	$x-1$	$x^4 + 4x^3 + x^2 - x - 5$
$x^3 + 5x^2 + 5x + 5$	$x-1$	$x^4 + 4x^3 - 5$
$x^3 + 5x^2 + 4x + 5$	$x-1$	$x^4 + 4x^3 - x^2 + x - 5$
$x^3 + 5x^2 + 3x + 5$	$x-1$	$x^4 + 4x^3 - 2x^2 + 2x - 5$
$x^3 + 5x^2 + 2x + 5$	$x-1$	$x^4 + 4x^3 - 3x^2 + 3x - 5$
$x^3 + 6x^2 + 9x + 5$	$(x^2+1)(x-1)^2$	$x^7 + 4x^6 - x^5 - 3x^4 - 3x^3 - 2x^2 - x + 5$
$x^3 + 6x^2 + 8x + 5$	$(x-1)^2(x^2+1)(x^3+1)$	$x^{10} + 4x^9 - 2x^8 - x^6 - 2x^5 - 3x^4 - 2x + 5$
$x^3 + 6x^2 + 7x + 5$	$(x-1)^2$	$x^5 + 4x^4 - 4x^3 - 3x^2 - 3x + 5$

TABLE 3. $g(x)$ for the case $\det A = -5$

The proof is given in [KLR], still the theorem falls short of property (*) on the leading coefficient a_k. We conjecture that the theorem is true for all expanding polynomials $f \in \mathbb{Z}[x]$ with $a_k = 1$. If this is true, then all the self-affine tiles generated by the consecutive collinear digit sets will be connected.

3. $\mathbb{Z}$-Similar Classification.

For the expanding $A \in M_2(\mathbb{Z})$ with $|\det A| = 2$, it was shown in [LW1] that the $\mathbb{Z}$-similar classes are uniquely determined by the charateristic polynomials, there are six of them by Proposition 2.5. In this section we consider the expanding $A \in M_2(\mathbb{Z})$ with $|\det A| = 3, 4, 5$. The basic techniques for the $\mathbb{Z}$-similar classification are outlined in the following. We will need some facts concerning the polynomials and the algebraic fields.

PROPOSITION 3.1. [KL1] *Let $f(x) = x^m + a_{m-1}x^{m-1} + \cdots \pm q$, where $a_i \in \mathbb{Z}$ and q is a prime. Suppose all the roots of $f(x)$ have moduli > 1, then $f(x)$ is irreducible in $\mathbb{Q}[x]$ and all the roots are simple.*

Let R be a ring. We say that the two ideals S and T of R are in the same class if there exist two non-zero elements $\alpha, \beta \in R$ such that $\alpha S = \beta T$. This relationship determines the ideal classes of R. The following theorem allows us to convert the $\mathbb{Z}$-similar classification to the classical ideal classification.

THEOREM 3.2. (Latimer and MacDuffee [**N**]) *Let $f(x) \in \mathbb{Z}[x]$ be an irreducible monic polynomial of degree n and let θ be a root of $f(x)$. Then there is a one-to-one correspondence between the ideal classes of the ring $\mathbb{Z}[\theta]$ and the $\mathbb{Z}$-similar classes of matrices $A \in M_n(\mathbb{Z})$ such that $f(A) = 0$.*

PROPOSITION 3.3. [M, p.15] *Let m be a square free integer and let $Q[\sqrt{m}]$ be the quadratic field (i.e. the rational field generated by $\sqrt{m}$). Then the set of algebraic integers R in $Q[\sqrt{m}]$ is*

$$R = \{a + b\sqrt{m} :\ a, b \in \mathbb{Z}\} \quad \text{if } m \equiv 2 \text{ or } 3 \ (\text{mod } 4), \tag{3.1}$$

$$R = \{\frac{a + b\sqrt{m}}{2} :\ a, b \in \mathbb{Z},\ a \equiv b \ (\text{mod } 2)\} \quad \text{if } m \equiv 1 \ (\text{mod } 4). \tag{3.2}$$

The number of the ideal classes of the above rings R ($\subseteq Q[\sqrt{m}]$) is tabled in [Mo, pp. 313-345]; we list those that we will need. Here m is a square-free integer and h_m is the class number.

m	h_m
3, 5, 6, 13, 17, 21, 29, −1, −2, −3, −7, −11, −19	1
−5, −15	2

TABLE 4. Class numbers

Let $A \in M_2(\mathbb{Z})$ be an expanding matrix and let $f(x)$ be its c.p. Then the root θ of $f(x)$ can be written as $\theta = \frac{1}{2}(u + v\sqrt{m})$, $u, v, m \in \mathbb{Z}$. Our $\mathbb{Z}$-similar classification is based on the following two methods.

Method I. If f is irreducible and $\mathbb{Z}[\theta] = R$ as in (3.1) or (3.2), then we can determine the number of $\mathbb{Z}$-similar classes of f by applying the theorem of Latimer and MacDuffee and Table 4.

Method II. If the above conditions on f are not satisfied, then we will use the following scheme from [LW1]. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and let $p(A) := -a_{11}a_{22}$.

(a) If $p(A) > 0$, we consider the unimodular matrices $P = \begin{bmatrix} 1 & -\epsilon \\ 0 & 1 \end{bmatrix}$ and $P' = \begin{bmatrix} 1 & 0 \\ -\epsilon' & 1 \end{bmatrix}$ where $\epsilon := \text{sign}(a_{11}a_{21})$, $\epsilon' := \text{sign}(a_{22}a_{12})$. Let $A_1 = PAP^{-1}$ or $P'A(P')^{-1}$, then

$$p(A_1) = p(A) + a_{21}^2 + \epsilon a_{21}(a_{22} - a_{11}) \tag{3.3}$$

or, respectively

$$p(A_1) = p(A) + a_{12}^2 + \epsilon' a_{12}(a_{11} - a_{22}). \tag{3.3)'$$

We aim to have $p(A_1) < p(A)$, so as to reduce A to a $\mathbb{Z}$-similar matrix with smaller $p(\cdot)$. We can repeat this for k times to obtain a $\mathbb{Z}$-similar matrix A_k with $p(A_k)$ equals to a few specific cases.

(b) For all these specific cases, we determine their $\mathbb{Z}$-similar classes individually.

For an expanding matrix $A \in M_2(\mathbb{Z})$ with $|\det(A)| = 3$, we can make use of Proposition 2.5 to write down all the characteristic polynomials:

(I) $x^2 - 3$, $x^2 \pm 2x + 3$,
(II) $x^2 \pm x - 3$, $x^2 \pm x + 3$, $x^2 \pm 3x + 3$,
(III) $x^2 + 3$.

THEOREM 3.4. *Suppose an expanding matrix $A \in M_2(\mathbb{Z})$ has c.p. $f(x)$ in (I) or (II). Then A is $\mathbb{Z}$-similar to the companion matrix C of $f(x)$.*

On the other hand if $f(x) = x^2 + 3$ in (III), then A has two $\mathbb{Z}$-similar classes: the companion matrix C and $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$.

Sketch of proof. The polynomial $f(x) = x^2 - 3$ has a root $\theta = \sqrt{3}$ and $\mathbb{Z}[\theta]$ is of the form (3.1). Proposition 3.3 asserts that $\mathbb{Z}[\theta]$ is the set of algebraic integers in the quadratic field $Q[\sqrt{3}]$, and it has only one ideal class (see Table 4). Therefore, there is only one $\mathbb{Z}$-similar class for f by Theorem 3.2, and it is represented by the companion matrix C.

For $f(x) = x^2 \pm 2x + 3$ in (I), we can consider the roots $\theta = \pm 1 + \sqrt{-2}$ and the same argument applies.

For the polynomials in (II), we consider the respective roots $\theta = \frac{1}{2}(\pm 1 + \sqrt{13})$, $\frac{1}{2}(\pm 1 + \sqrt{-11})$, $\frac{1}{2}(\pm 3 + \sqrt{-3})$. In each of the cases, $R = Z[\theta]$ is of the form (3.2). By Proposition 3.3, they are the sets of algebraic integers in the quadratic fields $Q[\sqrt{13}]$, $Q[\sqrt{-11}]$, $Q[\sqrt{-3}]$ respectively. Again there is only one ideal class of $\mathbb{Z}[\theta]$ from Table 4 and Theorem 3.2 applies.

For the remaining case in (III), it is direct to check that $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$ has x^2+3 as characteristic polynomial and are not $\mathbb{Z}$-similar to C. Let $\theta = \sqrt{-3}$, it is a root of $x^2 + 3$ and $\mathbb{Z}[\theta] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$ is not of the form R in (3.1) and (3.2). Hence the table in [Mo] does not give the number of ideal classes of $\mathbb{Z}[\theta]$. Therefore to determine that there are only two $\mathbb{Z}$-similar classes, we will use Method II as outlined above.

Let $A = [a_{ij}] \in M_2(\mathbb{Z})$ be expanding and has c.p. $f(x) = x^2 + 3$. Then $f(x) = x^2 + 3 = x^2 - (a_{11} + a_{22})x + \det(A)$, it follows that $a_{11} + a_{22} = 0$ so that $p(A) = a_{11}^2 \geq 0$. For the case $p(A) > 0$, we claim that

$$0 < |a_{21}| \leq |a_{11}| + 1 \quad \text{or} \quad 0 < |a_{12}| \leq |a_{11}|. \tag{3.4}$$

Indeed we observe that $a_{12}, a_{21} \neq 0$ (if otherwise, we will have $3 = \det(A) = -a_{11}^2$ which is impossible). For the remaining inequalities, we assume the contrary holds, i.e., $|a_{21}| > |a_{11}| + 1$, $|a_{12}| > |a_{11}|$, then

$$|a_{12}a_{21}| \geq (|a_{11}| + 2)(|a_{11}| + 1) = |a_{11}|^2 + 3|a_{11}| + 2.$$

It follows that $3 = |\det A| \geq |a_{12}a_{21}| - |a_{11}|^2 \geq 5$ and is impossible.

For $p(A) = 0$, it is not difficult to show that $A \sim C$. For $p(A) > 0$, we divide our consideration into two cases:

(i) If $|a_{11}| = 1$, A is $\mathbb{Z}$-similar to $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix}$. The second matrix is $\mathbb{Z}$-similar to C.

(ii) If $|a_{11}| > 1$, we consider the two cases in (3.4). For $0 < |a_{21}| \leq |a_{11}| + 1$, by observing that $a_{11} = -a_{22}$, we can rewrite (3.3) as

$$p(A_1) = p(A) + a_{21}^2 + \epsilon a_{21}(a_{22} - a_{11}) = p(A) + a_{21}^2 - 2|a_{21}||a_{11}|.$$

It is elementary to show that $p(A_1) < p(A)$. For the other case $0 < |a_{12}| \leq |a_{11}|$, we can draw the same conclusion by using the alternative form $(3.3)'$:

$$p(A_1) = p(A) + a_{12}^2 + \epsilon' a_{12}(a_{11} - a_{22}).$$

We continue this process to construct $\mathbb{Z}$-similar matrices A_k with decreasing $p(A_k)$ until $p(A_k) = 0$ or $p(A_k) = 1$ as in case (i), then we conclude that $f(x)$ can only have the two $\mathbb{Z}$-similar classes as listed. □

For the expanding matrices $A \in M_2(\mathbb{Z})$ with $|\det(A)| = 4$, there are 14 characteristic polynomials. They are

(I) $x^2 \pm 3x + 4, \quad x^2 \pm x - 4,$
(II) $x^2 + 4, \quad x^2 \pm x + 4, \quad x^2 \pm 2x + 4, \quad x^2 \pm 2x - 4,$
(III) $x^2 - 4, \quad x^2 \pm 4x + 4.$

THEOREM 3.5. *Each A with c.p. in (I) is $\mathbb{Z}$-similar to its companion matrix C.*

Each c.p. in (II) has two $\mathbb{Z}$-similar classes of expanding matrices. The representatives of these classes, besides the companion matrix C, are listed according to the ordering in (II) as: $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} \pm 1 & 2 \\ -3 & \mp 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & \mp 2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & \mp 2 \end{bmatrix}.$

For $x^2 - 4$, the $\mathbb{Z}$-similar classes are $\begin{bmatrix} 2 & n \\ 0 & -2 \end{bmatrix}$ *for $n = 0, 1, 2$; and for $x^2 \pm 4x + 4$, they are* $\begin{bmatrix} \mp 2 & |n| \\ 0 & \mp 2 \end{bmatrix}$ *for all $n \in \mathbb{Z}$.*

For the proof we apply Method I to the c.p. in group (I) and $x^2 \pm x + 4$ in group (II). For the rest of the cases, we need to use Method II.

For the expanding matrices $A \in M_2(\mathbb{Z})$ with $|\det A| = 5$, there are 18 possible c.p. $f(x)$.

(I) $x^2 \pm 3x - 5, \quad x^2 \pm 2x - 5, \quad x^2 \pm x - 5,$
$x^2 \pm x + 5, \quad x^2 \pm 3x + 5, \quad x^2 \pm 4x + 5, \quad x^2 \pm 5x + 5,$
(II) $x^2 \pm 5, \quad x^2 \pm 2x + 5.$

THEOREM 3.6. *Suppose an expanding matrix $A \in M_2(\mathbb{Z})$ has c.p. $f(x)$ in (I). Then A is $\mathbb{Z}$-similar to the companion matrix of $f(x)$.*

Each c.p. $f(x)$ in (II) corresponds to two $\mathbb{Z}$-similar classes. The representing matrices of the four polynomials, besides the companion matrix, are :

$$\begin{bmatrix} 2 & 3 \\ -3 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

listed according to the order in (II).

For the proof, we apply Method I to the c.p. in group (I) and $x^2 + 5$ and Method (II) for the rest.

In Table 5, we listed the degree 2, monic, expanding integer polynomials $f(x)$ with $f(0) = \pm 3, \pm 4, \pm 5$ that correspond to more than one $\mathbb{Z}$-similar classes of matrices besides the companion matrices. They appear in Theorems 3.4, 3.5, 3.6.

A transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ is called a *similitude* if there is a constant $\alpha > 0$ such that $|S(x) - S(y)| = \alpha |x - y|$ for all $x, y \in \mathbb{R}^n$. In this case, S is of the form

c.p. of A	Number of similarity classes	Representatives of classes besides C
x^2+3	2	$\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$
x^2-2x-4	2	$\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$
x^2-4	3	$\begin{bmatrix} 2 & n \\ 0 & -2 \end{bmatrix}, n=0,2$
x^2+2x-4	2	$\begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$
x^2-4x+4	∞	$\begin{bmatrix} 2 & \lvert n\rvert \\ 0 & 2 \end{bmatrix}, \quad \lvert n\rvert \neq 1$
x^2-2x+4	2	$\begin{bmatrix} 0 & 2 \\ -2 & 2 \end{bmatrix}$
x^2-x+4	2	$\begin{bmatrix} -1 & 2 \\ -3 & 2 \end{bmatrix}$
x^2+4	2	$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$
x^2+x+4	2	$\begin{bmatrix} 1 & 2 \\ -3 & -2 \end{bmatrix}$
x^2+2x+4	2	$\begin{bmatrix} 0 & 2 \\ -2 & -2 \end{bmatrix}$
x^2+4x+4	∞	$\begin{bmatrix} -2 & \lvert n\rvert \\ 0 & -2 \end{bmatrix}, \quad \lvert n\rvert \neq 1$
x^2-2x-4	2	$\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$
x^2-5	2	$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$
x^2+2x+5	2	$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$
x^2-2x+5	2	$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$
x^2+5	2	$\begin{bmatrix} 2 & 3 \\ -3 & -2 \end{bmatrix}$

TABLE 5. Representatives of the non-unique $\mathbb{Z}$-similar classes

$Sx = Mx + b$ where M is a self-similar matrix, i.e., $|Mx| = \alpha|x|$ for all $x \in \mathbb{R}^n$. The following proposition follows directly from the definition.

PROPOSITION 3.7. *A self-similar matrix $M \in M_2(\mathbb{Z})$ is of the form* $\begin{bmatrix} m & n \\ \pm n & \mp m \end{bmatrix}$.

We can use this proposition and the above results to check if the expanding matrices are $\mathbb{Z}$-similar to the self-similar matrices. They are listed in Table 6.

If T is a tile generated by a self-similar matrix M and $\mu(T) = 1$, then there are algorithms to calculate $\dim_H(\partial T)$, the boundary of T [SW]. We recall that if L is a bi-Lipschitz map on a set E, i.e. there exist $c_1, c_2 > 0$ such that

$$c_1|x-y| \leq L|x-y| \leq c_2|x-y| \quad \text{for all } x, y \in E,$$

c.p. of A	Number of similarity classes	The classes with a self-similar representative
x^2-2	1	$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
x^2-2x+2	1	$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
x^2+2x+2	1	$\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$
x^2-4x+4	∞	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
x^2-4	3	$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$
x^2+4	2	$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$
x^2+4x+4	∞	$\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$
x^2-5	2	$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$
x^2-2x+5	2	$\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$
x^2+2x+5	2	$\begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$
x^2-4x+5	1	$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$
x^2+4x+5	1	$\begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$

TABLE 6. $\mathbb{Z}$-Similar classes with self-similar representatives

then $\dim_H(LE) = \dim_H(E)$, where $\dim_H$ denotes the Hausdorff dimension. Now if A is $\mathbb{Z}$-similar to M, i.e. $A = P^{-1}MP$ with $P \in GL_2(\mathbb{Z})$, and if we let $D' = PD$, then $T(A,D) = P^{-1}T(M,D')$ and if $\mu(T(M,D')) = 1$, then $\dim_H(\partial T(A,D))$ can be calculated.

To conclude we consider the condition for $\mu(T) > 0$. By using the notations in the beginning of Section 2, we say that $\mathcal{D}$ is a *standard digit set* if $\mathcal{D}$ is complete in $L/A(L)$. In [LW3], Lagarias and Wang proved that if q is a prime, and

$$q\mathbb{Z}^n \not\subseteq A^2(\mathbb{Z}^n), \tag{3.5}$$

then $\mu(T(A,\mathcal{D})) > 0$ if and only if $\mathcal{D}$ is a standard digit set. They also conjecture that condition (3.5) is redundant [LW3]. For the expanding A with $\det A = 3$, the cases

$$f(x) = x^2 \pm 3 \qquad \text{and} \qquad x^2 \pm 3x + 3.$$

do not satisfy (3.5). However by making use of Theorem 3.4 and a criterion of [LW3], we show that $\mu(T) > 0$ if and only if $\mathcal{D}$ is a standard digit set [KL2]. This reinforce the conjecture in [LW3].

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