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On the L^p-Lipschitz Exponents of the Scaling Functions

Ka-Sing Lau and Mang-Fai Ma

ABSTRACT. Based on a necessary and sufficient condition for the existence of L^p-scaling function as solution of a dilation equation [**LW**], we derive a formula for the L^p-Lipschitz exponent. For p an even integer (or for all $p \geq 1$ in some special cases), the formula can be simplified to a computationally efficient matrix form.

1. Introduction

A nonzero function $f(x)$ is called a *scaling function* if it satisfies the *two-scale dilation equation*

$$f(x) = \sum_{n=0}^{N} c_n f(2x - n). \tag{1.1}$$

Such functions play significant roles in wavelet theory, constructive approximation theory and fractal geometry. Their existence and regularity, treated as compactly supported L^2 or continuous solutions of (1.1), have been studied in great detail (e.g. [**CD,CH,D,DL1,DL2,H,V**] and the references there). In this paper we will continue such investigation for the compactly supported L^p-solutions $1 \leq p < \infty$ (notation: L^p_c-solutions).

For the L^2-case the analysis depends heavily on the Fourier transformation of (1.1): $\hat{f}(\xi) = m_0(\xi/2)\hat{f}(\xi/2)$ where $m_0(\xi) = 1/2 \sum c_n e^{in\xi}$. For the L^p-case we rely on the following linear algebraic set-up used by Daubechies and Lagarias [**DL2**], Collela and Heil [**CH**], and independently by Micchelli and Prautzsch [**MP**]: For any g supported by $[0, N]$, we convert g into the vector-valued function

$$\mathbf{g}(x) = \big[g(x), g(x+1), \ldots, g\big(x + (N-1)\big)\big]^t, \quad x \in [0,1] \tag{1.2}$$

and use the right hand side of the dilation equation to construct two $N \times N$ matrices T_0 and T_1 (see Section 2). Let $\mathbf{v}$ be a N-vector, let $\mathbf{f}_0(x) = \mathbf{v}$ be the initial function, and let $\{\mathbf{f}_k(x)\}_{k=0}^{\infty}$ be obtained by iterating $\mathbf{f}_0(x)$ with the matrices T_0 and T_1 inductively (the *Cascade algorithm*). The limit, if it exists, will be the solution of the dilation equation. A natural question is: is there any difference in choosing the

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initial vector? Although there are many choices of $\mathbf{v}$ for the iterated sequence to converge, we find that the most special one is the right-eigenvector corresponding to the eigenvalue 2 (notation: 2-eigenvector) of the matrix T_0+T_1. We can use this eigenvector to obtain a necessary and sufficient condition for the existence of L^p_c-solution ([**LW**] and Theorem 2.2 in the following) and also the smoothness of the solution. In [**J**], Jia studied the same existence question by using a "hat" function as initial function and obtained a slightly different criterion.

To study the regularity of the L^p_c-scaling function f, we use the *L^p-Lipschitz exponent* defined by

$$\mathrm{Lip}_p(f) = \liminf_{h\to 0^+} \frac{\ln \|\Delta_h f\|_p}{\ln h}$$

where $\Delta_h f(x) = f(x+h) - f(x)$, equivalently,

$$\mathrm{Lip}_p(f) = \inf\left\{ s\colon 0 < \limsup_{h\to 0^+} h^{-s} \|\Delta_h f\|_p \right\}.$$

When $p = 2$, α equals the Sobolev exponent of f. Recently there is a great interest on the L^p-Lipschitz exponent (L^p-dimension if measures are concerned) in connection to the multifractal theory (e.g. [**FP,HJ,DL3,J1,2,LN**]). The function $\tau(p) = p\,\mathrm{Lip}_p(f)$ (called the *moment scaling exponent* of f) has a very elegant heuristic relationship with $\gamma(\alpha) = \dim_H\{x\colon \lim_{h\to 0^+}\big(\ln|\Delta_h f(x)|/\ln h\big) = \alpha\}$, the Hausdorff dimension of the set of all x with local Lipschitz exponent α, namely the *multifractal formalism*: the Legendre transformation (equivalently, the concave conjugate) of $\tau(p)$ is $\gamma(\alpha)$. This elegant relationship was proposed by physicists [**FP,HJ**] and attracts a lot of attention from mathematicians. Daubechies and Lagarias [**DL3**] showed that the formalism does not hold for the scaling function D_4. Nevertheless it was observed that the failure is due to the nondifferentiability of $\tau(p)$ at $p = 2$ (see Figure 2e) and it raises more interesting questions on the L^p-Lipschitz exponent.

For the scaling functions, the L^2-Lipschitz exponent has been studied in detail in [**CD,H**] and [**V**] and sharpened in [**LMW**]. In this note we will consider the L^p-case. We note that in the proof of existence, the iteration of $\mathbf{f}_k$, starting with the 2-eigenvector $\mathbf{v}$, converges to the solution $\mathbf{f}$ at a geometric rate. Amazingly, this rate actually gives us the L^p-Lipschitz exponent of f (Theorem 3.1). The rate is given by a series of matrix products and is therefore not computationally efficient. For some special cases (including those considered in [**DL3**]), we can reduce it to simple formulas. For the more general case we can still calculate the L^p-Lipschitz exponent when p is a positive integer. Note that in our consideration we only treat the Lipschitz exponent of order not greater that 1. For higher order of smoothness we need to consider the higher order difference $\Delta_h^k f$.

In the following we give the results and sketch some of the proofs. The detail will appear in [**LM**].

2. Existence of L^p_c-Scaling Functions

It is known that if (1.1) has an integrable solution, then $\sum c_n = 2^m$, $m > 0$ an integer (m is the order of zero of $\hat{f}(\xi)$ at 0, and $m = 1$ if $\int f \neq 0$) [**DL1**]. We will hence use the natural assumption $\sum c_n = 2$ on the coefficients unless otherwise specified. Let $T_0 = [c_{2i-j-1}]_{1\le i,j\le N}$ and $T_1 = [c_{2i-j}]_{1\le i,j\le N}$, i.e.,

$$T_0 = \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 \\ c_2 & c_1 & c_0 & \cdots & 0 \\ c_4 & c_3 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{N-1} \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_1 & c_0 & 0 & \cdots & 0 \\ c_3 & c_2 & c_1 & \cdots & 0 \\ c_5 & c_4 & c_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_N \end{pmatrix}.$$

For any $g \in L^p(\mathbb{R})$ with support in $[0, N]$, let $\mathbf{g}(x)$ be the vector-valued function representing g as in (1.2) and let

$$\mathbf{T}g(x) = \begin{cases} T_0\mathbf{g}(2x) & \text{if } x \in [0, 1/2), \\ T_1\mathbf{g}(2x-1) & \text{if } x \in [1/2, 1). \end{cases}$$

It is easy to show that f is a solution of (1.1) if and only if $\mathbf{f} = \mathbf{T}f$. With no confusion, we use $\|\cdot\|$ to denote the L^p-norm of g and also the vector-valued function $\mathbf{g}$. For any $k \ge 1$, we let $J = (j_1, \dots, j_k)$, $j_i = 0$ or 1, denote the multi-index ($J = \emptyset$ if $k = 0$), and let $|J|$ denote the length of J. The matrix T_J represents the product $T_{j_1} \cdots T_{j_k}$; $I_J = I_{(j_1,\dots,j_k)} = [a, b)$ denotes the dyadic interval where

$$a = \frac{j_1}{2} + \frac{j_2}{2^2} + \cdots + \frac{j_k}{2^k} \quad \text{and} \quad b = a + \frac{1}{2^k},$$

and g_I means the average $|I|^{-1}\int_I g(x)\,dx$ of g on an interval I. We first give a necessary condition for (1.1) to have an L^p_c-solution [**LW**].

PROPOSITION 2.1. *Assume* $\sum_{n=0}^N c_n = 2$. *For* $1 \le p < \infty$, *let* f *be an* L^p_c-*solution of* (1.1) *and let* $\mathbf{v} = [f_{[0,1]}, \dots, f_{[N-1,N]}]^t$ *be the vector defined by the average of* f *on the* N *subintervals. Then*

(i) $\mathbf{v}$ *is a* 2-*eigenvector of* $(T_0 + T_1)$.

(ii) *Let* $\mathbf{f}_0(x) = \mathbf{v}$, $x \in [0, 1)$, *and let* $\mathbf{f}_{n+1} = \mathbf{T}f_n$, $n = 0, 1, \dots$, *then* $\mathbf{f}_n(x) = \sum_{|J|=n}(T_J\mathbf{v})\chi_{I_J}(x)$, $x \in [0, 1)$, *and* $\mathbf{f}_n \to \mathbf{f}$ *in* $L^p([0,1], \mathbb{R}^N)$.

To look for a criterion for the existence of L^p_c-solution, the above proposition suggests that we should concentrate on the 2-eigenvector of $T_0 + T_1$. Let $\mathbf{v}$ be such a vector, then $(T_0 - I)\mathbf{v} = -(T_1 - I)\mathbf{v}$. Let $\tilde{\mathbf{v}} = (T_0 - I)\mathbf{v}$ and let $H(\tilde{\mathbf{v}})$ be the subspace in $\mathbb{R}^N$ spanned by $\{T_J\tilde{\mathbf{v}}: J \text{ a multi-index}\}$. Now, let $\{\mathbf{f}_k\}_{k=0}^\infty$ be defined as in the proposition, then $\mathbf{f}_n = \mathbf{f}_0 + \sum_{k=0}^{n-1}(\mathbf{f}_{k+1} - \mathbf{f}_k)$ and

$$\begin{aligned} \|\mathbf{f}_{k+1} - \mathbf{f}_k\|^p &= \frac{1}{2^{k+1}} \sum_{|J|=k} \big\|T_J(T_0 - I)\mathbf{v}\big\|^p + \big\|T_J(T_1 - I)\mathbf{v}\big\|^p \\ &= \frac{1}{2^k}\sum_{|J|=k} \|T_J\tilde{\mathbf{v}}\|^p. \end{aligned} \tag{2.1}$$

If $2^{-k}\sum_{|J|=k}\|T_J\tilde{\mathbf{v}}\|^p \to 0$, it can be shown that it actually converges at a geometric rate, hence $\{\mathbf{f}_n\}$ converges and the limit is the L^p_c-solution. Indeed we have the following stronger result [**LW**]:

THEOREM 2.2. *Suppose* $\sum_{n=0}^N c_n = 2$ *and* $1 \le p < \infty$. *Then the following statements are equivalent*:

(i) *Equation* (1.1) *has a nonzero* L^p_c-*solution.*

(ii) *There exists a 2-eigenvector* $\mathbf{v}$ *of* (T_0+T_1) *satisfying*

$$\lim_{n\to\infty}\frac{1}{2^n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p=0.$$

(iii) *There exist a 2-eigenvector* $\mathbf{v}$ *of* (T_0+T_1) *and an integer* $\ell\geq 1$ *such that*

$$\frac{1}{2^\ell}\sum_{|J|=\ell}\|T_J\mathbf{u}\|^p<1 \textit{ for all } \mathbf{u}\in H(\tilde{\mathbf{v}}),\quad \|\mathbf{u}\|\leq 1.$$

We remark that in [**J**], Jia used the initial function f_0 defined by

$$f_0(x)=\begin{cases}1+x & \text{if } x\in[-1,0),\\ 1-x & \text{if } x\in[0,1),\\ 0 & \text{otherwise}\end{cases}$$

and showed that the necessary and sufficient condition for the L^p-convergence of the iteration is the existence of ℓ such that

$$\frac{1}{2^\ell}\sum_{|J|=\ell}\|T_J\mathbf{u}\|^p<1 \text{ for all } \mathbf{u}\in H:=\left\{\mathbf{u}\in\mathbb{R}^N:\sum_{i=1}^N u_i=0\right\},\quad \|\mathbf{u}\|\leq 1.$$

Note that (iii) is a weaker condition and $H=H(\tilde{\mathbf{v}})$ in many cases.

3. Regularity of L^p_c-Scaling Functions

In this section we assume that f is an L^p_c-solution of (1.1). It follows from Theorem 2.2 that $\lim_{n\to\infty}2^{-n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p=0$. In fact the convergence is at a geometric rate [**LW**]. Under some slightly stronger conditions, this rate actually gives us the L^p-Lipschitz exponent of f.

THEOREM 3.1. *Assume* $\sum c_{2n}=\sum c_{2n+1}=1$ *and* 1 *is a simple eigenvalue of* T_0 *and* T_1. *Then for* $1\leq p<\infty$,

$$\operatorname{Lip}_p(f)=\liminf_{n\to\infty}\frac{\ln(2^{-n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p)}{p\ln(2^{-n})}. \tag{3.1}$$

We will give a sketch of the proof. First we need some basic facts concerning the eigenvectors of T_0 and T_1. Some of them can be found in [**H**]. Let $H=\{\mathbf{u}\in\mathbb{R}^N:\sum_{i=1}^N u_i=0\}$.

LEMMA 3.2. *Suppose* $\sum c_{2n}=\sum c_{2n+1}=1$. *Let* $\mathbf{v}$ *be a 2-eigenvector of* (T_0+T_1), $\tilde{\mathbf{v}}=(T_0-I)\mathbf{v}$ *and* $H(\tilde{\mathbf{v}})$ *be the subspace spanned by the* $T_J\tilde{\mathbf{v}}$*'s. Then*

(i) $H(\tilde{\mathbf{v}})$ *is a subspace of* H.

(ii) 1 *is an eigenvalue of* T_0 *and* T_1; *let* $\mathbf{v}_0$ *and* $\mathbf{v}_1$ *be the corresponding eigenvectors, then* $T_0\mathbf{v}_1=T_1\mathbf{v}_0$.

(iii) $\mathbf{v}$ *can be decomposed as* $\mathbf{v}=c\mathbf{v}_0+\mathbf{h}_0=c\mathbf{v}_1+\mathbf{h}_1$ *where* $c>0$ *and* $\mathbf{h}_0$, $\mathbf{h}_1\in H$; *furthermore if* 1 *is a simple eigenvalue of* T_0 *and* T_1, *then* $\mathbf{h}_0$, $\mathbf{h}_1\in H(\tilde{\mathbf{v}})$.

It is easy to see that the assumption $\sum c_{2n} = \sum c_{2n+1} = 1$ implies $\tilde{\mathbf{v}} \in H$ and H is invariant under T_0 and T_1. Hence (i) follows immediately. Note that 1 is an eigenvalue of the common submatrix $M = [c_{2i-j}]_{1\le i,j\le N-1}$ of T_0 and T_1. If $\mathbf{a} = [a_1, \dots, a_{N-1}]^t$ is an 1-eigenvector of M, then $\mathbf{v}_0 := [0, a_1, \dots, a_{N-1}]^t$ and $\mathbf{v}_1 := [a_1, \dots, a_{N-1}, 0]^t$ are 1-eigenvectors of T_0 and T_1 respectively, and a direct calculation yields (ii). For (iii) we need only choose $c = \sum_{i=1}^N v_i / \sum_{i=1}^{N-1} a_i$ where v_i are the coordinates of $\mathbf{v}$. To prove the last statement, we note that the assumption implies that $T_0 - I$ restricted on H is bijective; it is hence also bijective on the $(T_0 - I)$-invariant subspace $H(\tilde{\mathbf{v}})$. From the decomposition we have $(T_0 - I)\mathbf{h}_0 = \tilde{\mathbf{v}}$ and hence $\mathbf{h}_0$ must be in $H(\tilde{\mathbf{v}})$. The case for $\mathbf{h}_1$ is the same.

To prove the theorem, we let $\mathbf{v}$ be the 2-eigenvector of $T_0 + T_1$ satisfying condition (ii) of Theorem 2.2, $\mathbf{f}_n(x) = \sum_{|J|=n} (T_J \mathbf{v})\chi_{I_J}(x)$, and f_n the corresponding function on $\mathbb{R}$ supported by $[0, N]$. For fixed $0 < h < 1/2$, let n be such that $2^{-(n+1)} \le h < 2^{-n}$. We claim that $\|\Delta_h f_n\|$ has order $2^{-n} \sum_{|J|=n} \|T_J \tilde{\mathbf{v}}\|^p$. Indeed, let $E_n = \bigcup_{j=0}^{N-1} [j,\ j+1-2^{-n})$ and $\widetilde{E}_n = [-2^{-n}, N] \setminus E_n$. Since f_n is supported by $[0,\ N]$, we have

$$\begin{aligned}
\|\Delta_h f_n\|^p &= \int_{-2^{-n}}^{N} \left|\Delta_h f_n(x)\right|^p dx \\
&= \int_{E_n} \left|\Delta_h f_n(x)\right|^p dx + \int_{\widetilde{E}_n} \left|\Delta_h f_n(x)\right|^p dx \\
&= \int_0^{1-2^{-n}} \left\|\mathbf{f}_n(x+h) - \mathbf{f}_n(x)\right\|^p dx + h \sum_{k=0}^{N} \left| f_{[k,k+2^{-n})} - f_{[k-2^{-n},k)}\right|^p \\
&:= I_1 + I_2
\end{aligned}$$

where $f_{[a,b)}$ is the average of f on the interval $[a, b)$. By Lemma 3.2(ii), (iii) and using $\mathbf{f}_n(x) = \sum_{|J|=n} (T_J \mathbf{v})\chi_{I_J}(x)$, we have

$$I_1 \le C' \frac{1}{2^n} \left[\sum_{k=0}^{n-1} \sum_{|J|=n-k-1} \|T_J T_1 T_0^k \mathbf{h}_0\|^p + \sum_{k=0}^{n-1} \sum_{|J|=n-k-1} \|T_J T_0 T_1^k \mathbf{h}_1\|^p \right]$$

and

$$I_2 \le C'' \frac{1}{2^n} \left(\Big\| T_{\underbrace{0\cdots 0}_{n \text{ terms}}} \mathbf{h}_0 \Big\|^p + \Big\| T_{\underbrace{1\cdots 1}_{n \text{ terms}}} \mathbf{h}_1 \Big\|^p \right),$$

where C' and C'' are constants independent of n. Hence we have

$$\|\Delta_h f_n\|^p \le C_1 \frac{1}{2^n} \left(\sum_{|J|=n} \|T_J \mathbf{h}_0\|^p + \sum_{|J|=n} \|T_J \mathbf{h}_1\|^p \right)$$

for some constant C_1. The claim follows from observing that the terms on the right hand side have the desired order. Now note that $\|\Delta_h f\| \le 2\|f - f_n\| + \|\Delta_h f_n\|$ and that $\|f - f_n\|^p$ has order $2^{-n} \sum_{|J|=n} \|T_J \tilde{\mathbf{v}}\|^p$ as $n \to \infty$ (by (2.1) and Theorem 2.2). This together with the claim implies "$\ge$" of (3.1).

The reverse inequality follows from the following inequality which can be proved by the similar argument as above: there exists C_2 such that for $n, \ell \geq 1$,

$$C_2\left(\frac{1}{2^{n-1}}\sum_{|J|=n-1}\|T_J\tilde{\mathbf{v}}\|^p\right) \leq \|\Delta_{2^{-n}}f_{n+\ell}\|^p.$$

4. Reduction of the Criterion

In the following we consider solutions f of some special four-coefficient dilation equations with $c_0+c_2 = c_1+c_3 = 1$. We first calculate the $\mathrm{Lip}_p(f)$ for some special cases.

EXAMPLE 1. $c_3 = 0$. This is actually a 3-coefficient case with $c_0 + c_2 = 1$ and $c_1 = 1$. By a direct calculation, $\tilde{\mathbf{v}} = \left[c_0(c_0-1), -c_0(c_0-1)\right]^t$ is a common eigenvector of T_0 and T_1 with eigenvalues c_0 and $1 - c_0$ respectively. From this we show that $2^{-n}\sum_{|J|=n}\|T_J\tilde{\mathbf{v}}\|^p = 2^{-n}\left(|c_0|^p + |1-c_0|^p\right)^n\|\tilde{\mathbf{v}}\|^p$. It follows that

$$\mathrm{Lip}_p(f) = \frac{\ln(2^{-1}(|c_0|^p + |1-c_0|^p))}{-p\ln 2}.$$

EXAMPLE 2. $c_0 + c_3 = 1$. In this case $\tilde{\mathbf{v}} = [-c_0^2c_3, c_0^2c_3 - c_0c_3^2, c_0c_3^2]^t$ is again a common eigenvector of T_0 and T_1 with eigenvalues c_0 and $1 - c_0$ respectively. It follows that $\mathrm{Lip}_p(f)$ is the same as in Example 1 by using an identical calculation. (See Appendix, Figure 1 for some graphs.)

EXAMPLE 3. $c_0+c_3 = 1/2$. (This includes the Daubechies scaling function D_4 where $c_0 = (1+\sqrt{3})/4, c_3 = (1-\sqrt{3})/4$). Note that $\mathbf{u} = [0,1,-1]^t$, $\mathbf{h} = [1,-2,1]^t$ are eigenvectors of T_0 corresponding to the eigenvalues $1/2$ and c_0; also $T_1\mathbf{u} = 1/2\mathbf{u} + c_0\mathbf{h}$ and $T_1\mathbf{h} = (1/2 - c_0)\mathbf{h}$. By using $\{\mathbf{u}, \mathbf{h}\}$ as a basis of the subspace $H(\tilde{\mathbf{v}}) = H$, we can rewrite T_0 and T_1 on H as

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & c_0 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & 0 \\ c_0 & \frac{1}{2} - c_0 \end{pmatrix}.$$

Let $\beta_0 = c_0$ and $\beta_1 = \frac{1}{2} - c_0$. Then the corresponding matrix for T_J is

$$\begin{pmatrix} 2^{-n} & 0 \\ \lambda_J & \mu_J \end{pmatrix},$$

where $\lambda_J = \beta_0(j_1 2^{-(n-1)} + j_2 2^{-(n-2)}\beta_{j_1} + \cdots + j_n\beta_{j_1}\cdots\beta_{j_{n-1}})$, $\mu_J = \beta_{j_1}\beta_{j_2}\cdots\beta_{j_n}$, $j_i = 0$ or 1. A direct estimation yields

$$\mathrm{Lip}_p(f) = \min\left\{1, \frac{\ln(2^{-1}(|c_0|^p + |1/2 - c_0|^p))}{-p\ln 2}\right\}.$$

This formula was also proved by Daubechies and Lagarias [**DL3**] with another method and the additional assumption that $1/2 < c_0 < 3/4$. (See Appendix, Figure 2 for some graphs.)

In all the three examples the computation depends on the existence of a common eigenvector of T_0 and T_1 (which may be associated with different eigenvalues). This technique cannot be used for the general case. However we show in the following that if p is a positive even integer, then $\mathrm{Lip}_p(f)$ is related to the spectral radius of a matrix W_p whose entries are induced from the coefficients of the dilation equation. For simplicity, we only give the construction of W_p for the 4-coefficient dilation equation. It is not hard to extend this to the case with more coefficients.

We will first develop a simple expression for the sum $2^{-n}\sum_{|J|=n}\|T_J\mathbf{v}\|^p$ for p a positive even integer. Let $\mathbf{e}_1 = [0,1,-1]^t$ and $\mathbf{e}_2 = [1,-1,0]^t$ be a basis of $H = \{\mathbf{u}\in\mathbb{C}^3 : \sum u_i = 0\}$. Then

$$T_0 = \begin{pmatrix} 1-c_0-c_3 & c_3 \\ 0 & c_0 \end{pmatrix}, \quad T_1 = \begin{pmatrix} c_3 & 0 \\ c_0 & 1-c_0-c_3 \end{pmatrix} \text{ and } \tilde{\mathbf{v}} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

We define the vector $\mathbf{a}_n$ with the i-th entry by

$$(\mathbf{a}_n)_i = \sum_{|J|=n} (\mathbf{e}_0 T_J\mathbf{u})^{p-i}(\mathbf{e}_1 T_J\mathbf{u})^i, \quad i = 0,\ldots,p.$$

If p is an even integer, then

$$\begin{aligned}\sum_{|J|=n}\|T_J\mathbf{u}\|^p &= \sum_{|J|=n}\left(|\mathbf{e}_0T_J\mathbf{u}|^p + |\mathbf{e}_1T_J\mathbf{u}|^p\right) \\ &= (\mathbf{a}_n)_0 + (\mathbf{a}_n)_p = \left|(\mathbf{a}_n)_0\right| + \left|(\mathbf{a}_n)_p\right|.\end{aligned} \tag{4.1}$$

Note that $\mathbf{a}_0 = [\alpha^p, \alpha^{p-1}\beta, \ldots, \alpha\beta^{p-1}, \beta^p]^t$ for some α, β. If we let $d = 1-c_0-c_3$, we have, in view of the expression of T_0 and T_1 on H and by using simple iteration we have

PROPOSITION 4.1. *For any integer $p \geq 1$, we have*

$$\mathbf{a}_{n+1} = W_p\mathbf{a}_n = W_p^{n+1}\mathbf{a}_0$$

where W_p is a $(p+1)\times(p+1)$ matrix defined by

$$(W_p)_{ij} = \begin{cases} \binom{p-i}{j-i} c_0^i c_3^{j-i} d^{p-j} & \text{for } 0 \leq i < j \leq p \\ c_0^i d^{p-i} + c_3^{p-i} d^i & \text{for } i = j \\ \binom{i}{j} c_0^{i-j} c_3^{p-i} d^j & \text{for } 0 \leq j < i \leq p \end{cases}$$

where $d = 1-c_0-c_3$. In particular if p is an even integer, then

$$\sum_{|J|=n}\|T_J\mathbf{u}\|^p = [1,0,0,\ldots,0,1]W_p^n\mathbf{a}_0.$$

The matrix W_p can be written as $W_p = W_p^{(L)} + W_p^{(U)}$, where $W_p^{(L)}$ and $W_p^{(U)}$ are the lower and upper triangular part of W_p, in a very symmetric manner. For example,

$$W_2^{(L)} = \begin{pmatrix} \binom{0}{0}c_3^2 & 0 & 0 \\ \binom{1}{0}c_0c_3 & \binom{1}{1}c_3d & 0 \\ \binom{2}{0}c_0^2 & \binom{2}{1}c_0d & \binom{2}{2}d^2 \end{pmatrix}, \quad W_2^{(U)} = \begin{pmatrix} \binom{2}{0}d^2 & \binom{2}{1}c_3d & \binom{2}{2}c_3^2 \\ 0 & \binom{1}{0}c_0d & \binom{1}{1}c_0c_3 \\ 0 & 0 & \binom{0}{0}c_0^2 \end{pmatrix};$$

$$W_4^{(L)} = \begin{pmatrix} \binom{0}{0}c_3^4 & 0 & 0 & 0 & 0 \\ \binom{1}{0}c_0c_3^3 & \binom{1}{1}c_3^3d & 0 & 0 & 0 \\ \binom{2}{0}c_0^2c_3^2 & \binom{2}{1}c_0c_3^2d & \binom{2}{2}c_3^2d^2 & 0 & 0 \\ \binom{3}{0}c_0^3c_3 & \binom{3}{1}c_0^2c_3d & \binom{3}{2}c_0c_3d^2 & \binom{3}{3}c_3d^3 & 0 \\ \binom{4}{0}c_0^4 & \binom{4}{1}c_0^3d & \binom{4}{2}c_0^2d^2 & \binom{4}{3}c_0d^3 & \binom{4}{4}d^4 \end{pmatrix},$$

$$W_4^{(U)} = \begin{pmatrix} \binom{4}{0}d^4 & \binom{4}{1}c_3d^3 & \binom{4}{2}c_3^2d^2 & \binom{4}{3}c_3^3d & \binom{4}{4}c_3^4 \\ 0 & \binom{3}{0}c_0d^3 & \binom{3}{1}c_0c_3d^2 & \binom{3}{2}c_0c_3^2d & \binom{3}{3}c_0c_3^3 \\ 0 & 0 & \binom{3}{0}c_0^2d^2 & \binom{2}{1}c_0^2c_3d & \binom{2}{2}c_0^2c_3^2 \\ 0 & 0 & 0 & \binom{1}{0}c_0^3d & \binom{1}{1}c_0^3c_3 \\ 0 & 0 & 0 & 0 & \binom{0}{0}c_0^4 \end{pmatrix}.$$

For the 4-coefficient dilation equation, it is easy to check that $\dim H(\tilde{\mathbf{v}}) = 0$ if and only if $(c_0, c_3) \in \{(0,0), (1,0), (0,1), (1,1)\}$, and the solutions are characteristic functions ([**LW**], Lemma 3.3]). Hence $\mathrm{Lip}_p(f) = 1/p$. Also $\dim H(\tilde{\mathbf{v}}) = 1$ if and only if $c_0 + c_3 = 1$, and in Example 2 we have given a formula of $\mathrm{Lip}_p(f)$ for this case. It remains to consider the case $\dim H(\tilde{\mathbf{v}}) = 2$, which will complete all the cases for all 4-coefficient scaling functions.

THEOREM 4.2. *Consider the* 4*-coefficient dilation equation with the assumption that* $\dim H(\tilde{\mathbf{v}}) = 2$. *For p a positive even integer, the equation has a (unique) L_c^p-solution f if and only if $\rho(W_p)/2 < 1$, and in this case*

$$\mathrm{Lip}_p(f) = \frac{\ln(\rho(W_p)/2)}{-p \ln 2}.$$

The proof follows easily from Proposition 4.1, together with Theorems 2.2 and 3.1. We remark that when $p = 2$, the matrix W_2 is the same as the transition matrix obtained in [**DL3,H,LMW**] and [**V**] and the L^2-Lipschitz exponent and the Sobolev exponent coincide. If $c_0 > 0$, $c_3 > 0$, and $1 - c_0 - c_3 > 0$, then T_0 and T_1 are non-negative matrices. Hence (4.1) still holds if p is a positive odd integer. Consequently, we have

COROLLARY 4.3. *Consider the* 4*-coefficient dilation equation with $c_0 > 0$, $c_3 > 0$, and $1 - c_0 - c_3 > 0$. Suppose* $\dim H(\tilde{\mathbf{v}}) = 2$, *then the conclusion of Theorem* 4.2 *holds for p a positive integer.*

For the four coefficient case with $c_0 + c_2 = 1$, $c_1 + c_3 = 1$, we use (c_0, c_3) as the two free parameters. Figure 3 shows the domain of (c_0, c_3) for the existence of L_c^p-solutions for even integers using the above criterion $\rho(W_p)/2 < 1$. The curves are $\rho(W_p)/2 = 1$ corresponds to $p = 2, 4, 6, 10, 20$, and 40. Note that when $p \to \infty$ the limit is the triangular region which is the approximate region plotted in [**H**] for the existence of continuous 4-coefficient scaling functions using the joint spectral radius.

APPENDIX. We illustrated some of our results by the graphs. Figure 1 gives the graphs of the scaling functions together with the curve of L^p-Lipschitz exponent in Example 2, and Figure 2 is for Example 3.

Figure 4 is the graph of $\mathrm{Lip}_4(f)$ plotted against the (c_0, c_3)-plane, using Theorem 4.2. It shows the overall picture of $\mathrm{Lip}_4(f)$ for the 4-coefficient case. It looks similar to the graph of $\mathrm{Lip}_2(f)$ plotted in [**LMW**].

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DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, HONG KONG.
E-mail address: kslau@euler.math.cuhk.edu.hk

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA 15260, USA
E-mail address: mfma@vms.cis.pitt.edu

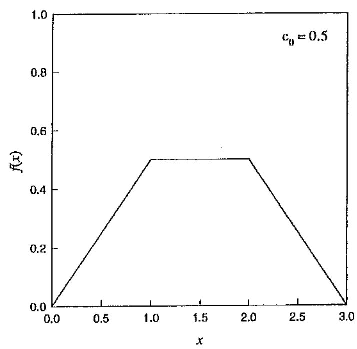


Figure 1a

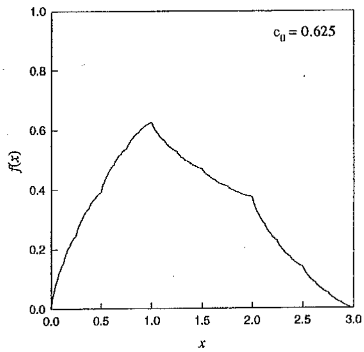


Figure 1b

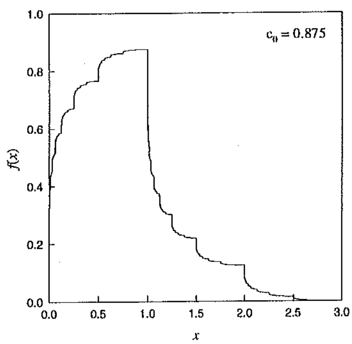


Figure 1c

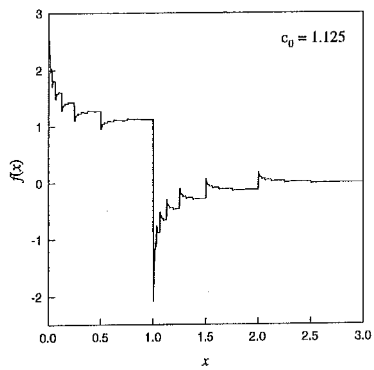
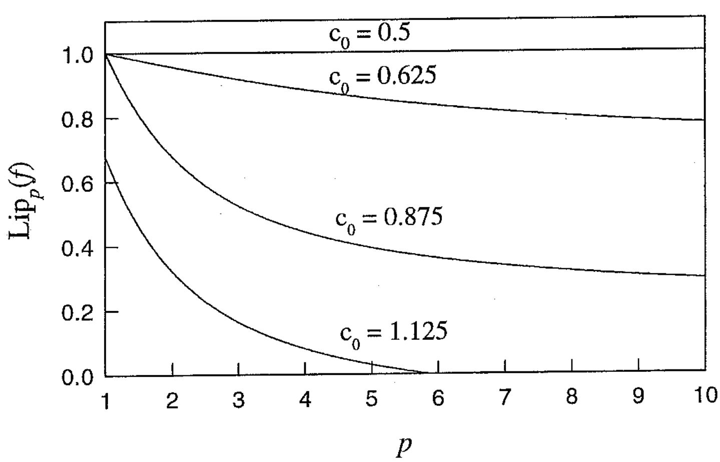


Figure 1d



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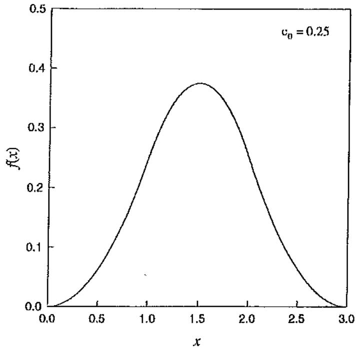
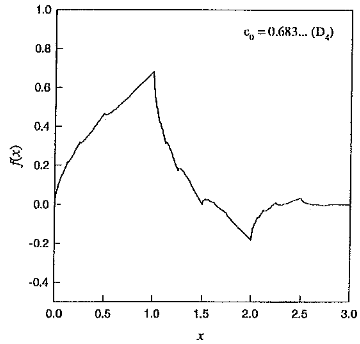


Figure 2a



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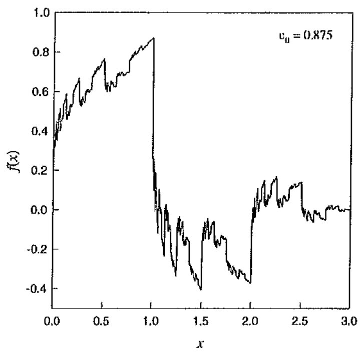

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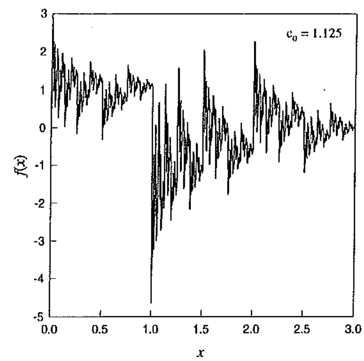

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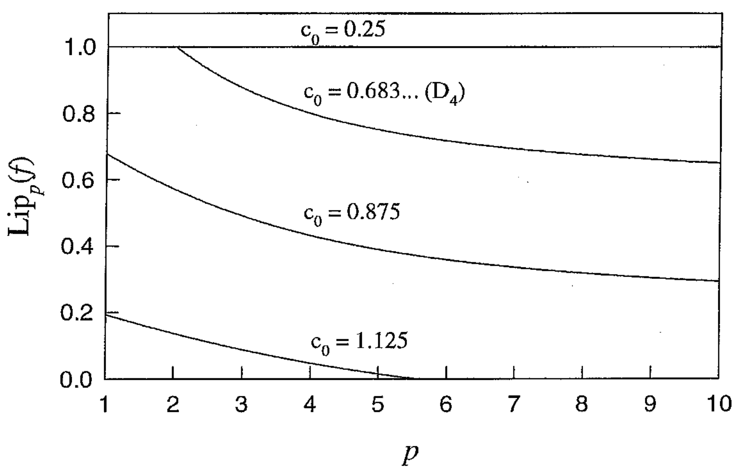

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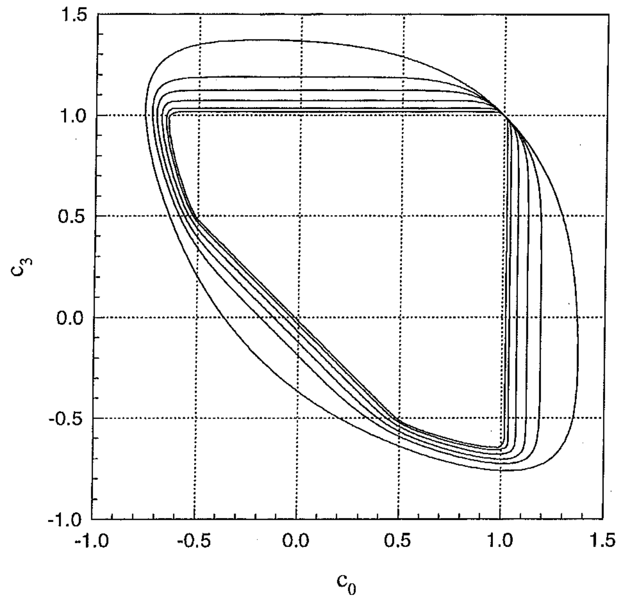

Figure 3

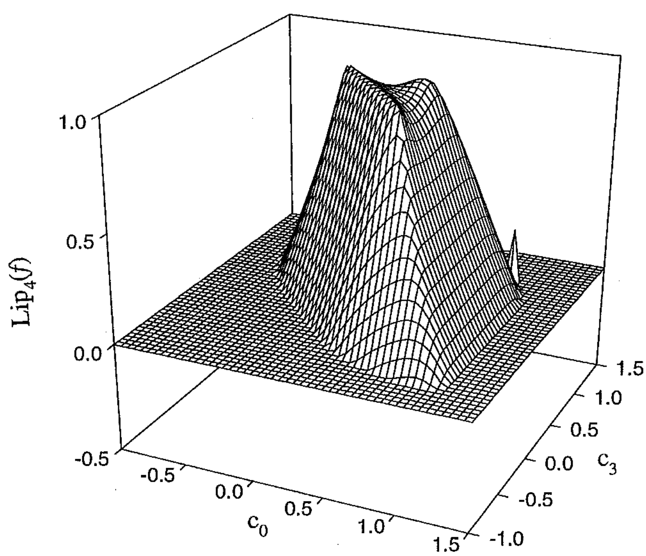

Figure 4