Available online at www.sciencedirect.com

ScienceDirect

J. Math. Pures Appl. 92 (2009) 407–428

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

Multifractal formalism for self-similar measures with weak separation condition

De-Jun Feng *, Ka-Sing Lau

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

Received 21 July 2008

Available online 29 May 2009

Abstract

For any self-similar measure μ on $\mathbb{R}^d$ satisfying the weak separation condition, we show that there exists an open ball U_0 with $\mu(U_0) > 0$ such that the distribution of μ, restricted on U_0, is controlled by the products of a family of non-negative matrices, and hence $\mu|_{U_0}$ satisfies a kind of quasi-product property. Furthermore, the multifractal formalism for $\mu|_{U_0}$ is valid on the whole range of dimension spectrum, regardless of whether there are phase transitions. Moreover the dimension spectra of μ and $\mu|_{U_0}$ coincide for $q \geqslant 0$. This result unifies and improves many of the recent works on the multifractal structure of self-similar measures with overlaps.

Résumé

On montre que pour toute mesure autosimilaire sur $\mathbb{R}^d$ satisfaisant la condition de séparation faible, il existe une boule U_0 telle que $\mu(U_0) > 0$ ainsi qu'une famille finie $\mathcal{F}$ de matrices positives telles que $\mu|_{U_0}$, la distribution de μ restreinte à U_0, soit contrôlée par des produits d'éléments de $\mathcal{F}$, de sorte que $\mu|_{U_0}$ satisfasse une propriété de type quasimultiplicativité. De plus, le formalisme multifractal est valide pour $\mu|_{U_0}$ sur tout l'intervalle de définition du spectre des singularités, qu'il y ait ou non des transitions de phases. Enfin, les spectres de singularités de μ et $\mu|_{U_0}$ coïncident pour $q \geqslant 0$. Ces résultats unifient et améliorent un grand nombre de travaux récents portant sur la structure multifractale des mesures autosimilaires avec recouvrements.

MSC: 28A78; 28A80

Keywords: Self-similar measures; Multifractal formalism; Weak separation condition; Moran structure

1. Introduction

Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. For any open set $V \subset \mathbb{R}^d$ with $\mu(V) > 0$, let $\tau_V(q)$, $q \in \mathbb{R}$, be the *L^q-spectrum of μ restricted on V*, which is defined by:

$$\tau_V(q) = \liminf_{r\to 0} \frac{\log \Theta_V(q;r)}{\log r},$$

* Corresponding author.
E-mail addresses: djfeng@math.cuhk.edu.hk (D.-J. Feng), kslau@math.cuhk.edu.hk (K.-S. Lau).

0021-7824/$ – see front matter © 2009 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.matpur.2009.05.009

where

$$\Theta_V(q;r)=\sup\sum_i \mu\big(B(x_i,r)\big)^q,\quad r>0,\ q\in\mathbb{R},$$

and the supremum is taken over all families of disjoint closed balls $\{B(x_i,r)\}_i$ contained in V with $x_i\in\operatorname{supp}(\mu)$. For any $\alpha\geqslant 0$, define:

$$E_V(\alpha)=\big\{x\in V\cap\operatorname{supp}\mu:\ \alpha(x)=\alpha\big\},$$

where $\alpha(x)$ is the local dimension of μ at x defined by

$$\alpha(x)=\lim_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}$$

provided that the limit exists. In particular, for $V=\mathbb{R}^d$, we write $\tau_{\mathbb{R}^d}(q)=\tau(q)$ and $E_{\mathbb{R}^d}(\alpha)=E(\alpha)$, and call them the *$L^q$-spectrum* and the *level set* of μ respectively. Moreover, we call $\dim_H E(\alpha)$ the *dimension spectrum* of μ, and $\dim_H E_V(\alpha)$ the *dimension spectrum of μ restricted on V*, where $\dim_H$ denotes the Hausdorff dimension.

In this paper we focus our consideration on self-similar measures. For $1\leqslant i\leqslant m$, let $S_i:\mathbb{R}^d\to\mathbb{R}^d$ be contractive similitudes,

$$S_i(x)=\rho_i R_i(x)+b_i, \tag{1.1}$$

where $0<\rho_i<1$, $b_i\in\mathbb{R}^d$ and R_i are orthogonal transformations. As usual, we call $\{S_i\}_{i=1}^m$ an *iterated function system* (IFS). It follows that there is a unique non-empty compact set $K\subset\mathbb{R}^d$ such that $K=\bigcup_{i=1}^m S_i(K)$ [20]. The set K is called the *self-similar set generated by* $\{S_i\}_{i=1}^m$. Furthermore, for any given probability vector $(p_1,\dots,p_m)$, i.e., $p_i>0$ for $1\leqslant i\leqslant m$ and $\sum_{i=1}^m p_i=1$, there is a unique Borel probability measure μ on $\mathbb{R}^d$ satisfying the self-similar relation:

$$\mu=\sum_{i=1}^m p_i\,\mu\circ S_i^{-1}.$$

The measure μ is supported by K and is called a *self-similar measure*.

One of the main objectives of multifractal analysis is to study the dimension spectrum and its relation with the L^q-spectrum for a given measure. Usually it is difficult or impossible to calculate the dimension spectrum of a given measure directly. The celebrated heuristic principle known as the *multifractal formalism* which was first introduced by some physicists [16], states that the dimension spectrum $\dim_H E(\alpha)$ can be recovered by the L^q-spectrum $\tau(q)$ through the Legendre transform:

$$\dim_H E(\alpha)=\tau^*(\alpha):=\inf\big\{\alpha q-\tau(q):\ q\in\mathbb{R}\big\}. \tag{1.2}$$

For backgrounds and the rigorous mathematical foundations of the multifractal formalism, we refer to [7,31,28]. The formalism has been verified to hold for many natural measures, including for example, Gibbs measures [33,32], weak Gibbs measures [15,40], quasi-Bernoulli measures [2,18,1], and in particular, self-similar measures satisfying the well-known *open set condition* [3,29] (see also [6,8,34,25,17]).

In [21], Lau and Ngai introduced the notion of "weak separation condition" (WSC) which is weaker than the open set condition and includes many interesting overlapping IFS. They proved that under this condition, the multifractal formalism (1.2) holds at those α such that $\alpha=\tau'(q)$ for some $q>0$. Recently, Feng showed that for any self-similar measures *without any separation conditions*, formula (1.2) still holds if $\alpha=\tau'(q)$ for some $q>1$ [12]. It remains unknown whether $\tau(q)$ is always differentiable over $(0,+\infty)$ for any self-similar measures.

In recent years there has been a large literature concerning concrete classes of self-similar measures with the WSC, and many exceptional multifractal phenomena have been found at $q<0$ (see, e.g., [19,10,15,9,23,13,36,38,39,11]). For example, the L^q-spectra $\tau(q)$ may be non-differentiable for some $q<0$ (the so-called *phase transition* behavior), and this may lead to the break down of the multifractal formalism. The phase transition was first found in the case of the Bernoulli convolution associated with golden ratio, in which τ is analytic on $\mathbb{R}\backslash\{q_0\}$ except for a negative point q_0 at which τ is non-differentiable [10]. Nevertheless, this measure is proved to be weak-Gibbs and hence the multifractal formalism still holds [15]. Another striking example, which has a similar phase transition behavior, is the 3-fold convolution of the standard Cantor measure, for which the set of local dimensions is the union of an interval and an isolated point [19], and the multifractal formalism (1.2) does not hold on an interval corresponding to the non-differentiable point [23] of $\tau(q)$, whilst a modified multifractal formalism holds [13]. The more extensive class of examples of this sort was studied by Shmerkin [36] and Testud [38]. In particular Testud constructed some simple self-similar measures on $\mathbb{R}$ satisfying the WSC such that the dimension spectra are very wild and not concave [39].

In this paper, we prove a more complete and unified result about the multifractal structure of self-similar measures with the WSC, regardless of whether there are phase transitions. The definition of the WSC is given in Definition 2.3. We prove (see Theorems 1.1, 1.2) that under the WSC assumption, the multifractal formalism always holds for $q \geqslant 0$. Furthermore, there is a tractable open ball U_0 with $\mu(U_0) > 0$ such that the multifractal formalism holds for $\mu|_{U_0}$ for all $q \in \mathbb{R}$, the dimension spectra of $\mu|_{U_0}$ and μ coincide for $q \geqslant 0$. Intuitively, that μ behaves more regularly on $U_0 \cap K$ than on K is due to the fact that in our construction, U_0 does not contain points with very small measures in neighborhoods which affect the formalism corresponding to $q < 0$.

We first obtain the following structural theorem for the WSC. For any IFS $\{S_i\}_{i=1}^m$ and for any finite word $u = i_1 \dots i_k$ over the alphabet $\{1, \dots, m\}$, we let $S_u = S_{i_1} \dots S_{i_k}$. Let ϑ denote the empty word.

Theorem 1.1. *Let $\{S_i\}_{i=1}^m$ be an IFS which satisfies the WSC. Then there exists an open ball U_0 with $\mu(U_0) > 0$ and a positive integer ℓ such that for any finite or empty word u, we can associate an ℓ-dimensional row vector $\mathbf{p}(u)$ of positive entries such that $\mu(S_u(U_0)) \approx \|\mathbf{p}(u)\|$.*

Furthermore, for the above u and for any finite word v so that $S_v(U_0) \subset U_0$, there exists an $\ell \times \ell$ matrix $M(v)$ of non-negative entries such that

$$\mathbf{p}(uv) = \mathbf{p}(u)M(v).$$

The above $\approx$ means that the two terms are bounded from above and below by two positive constants independent of u. The construction of U_0 is by the definition of WSC (see (2.5)); the proof of the theorem is in Lemma 2.6 and Theorem 2.8. It follows that

$$\mathbf{p}(v_1 \dots v_k) = \mathbf{p}(\vartheta)M(v_1) \dots M(v_k),$$

whenever $S_{v_i}(U_0) \subset U_0$ for $1 \leqslant i \leqslant k$. That is, the distribution of μ restricted on U_0 is controlled by the products of non-negative matrices. As a consequence, we see that

$$a_1 \dots a_k \mathbf{p}(\vartheta) \preccurlyeq \mathbf{p}(v_1 \dots v_k) \preccurlyeq b_1 \dots b_k \mathbf{p}(\vartheta) \tag{1.3}$$

whenever $S_{v_i}(U_0) \subset U_0$ and $a_i \mathbf{p}(\vartheta) \preccurlyeq \mathbf{p}(v_i) \preccurlyeq b_i \mathbf{p}(\vartheta)$ $(i = 1, \dots, k)$. (Here for vectors $\mathbf{c} = (c_i)$ and $\mathbf{d} = (d_i)$, $\mathbf{c} \preccurlyeq \mathbf{d}$ means that $c_i \leqslant d_i$ for all i.) We call (1.3) the *quasi-product property* of $\mu|_{U_0}$. This property plays a key role in our multifractal analysis of $\mu|_{U_0}$ (it is used in the proof of Propositions 5.3 and 5.2).

Theorem 1.2. *Let μ be a self-similar measure on $\mathbb{R}^d$ generated by an IFS $\{S_i\}_{i=1}^m$ which satisfies the WSC and let U_0 be the open ball in Theorem* 1.1. *Then*

(i) *$E_{U_0}(\alpha) \neq \emptyset$ if and only if $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, where*

$$\alpha_{\min} = \lim_{q \to \infty} \frac{\tau_{U_0}(q)}{q}, \qquad \alpha_{\max} = \lim_{q \to -\infty} \frac{\tau_{U_0}(q)}{q}.$$

(ii) *For any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$,*

$$\dim_H E_{U_0}(\alpha) = \tau_{U_0}^*(\alpha) := \inf\{\alpha q - \tau_{U_0}(q) \colon q \in \mathbb{R}\},$$

and there exists a Borel probability measure ν supported on $E_{U_0}(\alpha)$ such that

$$\liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} = \dim_H E_{U_0}(\alpha) \quad \text{for } \nu\text{-a.e. } x.$$

(iii) *Moreover, $\tau(q) = \tau_{U_0}(q)$ for $q \geqslant 0$, and*

$$\dim_H E(\alpha) = \tau^*(\alpha) := \inf\{\alpha q - \tau(q) \colon q \in \mathbb{R}\}, \quad \forall \alpha \in [\alpha_{\min}, \tau'(0-)],$$

where $\tau'(0-)$ denotes the left derivative of τ at 0.

We remark that U_0 in Theorems 1.1, 1.2 is not unique and can be more general bounded open set (see Section 6). Furthermore under some mild conditions, the last conclusion in Theorem 1.2 can be strengthened to $\dim_H E(\alpha) = \tau^*_{U_0}(\alpha)$ for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$ (see Corollary 5.8 and Examples 6.1–6.2 for details). We also remark that Theorems 1.1, 1.2 can be extended to some special self-affine measures (see Example 6.5).

Part (ii) is the main component of Theorem 1.2. Since the estimate $\dim_H E_{U_0}(\alpha) \leqslant \tau^*_{U_0}(\alpha)$ holds for any compactly supported probability measures (see, e.g., Theorem 4.1 in [21]), the difficult part is the reverse inequality. By using the quasi-product property and a generalized box-counting principle for measures (Proposition 3.3), for any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, we give a delicate construction of a Cantor-type subset of $E_{U_0}(\alpha)$ with Moran structure such that its Hausdorff dimension is $\geqslant \tau^*_{U_0}(\alpha)$. This gives a lower bound estimate of $\dim_H E_{U_0}(\alpha)$. In the already known results, the box-counting principle holds only for those α that are equal to the derivative $\tau'_{U_0}(q)$ for some $q \in \mathbb{R}$. The subtleness of our construction is on the Cantor sets with Moran structure which gets by the derivative and provides a new way to extend the desired result to all the $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

The paper is arranged in the following manner: in Section 2, we prove the matrix-product form and the quasi-product property of self-similar measures with the WSC; we formulate a box-counting principle for general measures through the Legendre transform of the L^q-spectra in Section 3 and refine this principle for self-similar measures with the WSC in Section 4; in Section 5, we prove Theorem 1.2 through the Moran construction. Finally in Section 6, we give some examples and remarks related to the main theorem.

2. Preliminaries and the WSC

Let μ be the self-similar measure generated by an IFS $\{S_i\}_{i=1}^m$ on $\mathbb{R}^d$ of the form (1.1) and a probability vector $(p_1, \ldots, p_m)$, $p_i > 0$ and let K denote the associated self-similar set. For the index sets, we let $\mathcal{A} = \{1, \ldots, m\}$, let $\mathcal{A}^* = \bigcup_{k=1}^{\infty} \mathcal{A}^k$ be the collection of all finite non-empty words over $\mathcal{A}$ and let ϑ denote the empty word. For $u = u_1 \ldots u_k \in \mathcal{A}$, we write

$$S_u = S_{u_1} \circ \cdots \circ S_{u_k}, \qquad K_u = S_u(K),$$
$$\rho_u = \rho_{u_1} \ldots \rho_{u_k}, \qquad p_u = p_{u_1} \ldots p_{u_k}$$

and

$$[u] = \big\{(x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}:\ x_i = u_i \text{ for } 1 \leqslant i \leqslant k\big\}.$$

In particular, we write $S_\vartheta = \mathrm{id}$, $K_\vartheta = K$ and $\rho_\vartheta = p_\vartheta = 1$. For $u \in \mathcal{A}^*$, let $\tilde{u}$ be the word obtained from u by dropping the last letter. For any $0 < r \leqslant 1$ and $E \subset \mathbb{R}^d$, define:

$$\Gamma_r = \{u \in \mathcal{A}^*:\ \rho_u < r \leqslant \rho_{\tilde{u}}\}, \tag{2.1}$$
$$\Gamma_r(E) = \{u \in \Gamma_r:\ K_u \cap E \neq \emptyset\} \quad \text{and}$$
$$\mathcal{S}_r(E) = \big\{S_u:\ u \in \Gamma_r(E)\big\}. \tag{2.2}$$

We point out that there can be repetitions among the S_u, $u \in \Gamma_r(E)$, so that possibly $\#\mathcal{S}_r(E) < \#\Gamma_r(E)$. Let $\rho_{\min} = \min\{\rho_i:\ 1 \leqslant i \leqslant m\}$. The following result is well known.

Lemma 2.1. *Let $0 < r \leqslant 1$. Then*

(i) $\{[u]:\ u \in \Gamma_r\}$ *is a partition of the space* $\mathcal{A}^{\mathbb{N}}$.
(ii) $K = \bigcup_{u \in \Gamma_r} S_u(K)$ *and* $\mu = \sum_{u \in \Gamma_r} p_u \mu \circ S_u^{-1}$.
(iii) $\mu(E) = \sum_{u \in \Gamma_r(E)} p_u \mu \circ S_u^{-1}(E)$ *for any Borel set* $E \subset \mathbb{R}^d$.

Proposition 2.2. *Assume that K is not a singleton. Then there exist constants $C_1, C_2, \delta > 0$ and $0 < s_2 < s_1$ such that*

$$C_1 r^{s_1} \leqslant \mu\big(B(x, r)\big) \leqslant C_2 r^{s_2}, \quad \forall x \in K,\ 0 < r \leqslant \delta.$$

Proof. Since K is not a singleton, there exist $0 < \eta \leqslant 1$ and two words $\omega_1, \omega_2 \in \Gamma_\eta$ such that $K_{\omega_1} \cap K_{\omega_2} = \emptyset$. Therefore there exists $\delta \in (0, \rho_{\min})$ such that for any $x \in \mathbb{R}^d$, $B(x, \delta)$ intersects at most one of K_{ω_1} and K_{ω_2}. Define for $0 < r \leqslant \delta$,

$$\phi(r) = \sup_{x \in \mathbb{R}^d} \mu\big(B(x, r)\big).$$

Let $c = \min\{\rho_v\colon\ v \in \Gamma_\eta\}$. Then for $x \in \mathbb{R}^d$ and $0 < r < \delta$, either $B(x,r) \cap K_{\omega_1} = \emptyset$ or $B(x,r) \cap K_{\omega_2} = \emptyset$. If the former case occurs, then by Lemma 2.1(iii), we have:

$$\begin{aligned}\mu\big(B(x,r)\big) &= \sum_{v \in \Gamma_\eta(B(x,r))} p_v \mu\big(S_v^{-1}\big(B(x,r)\big)\big) \leqslant \sum_{v \in \Gamma_\eta,\ v \neq \omega_1} p_v \mu\big(S_v^{-1}\big(B(x,r)\big)\big) \\ &\leqslant \sum_{v \in \Gamma_\eta,\ v \neq \omega_1} p_v \phi(r/c) = (1 - p_{\omega_1})\phi(r/c).\end{aligned}$$

Similarly if the latter case occurs, we have $\mu(B(x,r)) \leqslant (1 - p_{\omega_2})\phi(r/c)$. Hence we always have $\mu(B(x,r)) \leqslant t\phi(r/c)$, where $t = \max\{1 - p_{\omega_1}, 1 - p_{\omega_2}\}$. It follows that $\phi(r) \leqslant t\phi(r/c)$ for $0 < r \leqslant \delta$. In particular letting $c^n\delta < r \leqslant c^{n-1}\delta$ for some $n \in \mathbb{N}$, then

$$\begin{aligned}\mu\big(B(x,r)\big) &\leqslant \mu\big(B\big(x, c^{n-1}\delta\big)\big) \leqslant \phi\big(c^{n-1}\delta\big) \leqslant t^{n-1}\phi(\delta) \\ &\leqslant \phi(\delta)\big(c^n\delta\big)^{(\log t^{n-1})/(\log(c^n\delta))} \leqslant C_2 r^{s_2},\end{aligned} \tag{2.3}$$

with $C_2 := \phi(\delta)$ and $s_2 := \inf_{n \in \mathbb{N}} \log t^{n-1}/\log(c^n\delta)$.

Let $|K|$ denote the diameter of K. For $x \in K$, there exists $v \in \Gamma_{r/|K|}$ such that $x \in K_v$. It follows that $|K_v| < r$ and $K_v \subset B(x,r)$. Therefore

$$\mu\big(B(x,r)\big) \geqslant \mu(K_v) \geqslant p_v = (\rho_v)^{\log p_v/\log \rho_v} \geqslant \left(\frac{\rho_{\min} r}{2|K|}\right)^{\log p_v/\log \rho_v} \geqslant C_1 r^{s_1}$$

with $s_1 := \max\{\log p_i/\log \rho_i\colon\ 1 \leqslant i \leqslant m\}$ and $C_1 := (\min\{1, \rho_{\min}/(2|K|)\})^{s_1}$. This together with (2.3) proves the proposition. □

For any $x \in \mathbb{R}^d$ and $r > 0$, let $U(x,r)$ denote the open ball of radius r centered at x.

Definition 2.3. The IFS $\{S_i\}_{i=1}^m$ is said to satisfy the *weak separation condition* (WSC) if,

$$\sup_{x \in \mathbb{R}^d,\ 0 < r \leqslant 1} \#\mathcal{S}_r\big(U(x,r)\big) =: \ell < \infty, \tag{2.4}$$

where $\mathcal{S}_r(\cdot)$ is defined as in (2.2).

We remark that the above definition for the WSC is equivalent to that given by Lau and Ngai in [21], provided that K is not contained in a hyperplane of $\mathbb{R}^d$. For a proof, see Zerner [41, Theorem 1]. It is known that the open set condition implies the WSC [21]. There are many interesting examples of overlapping IFS that satisfy the WSC (see, e.g., [4,10,21,22]).

In the remaining part of this section, we always assume that $\{S_i\}_{i=1}^m$ satisfies the WSC. We have the following important observation which will be the basis of our analysis: let $x_0 \in \mathbb{R}^d$ and $r_0 \in (0,1]$ such that the supremum in (2.4) is attained, i.e.,

$$\#\mathcal{S}_{r_0}\big(U(x_0, r_0)\big) = \ell. \tag{2.5}$$

For convenience we let:

$$U_0 := U(x_0, r_0), \qquad \mathcal{S}_{r_0}(U_0) = \{S_{\omega_i}\colon\ 1 \leqslant i \leqslant \ell\},$$

where $\omega_i \in \Gamma_{r_0}$ and the S_{ω_i} are all distinct. The following proposition states that $\mathcal{S}_{r_0}(U_0)$ determines the corresponding families of maps in the iteration.

Proposition 2.4. *For any $u \in \mathcal{A}^* \cup \{\vartheta\}$, write $U_u = S_u(U_0)$, then we have*:

$$\mathcal{S}_{\rho_u r_0}(U_u) = \{S_{u\omega_i}\colon 1 \leqslant i \leqslant \ell\}.$$

Proof. Note that $U_u = S_u(U_0)$ has radius $\rho_u r_0$, and $K_v \cap U_0 \neq \emptyset$ for $v \in \Gamma_{r_0}$ implies $K_{uv} \cap U_u \neq \emptyset$. It follows that $u\omega_i \in \Gamma_{\rho_u r_0}(U_u)$ and hence $\mathcal{S}_{\rho_u r_0}(U_u) \supseteq \{S_{u\omega_i}\colon 1 \leqslant i \leqslant \ell\}$. The equality holds by the maximality of ℓ. □

Define a map $\mathbf{p}\colon \mathcal{A}^* \cup \{\vartheta\} \to \mathbb{R}^\ell$ by $\mathbf{p}(u) = (t_1, \ldots, t_\ell)$, where

$$t_i = \sum_{v \in \Gamma_u^{(i)}} p_v, \quad i = 1, \ldots, \ell,$$

and $\Gamma_u^{(i)} = \{v \in \Gamma_{\rho_u r_0}(U_u)\colon S_v = S_{u\omega_i}\}$. By Proposition 2.4,

$$\Gamma_u^{(i)} = \{v \in \Gamma_{\rho_u r_0}\colon S_v = S_{u\omega_i}\}. \tag{2.6}$$

It is easy to see that $\mathbf{p}(u)$ is a strictly positive vector in $\mathbb{R}^\ell$ for each $u \in \mathcal{A}^* \cup \{\vartheta\}$, since $(\mathbf{p}(u))_i \geqslant p_u p_{\omega_i}$ for $1 \leqslant i \leqslant \ell$. Let $\|\mathbf{p}(u)\| = \sum_{i=1}^{\ell} t_i$, then for any $u \in \mathcal{A}^* \cup \{\vartheta\}$,

$$\|\mathbf{p}(u)\| = \sum_{v \in \Gamma_{\rho_u r_0}(U_u)} p_v. \tag{2.7}$$

The following two lemmas follow easily from Proposition 2.4.

Lemma 2.5. *For any $u \in \mathcal{A}^* \cup \{\vartheta\}$, there exist $c_1, c_2 > 0$ (c_1, c_2 depend on u) such that*

$$c_1 \mu(E) \leqslant \mu\big(S_u(E)\big) \leqslant c_2 \mu(E), \tag{2.8}$$

for any Borel subset $E \subseteq U_0$.

Proof. Let $u \in \mathcal{A}^* \cup \{\vartheta\}$ and let $E \subseteq U_0$. Then by Lemma 2.1(iii) and Proposition 2.4,

$$\mu\big(S_u(E)\big) = \sum_{v \in \Gamma_{\rho_u r_0}(S_u(U_0))} p_v \mu\big(S_v^{-1}\big(S_u(E)\big)\big) = \sum_{i=1}^{\ell} \sum_{v \in \Gamma_u^{(i)}} p_v \mu\big(S_{\omega_i}^{-1}(E)\big) = \sum_{i=1}^{\ell} \big(\mathbf{p}(u)\big)_i \mu\big(S_{\omega_i}^{-1}(E)\big).$$

In particular, we have $\mu(E) = \sum_{i=1}^{\ell} (\mathbf{p}(\vartheta))_i \mu(S_{\omega_i}^{-1}(E))$. Hence (2.8) follows by setting $c_1 = \min\{(\mathbf{p}(u))_i / (\mathbf{p}(\vartheta))_i\colon 1 \leqslant i \leqslant \ell\}$ and $c_2 = \max\{(\mathbf{p}(u))_i / (\mathbf{p}(\vartheta))_i\colon 1 \leqslant i \leqslant \ell\}$. □

Lemma 2.6. *There exists a constant $c > 0$ (depending on U_0) such that*

$$c\|\mathbf{p}(u)\| \leqslant \mu\big(S_u(U_0)\big) \leqslant \|\mathbf{p}(u)\|, \quad \forall u \in \mathcal{A}^* \cup \{\vartheta\}.$$

Proof. Let $u \in \mathcal{A}^* \cup \{\vartheta\}$, we write $U_u = S_u(U_0)$. By Lemma 2.1(iii) and (2.7), we have:

$$\mu(U_u) = \sum_{v \in \Gamma_{\rho_u r_0}(U_u)} p_v \mu \circ S_v^{-1}(U_u) \leqslant \sum_{v \in \Gamma_{\rho_u r_0}(U_u)} p_v = \|\mathbf{p}(u)\|.$$

To prove the reverse inequality, we observe that U_0 is open and $K_{\omega_i} \cap U_0 \neq \emptyset$, we can choose $\omega_i^* \in \mathcal{A}^*$ such that $K_{\omega_i \omega_i^*} \subset U_0$ for $1 \leqslant i \leqslant \ell$. Set

$$c = \min\{p_{\omega_i^*}\colon 1 \leqslant i \leqslant \ell\}.$$

Let $v \in \Gamma_{\rho_u r_0}(S_u(U_0))$. By Proposition 2.4, $S_v = S_{u\omega_i}$ for some $1 \leqslant i \leqslant \ell$. Hence $S_{v\omega_i^*} = S_{u\omega_i\omega_i^*}$ and $K_{v\omega_i^*} = K_{u\omega_i\omega_i^*} = S_u(K_{\omega_i\omega_i^*}) \subset S_u(U_0)$. It follows that

$$\mu \circ S_v^{-1}\big(S_u(U_0)\big) \geqslant \mu \circ S_v^{-1}(K_{v\omega_i^*}) = \mu(K_{\omega_i^*}) \geqslant p_{\omega_i^*} \geqslant c.$$

Summing over all such words v, we obtain:

$$\mu(U_u)=\sum_{v\in\Gamma_{\rho_u r_0}(U_u)} p_v\mu\circ S_v^{-1}\big(S_u(U_0)\big)\geqslant c\sum_{v\in\Gamma_{\rho_u r_0}(U_u)} p_v=c\big\|\mathbf{p}(u)\big\|.$$

This completes the proof of the lemma. □

In the remaining part of this section, we derive the matrix-product structure of $\mathbf{p}$. Let

$$\Omega=\big\{v\in\mathcal{A}^*\cup\{\vartheta\}\colon\ S_v(U_0)\subset U_0\big\}. \tag{2.9}$$

For $u\in\mathcal{A}^*\cup\{\vartheta\}$, $v\in\Omega$ and $1\leqslant i,j\leqslant\ell$, we set

$$\Omega_{u,v,j}=\big\{\gamma\in\mathcal{A}^*\colon\ \gamma\in\Gamma_{\rho_{uv}r_0},\ S_\gamma=S_{uv\omega_j}\big\},$$
$$\Omega_{u,i}^{(1)}=\big\{\gamma\in\mathcal{A}^*\colon\ \gamma\in\Gamma_{\rho_u r_0},\ S_\gamma=S_{u\omega_i}\big\}$$

and

$$\Omega_{i;v,j}^{(2)}=\begin{cases}\{\vartheta\} & \text{if } S_{\omega_i}=S_{v\omega_j},\\ \{\gamma\in\mathcal{A}^*\colon\ \gamma\in\Gamma_{\rho_v r_0/\rho_{\omega_i}},\ S_{\omega_i\gamma}=S_{v\omega_j}\} & \text{otherwise.}\end{cases}$$

Lemma 2.7. *Let $u\in\mathcal{A}^*\cup\{\vartheta\}$, $v\in\Omega$ and $1\leqslant j\leqslant\ell$. Then $\gamma\in\Omega_{u,v,j}$ if and only if there exists $i\in\{1,\dots,\ell\}$ such that γ can be written as $\gamma=\gamma_1\gamma_2$ with $\gamma_1\in\Omega_{u,i}^{(1)}$ and $\gamma_2\in\Omega_{i;v,j}^{(2)}$.*

Proof. It is routine to verify the sufficiency. In the following we show the necessity.

Let $\gamma\in\Omega_{u,v,j}$. Then we have $K_\gamma\cap S_{uv}(U_0)\neq\emptyset$, because $K_\gamma=K_{uv\omega_j}=S_{uv}(K_{\omega_j})$ and $K_{\omega_j}\cap U_0\neq\emptyset$. Meanwhile, $S_{uv}(U_0)\subset S_u(U_0)$ by the assumption $v\in\Omega$. We hence have:

$$K_\gamma\cap S_u(U_0)\neq\emptyset. \tag{2.10}$$

Since $\rho_{uv}r_0\leqslant\rho_u r_0$, we can decompose γ uniquely as $\gamma=\gamma_1\gamma_2$ with $\gamma_1\in\Gamma_{\rho_u r_0}$ and $\gamma_2\in\mathcal{A}^*\cup\{\vartheta\}$. Due to (2.10), $K_{\gamma_1}\cap S_u(U_0)\neq\emptyset$. This together with $\gamma_1\in\Gamma_{\rho_u r_0}$ and Proposition 2.4 yields $S_{\gamma_1}=S_{u\omega_i}$ for some $i\in\{1,\dots,\ell\}$, and hence $\gamma_1\in\Omega_{u,i}^{(1)}$. To show that $\gamma_2\in\Omega_{i;v,j}^{(2)}$, note that $S_{uv\omega_j}=S_\gamma=S_{\gamma_1\gamma_2}=S_{u\omega_i\gamma_2}$. It follows that

$$S_{v\omega_j}=S_{\omega_i\gamma_2}. \tag{2.11}$$

If $\gamma_2=\vartheta$, by (2.11) we obtain $\gamma_2\in\Omega_{i;v,j}^{(2)}$ directly. Otherwise $\gamma_2\neq\vartheta$. Then $\widetilde{\gamma}=\widetilde{\gamma_1\gamma_2}=\gamma_1\widetilde{\gamma_2}$. Since $\gamma_1\gamma_2=\gamma\in\Gamma_{\rho_{uv}r_0}$, we have:

$$\rho_{uv}r_0\leqslant\rho_{\widetilde{\gamma_1\gamma_2}}=\rho_{\gamma_1}\rho_{\widetilde{\gamma_2}}=\rho_{u\omega_i}\rho_{\widetilde{\gamma_2}}.$$

It follows that $\rho_v r_0/\rho_{\omega_i}\leqslant\rho_{\widetilde{\gamma_2}}$. Also we have $\rho_{\gamma_2}<\rho_v r_0/\rho_{\omega_i}$ by the fact that $\rho_{\gamma_1\gamma_2}<\rho_{uv}r_0$ and $\rho_{\gamma_1}=\rho_u\rho_{\omega_i}$. Hence we have $\gamma_2\in\Gamma_{\rho_v r_0/\rho_{\omega_i}}$. Combining this with (2.11) we obtain $\gamma_2\in\Omega_{i;v,j}^{(2)}$. This finishes the proof of the lemma. □

We now define a matrix-valued function M on Ω, taking values in the set of non-negative $\ell\times\ell$ matrices in the following way. For $v\in\Omega$, set $M(v)=(s_{i,j})_{1\leqslant i,j\leqslant\ell}$ by:

$$s_{i,j}=\begin{cases}0 & \text{if } \Omega_{i;v,j}^{(2)}=\emptyset,\\ \sum_{\gamma\in\Omega_{i;v,j}^{(2)}} p_\gamma & \text{otherwise.}\end{cases}$$

The main result in this section is the following theorem.

Theorem 2.8. *For any $u\in\mathcal{A}^*\cup\{\vartheta\}$ and $v\in\Omega$, we have*:

$$\mathbf{p}(uv)=\mathbf{p}(u)M(v). \tag{2.12}$$

Proof. By the definition of $\mathbf{p}$ and (2.6), we have for $1\leqslant j\leqslant\ell$,

$$\big(\mathbf{p}(uv)\big)_j=\sum_{\gamma\in\Gamma_{\rho_{uv}r_0}\colon S_\gamma=S_{uv\omega_j}} p_\gamma=\sum_{\gamma\in\Omega_{u,v,j}} p_\gamma.$$

Combining this with Lemma 2.7, we obtain

$$\big(\mathbf{p}(uv)\big)_j=\sum_{i=1}^{\ell}\sum_{\gamma_1\in\Omega_{u,i}^{(1)}}\sum_{\gamma_2\in\Omega_{i;v,j}^{(2)}}p_{\gamma_1}p_{\gamma_2}=\sum_{i=1}^{\ell}\Big(\sum_{\gamma_1\in\Omega_{u,i}^{(1)}}p_{\gamma_1}\Big)\Big(\sum_{\gamma_2\in\Omega_{i;v,j}^{(2)}}p_{\gamma_2}\Big)=\sum_{i=1}^{\ell}\big(\mathbf{p}(u)\big)_i\big(M(v)\big)_{ij}.$$

This completes the proof of (2.12). □

According to Theorem 2.8, we have directly the:

Theorem 2.9. *For any $v_1,\dots,v_k\in\Omega$, we have*:

$$\mathbf{p}(v_1\dots v_k)=\mathbf{p}(\vartheta)M(v_1)\dots M(v_k).$$

The above result, together with Lemma 2.6, shows that the distribution of μ on some specific subsets of U_0 is controlled by the product of non-negative matrices in the collection $\{M(v)\colon v\in\Omega\}$. This fact is important for us to understand the local structure of μ.

For any two vectors $\mathbf{p}_1,\mathbf{p}_2\in\mathbb{R}^\ell$, we write $\mathbf{p}_1\preccurlyeq\mathbf{p}_2$ if $(\mathbf{p}_1)_i\leqslant(\mathbf{p}_2)_i$ for all $1\leqslant i\leqslant\ell$. As a corollary of Theorem 2.8, we have:

Corollary 2.10. *Suppose $a\mathbf{p}(\vartheta)\preccurlyeq\mathbf{p}(u)\preccurlyeq b\mathbf{p}(\vartheta)$ holds for some $u\in\mathcal{A}^*$ and $a,b\geqslant 0$, then for any $v\in\Omega$, we have $a\mathbf{p}(v)\preccurlyeq\mathbf{p}(uv)\preccurlyeq b\mathbf{p}(v)$.*

Proof. By Theorem 2.8, $\mathbf{p}(uv)=\mathbf{p}(u)M(v)$. Since $\mathbf{p}(u)$ and $M(v)$ are non-negative, and $a\mathbf{p}(\vartheta)\preccurlyeq\mathbf{p}(u)\preccurlyeq b\mathbf{p}(\vartheta)$, we have:

$$a\mathbf{p}(\vartheta)M(v)\preccurlyeq\mathbf{p}(uv)\preccurlyeq b\mathbf{p}(\vartheta)M(v).$$

Since $\mathbf{p}(\vartheta)M(v)=\mathbf{p}(v)$, we obtain the desired result. □

According to Theorem 2.8 and Corollary 2.10, we obtain the following *quasi-product property* of μ by induction, which will be used to estimate the lower bound of the dimension spectrum (see the proofs of Propositions 5.3 and 5.2).

Corollary 2.11. *Suppose $v_1,\dots,v_k\in\Omega$, and $a_i\mathbf{p}(\vartheta)\preccurlyeq\mathbf{p}(v_i)\preccurlyeq b_i\mathbf{p}(\vartheta)$ for each $1\leqslant i\leqslant k$, where $a_i,b_i\geqslant 0$. Then $a_1\dots a_k\mathbf{p}(\vartheta)\preccurlyeq\mathbf{p}(v_1\dots v_k)\preccurlyeq b_1\dots b_k\mathbf{p}(\vartheta)$.*

3. L^q-spectrum and Legendre transform

Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. For any open set $V\subset\mathbb{R}^d$ with $\mu(V)>0$, we let:

$$\Theta_V(q;r)=\sup\sum_i\mu\big(B(x_i,r)\big)^q,\quad r>0,\ q\in\mathbb{R},$$

where the supremum is taken over all families of disjoint closed balls $\{B(x_i,r)\}_i$ contained in V with $x_i\in\operatorname{supp}(\mu)$. The L^q-spectrum $\tau_V(q)$ of μ on V is defined by:

$$\tau_V(q)=\liminf_{r\to 0}\frac{\log\Theta_V(q;r)}{\log r}.$$

When $V=\mathbb{R}^d$, we write $\Theta(q;r)=\Theta_{\mathbb{R}^d}(q;r)$ and $\tau(q)=\tau_{\mathbb{R}^d}(q)$ for short. It was proved by Peres and Solomyak [30] that for any self-similar measure μ (without assuming any separation condition), the limit $\tau(q)=\lim_{r\to 0}\frac{\log\Theta(q;r)}{\log r}$ exists for $q\geqslant 0$.

Proposition 3.1. *Let V be an open set with $\mu(V)>0$, then $\tau_V(\cdot)$ is a concave function over $\mathbb{R}$. If in addition μ is a self-similar measure defined by an IFS $\{S_j\}_{j=1}^m$ and the attractor K is not a singleton, then $\tau_V(q)=\tau(q)$ for any $q\geqslant 0$.*

Proof. The concavity of $\tau_V(\cdot)$ follows by a standard argument (see, e.g., [21, Proposition 3.2]). To prove the second statement, we fix a $q \geqslant 0$, then it is clear that $\Theta_V(q;r) \leqslant \Theta(q;r)$. Hence we have $\tau_V(q) \geqslant \tau(q)$. For the reverse inequality, we choose $\eta > 0$ and $\omega \in \mathcal{A}^*$ such that $B(K_\omega, \eta) \subset V$. Let $0 < r < \eta$ and suppose $\{B(x_i, r)\}$ is a family of disjoint balls of radius r with centers $x_i \in K$. Then $\{S_\omega(B(x_i, r))\}$ are disjoint balls contained in V with centers $S_\omega(x_i) \in K$. Hence

$$\Theta_V(q; \rho_\omega r) \geqslant \sum_i \mu\big(S_\omega\big(B(x_i, r)\big)\big)^q \geqslant (p_\omega)^q \sum_i \mu\big(B(x_i, r)\big)^q,$$

it follows that $\Theta_V(q; \rho_\omega r) \geqslant (p_\omega)^q \Theta(q; r)$, from which we conclude that $\tau_V(q) \leqslant \tau(q)$. □

Corollary 3.2. *Let μ be a self-similar measure on $\mathbb{R}^d$ as in Proposition 3.1, and let:*

$$\alpha_{\min} = \lim_{q\to+\infty} \tau_V(q)/q, \qquad \alpha_{\max} = \lim_{q\to-\infty} \tau_V(q)/q. \tag{3.1}$$

Then $0 < \alpha_{\min} \leqslant \alpha_{\max} < +\infty$. Moreover if we let:

$$\alpha(x) = \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

Then for any $x \in V \cap \operatorname{supp}(\mu)$, $\alpha(x) \in [\alpha_{\min}, \alpha_{\max}]$ if the limit exists.

Proof. The two limits in (3.1) exist by the concavity of $\tau_V(q)$, $q \in \mathbb{R}$. By Proposition 2.2, we have for $q \geqslant 0$,

$$\big(C_1 r^{s_1}\big)^q \leqslant \Theta_V(q; r) \leqslant N_r(K)\big(C_2 r^{s_2}\big)^q$$

for small r, where $N_r(K)$ denotes the largest number of disjoint balls of radius r centered in K. Therefore $s_2 q - d \leqslant \tau_V(q)$ so that $0 < s_2 \leqslant \lim_{q\to+\infty} \tau_V(q)/q = \alpha_{\min}$. For $q < 0$, we have,

$$\big(C_1 r^{s_2}\big)^q \leqslant \Theta_V(q; r) \leqslant N_r(K)\big(C_2 r^{s_1}\big)^q$$

instead, and a similar argument implies that $\alpha_{\max} \leqslant s_1 < \infty$.

For the last statement, we observe that for $x \in V \cap \operatorname{supp}(\mu)$ such that $\alpha(x)$ exists, then for small $r > 0$, $\Theta_V(q; r) \geqslant \mu(B(x, r))^q$ for $q \in \mathbb{R}$. It implies that $\tau_V(q) \leqslant q\alpha(x)$ for any $q \in \mathbb{R}$. Hence we have $\alpha(x) \in [\alpha_{\min}, \alpha_{\max}]$. □

For the concave function $\tau_V(q)$, we define its Legendre transform $\tau_V^*(\alpha)$ by:

$$\tau_V^*(\alpha) = \inf\big\{\alpha q - \tau_V(q) \colon q \in \mathbb{R}\big\}. \tag{3.2}$$

From convex function theory [35], it is well known that $\tau_V^*(\alpha)$ is also a concave function, $0 \leqslant \tau_V^*(\alpha) < \infty$ for $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, and $\tau_V^*(\alpha) = -\infty$ otherwise. We will only consider the interval $[\alpha_{\min}, \alpha_{\max}]$ as the effective domain of $\tau_V^*(\alpha)$. Moreover, if the derivative $\tau_V'(q) = \alpha$ exists, then the infimum in (3.2) is attained at q, and

$$\tau_V^*(\alpha) = \alpha q - \tau_V(q).$$

The following proposition for a general measure is a refinement of the standard box-counting principle originated in [16] and considered in a number of papers (see, e.g., [7,21]).

Proposition 3.3. *Let μ be a finite Borel measure on $\mathbb{R}^d$ and let V be an open subset with $\mu(V) > 0$. Assume that there exists an open set $U \subset V$ such that $\overline{U} \subset V$ and $\tau_U(q) = \tau_V(q)$ for all $q > 0$. Suppose that $\alpha \in \mathbb{R}$ is such that $\alpha = \tau_V'(q)$ for some $q \in \mathbb{R}$. Then for any integer $N > 0$, and $\delta, r_0 > 0$, there exist $0 < r < r_0$ and a family of disjoint balls $\{B(x_i, r)\}_{i=1}^k$ contained in V with $x_i \in \operatorname{supp}(\mu)$ such that*

$$k \geqslant r^{-\tau_V^*(\alpha)+\delta(|q|+1)} \tag{3.3}$$

and

$$r^{\alpha+2\delta} \leqslant \mu\big(B(x_i, r/N)\big) \leqslant \mu\big(B(x_i, r)\big) \leqslant r^{\alpha-2\delta}, \quad \forall 1 \leqslant i \leqslant k. \tag{3.4}$$

The proof follows the classical approach with a few subtle modifications. Since the proof is quite long and is away from our central development, we put it in Appendix A. We remark that this proposition differs from the usual version of the box-counting principle (see, e.g., Theorem 5.1 of [21]) in the following two aspects: first, $\tau_V(q)$ is considered for any open set V rather than $\tau_{\mathbb{R}^d}(q)$; secondly, the simultaneous estimate (3.4) for $\mu(B(x_i,r))$ and $\mu(B(x_i,r/N))$ is obtained rather than the single estimate of $\mu(B(x_i,r))$. The estimate is not direct since $\mu(B(x,r))$ may be very different from $\mu(B(x,r/N))$ for measures such as self-similar measures with the WSC.

We do not know if the existence condition of U for V in Proposition 3.3 can be removed. However it is satisfied in most of interesting situations. For example, it is automatically satisfied if $V=\mathbb{R}^d$. Also from Proposition 3.1, we see that if μ is a self-similar measure, then $\tau_U(q)=\tau_{\mathbb{R}^d}(q)$ for any $q\geqslant 0$ and any open set U with $U\cap \operatorname{supp}(\mu)\neq\emptyset$. Hence the condition on V is satisfied for self-similar measures.

4. A counting result with the WSC

In this section we give a refinement of Proposition 3.3 for the self-similar measure μ generated by an IFS $\{S_i\}_{i=1}^m$ that satisfies the WSC. We will fix the open ball $U_0=U(x_0,r_0)$ that determines the *quasi-product structure* of the self-similar measure μ. Let Ω be defined as in (2.9).

Proposition 4.1. *Let $q\in\mathbb{R}$ and let $U_0=U(x_0,r_0)$ be the open ball in the definition of WSC in* (2.5). *Suppose that $\tau'_{U_0}(q):=\alpha$ exists. Then for any $\delta,\eta>0$, there exist $r\in(0,\eta)$, $k\geqslant r^{-\tau^*_{U_0}(\alpha)+\delta(|q|+1)}$ and $u_1,\dots,u_k\in\Omega$ satisfying the following properties*:

(i) $r^{1+\delta}\leqslant\rho_{u_i}\leqslant r^{1-\delta}$ *for all* $1\leqslant i\leqslant k$.
(ii) $S_{u_i}(4U_0)$ *are disjoint subsets of* U_0, *where* $4U_0:=U(x_0,4r_0)$.
(iii) $r^{\alpha+3\delta}\mathbf{p}(\vartheta)\preccurlyeq\mathbf{p}(u_i)\preccurlyeq r^{\alpha-3\delta}\mathbf{p}(\vartheta)$ *for all* $1\leqslant i\leqslant k$.

Proof. Recall the notation $\mathcal{S}_{r_0}(U_0)=\{S_{\omega_1},\dots,S_{\omega_\ell}\}$ in Section 2 and note that the hypothesis of Proposition 3.3 are satisfied (due to Proposition 3.1). Let $V=U_0$ and let N be the least integer $>8(1+|K|+|U_0|)$. Hence for $\delta,\eta>0$, we can find a family of disjoint balls $\{B(x_i,r)\}_{i=1}^k$ contained in U_0 such that $x_i\in\operatorname{supp}(\mu)$,

$$0<r<\eta,\qquad r^\delta<\min\Big\{\rho_{\min}/N,\ \min_{1\leqslant j\leqslant\ell}p_{\omega_j}/\ell,\ \min_{1\leqslant j\leqslant\ell}\big(\mathbf{p}(\vartheta)\big)_j\Big\},\tag{4.1}$$

$$k\geqslant r^{-\tau^*_{U_0}(\alpha)+\delta(|q|+1)},\quad\text{and}$$

$$r^{\alpha+2\delta}\leqslant\mu\big(B(x_i,r/N)\big)\leqslant\mu\big(B(x_i,r)\big)\leqslant r^{\alpha-2\delta},\quad\forall 1\leqslant i\leqslant k.\tag{4.2}$$

In the following we construct the words u_i which satisfy the desired properties.

Choose $0<\epsilon<r$ such that

$$\mu\big(\partial B\big(x_i,(r+\epsilon)/N\big)\big)=0,\quad i=1,\dots,k.$$

($\partial(E)$ denotes the boundary of E.) This can be done since $\mu(\partial B(x_i,t))=0$ except for countably many t. For convenience we denote $r'=(r+\epsilon)/N$, therefore we have:

$$\mu\big(B(x_i,r')\big)=\mu\big(U(x_i,r')\big),\quad i=1,\dots,k.\tag{4.3}$$

By Lemma 2.1(iii),

$$\mu\big(U(x_i,r')\big)=\sum_{\gamma\in\Gamma_{r'}(U(x_i,r'))}p_\gamma\mu\circ S_\gamma^{-1}\big(U(x_i,r')\big)\leqslant\sum_{\gamma\in\Gamma_{r'}(U(x_i,r'))}p_\gamma.$$

By the WSC (2.4), there exists $u\in\Gamma_{r'}$ such that $K_u\cap U(x_i,r')\neq\emptyset$, and

$$\sum_{\gamma\in\Gamma_{r'}(U(x_i,r'))}p_\gamma\leqslant\ell\sum_{\gamma\in\Gamma_{r'}(U(x_i,r')):\ S_\gamma=S_u}p_\gamma=\ell\sum_{\gamma\in\Gamma_{r'}:\ S_\gamma=S_u}p_\gamma$$

(take u such that the second sum is the largest). We fix this u and denote it by u_i. By combining the above two inequalities, we have:

$$\mu\big(U(x_i, r')\big) \leqslant \ell \sum_{\gamma\in\Gamma_{r'}:\ S_\gamma=S_{u_i}} p_\gamma. \tag{4.4}$$

Since u_i, $1 \leqslant i \leqslant k$, are in $\Gamma_{r'}$, we have $\rho_{\min} r' \leqslant \rho_{u_i} < r'$. Then by the choice of r (i.e., $r^\delta < \rho_{\min/N}$), u_i satisfies property (i).

To prove property (ii), it suffices to show that $S_{u_i}(4U_0) \subset B(x_i, r)$. We note that $K_{u_i} \cap U(x_i, r') \neq \emptyset$ and $K_{u_i} \cap S_{u_i}(U_0) \neq \emptyset$. This implies that

$$K_{u_i} \subset U\big(x_i, r' + |K_{u_i}|\big), \qquad S_{u_i}(U_0) \subset U\big(x_i, r' + |K_{u_i}| + \big|S_{u_i}(U_0)\big|\big).$$

Since

$$r' + |K_{u_i}| + \big|S_{u_i}(U_0)\big| \leqslant r'\big(1 + |K| + |U_0|\big) = \frac{r+\epsilon}{N}\big(1 + |K| + |U_0|\big) \leqslant (r+\epsilon)/8 \leqslant r/4,$$

we obtain $S_{u_i}(U_0) \subset U(x_i, r/4)$ and hence $S_{u_i}(4U_0) \subset U(x_i, r) \subset B(x_i, r)$. This also implies $u_i \in \Omega$ since $S_{u_i}(U_0) \subset B(x_i, r) \subset U_0$.

Finally we prove property (iii). By the definition of $\mathbf{p}$ and (2.6), we have for $1 \leqslant j \leqslant \ell$,

$$\big(\mathbf{p}(u_i)\big)_j = \sum_{v\in\Gamma_{\rho_{u_i} r_0}:\ S_v=S_{u_i\omega_j}} p_v. \tag{4.5}$$

Observe that if $\gamma \in \Gamma_{r'}$ satisfies $S_\gamma = S_{u_i}$, then $S_{\gamma\omega_j} = S_{u_i\omega_j}$ and $\gamma\omega_j \in \Gamma_{\rho_{u_i} r_0}$. It follows that

$$\begin{aligned}\big(\mathbf{p}(u_i)\big)_j &\geqslant \sum_{\gamma\in\Gamma_{r'}:\ S_\gamma=S_{u_i}} p_\gamma p_{\omega_j} \geqslant p_{\omega_j}\mu\big(U(x_i, r')\big)/\ell \quad \text{(by (4.4))}\\ &\geqslant p_{\omega_j} r^{\alpha+2\delta}/\ell \geqslant r^{\alpha+3\delta} \quad \text{(by (4.2), (4.3) and (4.1))}\\ &\geqslant r^{\alpha+3\delta}\big(\mathbf{p}(\vartheta)\big)_j.\end{aligned}$$

That is, $r^{\alpha+3\delta}\mathbf{p}(\vartheta) \preccurlyeq \mathbf{p}(u_i)$. To see the other direction, observe that if $S_v = S_{u_i\omega_j}$, then $K_v = K_{u_i\omega_j} \subset K_{u_i} \subset B(x_i, r)$. By (4.5), we have:

$$\begin{aligned}\big(\mathbf{p}(u_i)\big)_j &= \sum_{v\in\Gamma_{\rho_{u_i} r_0}:\ S_v=S_{u_i\omega_j}} p_v \mu\circ S_v^{-1}\big(B(x_i, r)\big) \leqslant \mu\big(B(x_i, r)\big) \quad \text{(by Lemma 2.1(iii))}\\ &\leqslant r^{\alpha-2\delta} \leqslant r^{\alpha-3\delta}\big(\mathbf{p}(\vartheta)\big)_j \quad \text{(by (4.2)).}\end{aligned}$$

That is, $\mathbf{p}(u_i) \preccurlyeq r^{\alpha-3\delta}\mathbf{p}(\vartheta)$. This finishes the proof of the proposition. □

5. Multifractal formalism and Moran constructions

In this section we are aiming to prove the multifractal formalism for the self-similar measure with respect to the open ball U_0 from the definition of WSC, i.e.,

$$\dim_H E_{U_0}(\alpha) = \tau^*_{U_0}(\alpha),$$

where $E_{U_0}(\alpha)$ is the set of $x \in U_0$ with local dimension $\alpha(x) = \alpha$. It is known that for any probability μ with compact support, $\dim_H E(\alpha) \leqslant \tau^*(\alpha)$ whenever $E(\alpha) \neq \emptyset$ (see, e.g., Theorem 4.1 in [21]) and it is straightforward to extend this to open subsets. The difficulty is to prove the reverse inequality. For this we need to use the Cantor-type sets with a special Moran construction (by applying Proposition 4.1) to get the lower bound estimate.

Let $B \subset \mathbb{R}^d$ be a closed ball. Let $\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers. Let $D = \bigcup_{k\geqslant 0} D_k$ with $D_0 = \{\emptyset\}$ and $D_k = \{\omega = (j_1 j_2 \cdots j_k):\ 1 \leqslant j_i \leqslant n_i,\ 1 \leqslant i \leqslant k\}$. Suppose that $\mathcal{G} = \{B_\omega:\ \omega \in D\}$ is a collection of closed balls of radius r_ω in $\mathbb{R}^d$. We say that $\mathcal{G}$ fulfills the *Moran structure* provided it satisfies the following conditions:

(1) $B_\emptyset = B$, $B_{\omega j} \subset B_\omega$ for any $\omega \in D_{k-1}$, $1 \leqslant j \leqslant n_k$;
(2) $B_\omega \cap B_{\omega'} = \emptyset$ for $\omega, \omega' \in D_k$ with $\omega \neq \omega'$;
(3) $\lim_{k\to\infty} \max_{\omega\in D_k} r_\omega = 0$;
(4) For all $\omega\eta \neq \omega'\eta$, $\omega, \omega' \in D_m$, $\omega\eta, \omega'\eta \in D_n$, $m \leqslant n$,

$$\frac{r_{\omega\eta}}{r_\omega} = \frac{r_{\omega'\eta}}{r_{\omega'}}.$$

If $\mathcal{G}$ fulfills the above Moran structure, we call:

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\omega\in D_n} B_\omega$$

the *Moran set* associated with $\mathcal{G}$.

For $k \in \mathbb{N}$, let

$$c_k = \min_{(i_1\cdots i_k)\in D_k} \frac{r_{i_1\cdots i_k}}{r_{i_1\cdots i_{k-1}}}, \qquad M_k = \max_{(i_1\cdots i_k)\in D_k} r_{i_1\cdots i_k}.$$

Proposition 5.1. *(See [14, Proposition 3.1].) For a Moran set F defined as above, suppose furthermore*

$$\lim_{k\to\infty} \frac{\log c_k}{\log M_k} = 0. \tag{5.1}$$

Then we have:

$$\dim_H F = \liminf_{k\to\infty} s_k,$$

where s_k satisfies the equation $\sum_{\omega\in D_k} r_\omega^{s_k} = 1$ for each k. Moreover, there exists a Borel probability measure ν on F such that

$$\liminf_{\delta\to 0} \frac{\log \nu(B(x,\delta))}{\log \delta} = \dim_H F$$

for all $x \in F$, where $B(x,\delta)$ denotes the closed ball in $\mathbb{R}^d$ of radius r centered at x.

We remark that the existence of ν in the above proposition is only implicit in the proof of [14, Proposition 3.1]. Of course, $\dim_H \nu = \dim_H F$, where $\dim_H \nu = \inf\{\dim_H E\colon E \text{ is Borel with } \nu(E) = 1\}$.

Let μ be a self-similar measure satisfying the WSC and assume that U_0 is an open ball satisfying (2.5).

Proposition 5.2. *Let $\alpha_{\min} = \lim_{q\to\infty} \tau_{U_0}(q)/q$ and $\alpha_{\max} = \lim_{q\to-\infty} \tau_{U_0}(q)/q$. Then $E_{U_0}(\alpha_{\min}) \neq \emptyset$ and $E_{U_0}(\alpha_{\max}) \neq \emptyset$. Furthermore*

$$\dim_H E_{U_0}(\alpha_{\min}) \geqslant \tau^*_{U_0}(\alpha_{\min}), \qquad \dim_H E_{U_0}(\alpha_{\max}) \geqslant \tau^*_{U_0}(\alpha_{\max}).$$

Proof. It is known that $\tau_{U_0}(q)$ is a concave real-valued function of q on $\mathbb{R}$ and the limits $\alpha_{\min}$, $\alpha_{\max}$ exist, and $\alpha_{\min}, \alpha_{\max} \in (0,\infty)$ (Corollary 3.2). In the following, we only prove that $E_{U_0}(\alpha_{\min}) \neq \emptyset$ and $\dim_H E_{U_0}(\alpha_{\min}) \geqslant \tau^*_{U_0}(\alpha_{\min})$. The corresponding results for $E_{U_0}(\alpha_{\max})$ can be proved similarly.

Let $\{q_n\} \uparrow \infty$ such that the derivative $\tau'_{U_0}(q_n) =: \alpha_n$ exists for each n. By the concavity of $\tau_{U_0}(\cdot)$, the sequence $\{\alpha_n\}$ is non-increasing, and $\alpha_{\min} = \lim_{n\to\infty} \alpha_n$. Furthermore the function $\tau^*_{U_0}$ is concave on $[\alpha_{\min}, \alpha_{\max}]$, and hence it is lower semi-continuous (see [35, Theorem 10.2]). Therefore we have:

$$\tau^*_{U_0}(\alpha_{\min}) \leqslant \liminf_{n\to\infty} \tau^*_{U_0}(\alpha_n) = \liminf_{n\to\infty} \bigl(\alpha_n q_n - \tau_{U_0}(q_n)\bigr).$$

We choose a positive sequence $\{\delta_i\}_{i=1}^{\infty} \downarrow 0$ such that $\lim_{n\to\infty} \delta_n q_n = 0$. For each $i \in \mathbb{N}$, using Proposition 4.1 we construct $r_i > 0$, $k_i \in \mathbb{N}$ and $\mathcal{B}_i = \{u_{i,s}\colon 1 \leqslant s \leqslant k_i\} \subset \Omega$ such that

(a) $1 > r_1 > r_2 > \cdots$.
(b) $k_i \geqslant (r_i)^{-\tau^*_{U_0}(\alpha_i)+\delta_i(q_i+1)}$.
(c) $(r_i)^{1+\delta_i} \leqslant \rho_{u_{i,s}} \leqslant (r_i)^{1-\delta_i}$, $\forall 1 \leqslant s \leqslant k_i$.
(d) $S_{u_{i,s}}(4U_0)$ $(s = 1, \dots, k_i)$ are disjoint subsets of U_0.
(e) $(r_i)^{\alpha_i+3\delta_i}\mathbf{p}(\vartheta) \preccurlyeq \mathbf{p}(u_{i,s}) \preccurlyeq (r_i)^{\alpha_i-3\delta_i}\mathbf{p}(\vartheta)$, $\forall 1 \leqslant s \leqslant k_i$.

Also we let $\{N_i\}_{i=1}^{\infty}$ be a sequence of integers large enough such that

(f) $r_i^{N_i} < (r_{i+1})^{2^i}$ for each $i \in \mathbb{N}$.

Now we define a sequence of subsets of $\mathcal{A}^*$ in the following manner,

$$\underbrace{\mathcal{B}_1, \dots, \mathcal{B}_1}_{N_1}, \underbrace{\mathcal{B}_2, \dots, \mathcal{B}_2}_{N_2}, \dots, \underbrace{\mathcal{B}_i, \dots, \mathcal{B}_i}_{N_i}, \dots$$

(i.e., $\mathcal{B}_1$ is repeated N_1 times, follow by $\mathcal{B}_2$ repeated N_2 times and so on), and relabel them as $\{\mathcal{B}^*_n\}_{n=1}^{\infty}$. Let

$$\mathcal{G} = \big\{S_{v_1\dots v_k}(\overline{U_0}):\ k \in \mathbb{N},\ v_i \in \mathcal{B}^*_i \text{ for } 1 \leqslant i \leqslant k\big\}. \tag{5.2}$$

It is easy to check that $\mathcal{G}$ fulfills the Moran structure. Let

$$F = \bigcap_{n=1}^{\infty} \bigcup_{v_1 \in \mathcal{B}^*_1, \dots, v_n \in \mathcal{B}^*_n} S_{v_1 v_2 \dots v_n}(\overline{U_0}).$$

Then F is the Moran set associated with $\mathcal{G}$.

Next we show that

$$\alpha(x) = \lim_{n\to\infty} \alpha_n, \quad \forall x \in F, \tag{5.3}$$

and

$$\dim_H F \geqslant \liminf_{n\to\infty} \tau^*_{U_0}(\alpha_n). \tag{5.4}$$

Let $x \in F$, then there exist $v_i \in \mathcal{B}^*_i$ $(i = 1, 2, \dots)$ such that

$$\{x\} = \lim_{n\to\infty} S_{v_1\dots v_n}(\overline{U_0}).$$

For $r > 0$ small enough, there is a unique large integer n such that

$$\big|S_{v_1\dots v_{n+1}}(\overline{U_0})\big| \leqslant 2r < \big|S_{v_1\dots v_n}(\overline{U_0})\big|.$$

Note that $S_{v_{n+2}}(4U_0) \subset U_0$ (by (d)), we have $|S_{v_1\dots v_{n+2}}(U_0)| \leqslant |S_{v_1\dots v_{n+1}}(\overline{U_0})|/4 \leqslant r/2$. Hence

$$S_{v_1\dots v_{n+2}}(U_0) \subset B(x, r) \subset S_{v_1\dots v_n}(4U_0) \subset S_{v_1\dots v_{n-1}}(U_0). \tag{5.5}$$

On the other hand by Lemma 2.6, there exists a constant $c > 0$ (depending on U_0) such that

$$c\big\|\mathbf{p}(v_1 \dots v_j)\big\| \leqslant \mu\big(S_{v_1\dots v_j}(U_0)\big) \leqslant \big\|\mathbf{p}(v_1 \dots v_j)\big\|, \quad \forall j \in \mathbb{N}. \tag{5.6}$$

Combining (5.5) and (5.6), we have:

$$\frac{\log\|\mathbf{p}(v_1 \dots v_{n-1})\|}{\log(\rho_{v_1\dots v_{n+2}}|U_0|)} \leqslant \frac{\log\mu(B(x,r))}{\log r} \leqslant \frac{\log(c\|\mathbf{p}(v_1 \dots v_{n+2})\|)}{\log(\rho_{v_1\dots v_{n-1}}|U_0|)}. \tag{5.7}$$

Thus to calculate $\alpha(x)$, we need to estimate $\mathbf{p}(v_1 \dots v_m)$ and $\rho_{v_1\dots v_m}$ for $m = n-1, n+2$. For large n, write $n = \sum_{i=1}^{k} N_i + p$ with $1 \leqslant p \leqslant N_{k+1}$. In the case that $1 \leqslant p \leqslant N_{k+1} - 2$, by (e), (c), (b) and Corollary 2.11, we obtain:

$$\begin{aligned}&\left(\prod_{i=1}^{k}(r_i)^{N_i(\alpha_i+3\delta_i)}\right)(r_{k+1})^{(p+2)(\alpha_{k+1}+3\delta_{k+1})}\mathbf{p}(\vartheta)\\ &\quad\preccurlyeq \mathbf{p}(v_1 \dots v_{n+2}) \preccurlyeq \mathbf{p}(v_1 \dots v_{n-1}) \preccurlyeq \left(\prod_{i=1}^{k}(r_i)^{N_i(\alpha_i-3\delta_i)}\right)(r_{k+1})^{(p-1)(\alpha_{k+1}-3\delta_{k+1})}\mathbf{p}(\vartheta),\end{aligned} \tag{5.8}$$

$$\left(\prod_{i=1}^{k}(r_i)^{N_i(1+\delta_i)}\right)(r_{k+1})^{(p+2)(1+\delta_{k+1})} \leqslant \rho_{v_1\ldots v_{n+2}} \leqslant \rho_{v_1\ldots v_{n-1}} \leqslant \left(\prod_{i=1}^{k}(r_i)^{N_i(1-\delta_i)}\right)(r_{k+1})^{(p-1)(1-\delta_{k+1})}, \tag{5.9}$$

and

$$\prod_{s=1}^{n}\#\mathcal{B}_s^* \geqslant \left(\prod_{i=1}^{k}(r_i)^{N_i(-\tau_{U_0}^*(\alpha_i)+\delta_i(q_i+1))}\right)(r_{k+1})^{p(-\tau_{U_0}^*(\alpha_{k+1})+\delta_{k+1}(q_{k+1}+1))}. \tag{5.10}$$

In the other case $N_{k+1}-1 \leqslant p \leqslant N_{k+1}$, we have the similar inequalities where the lower bounds for $\mathbf{p}(v_1\ldots v_{n+2})$ and $\rho_{v_1\ldots v_{n+2}}$ in (5.8) and (5.9) are replaced respectively by:

$$\left(\prod_{i=1}^{k+1}(r_i)^{N_i(\alpha_i+3\delta_i)}\right)(r_{k+2})^{(p-N_{k+1}+2)(\alpha_{k+2}+3\delta_{k+2})}\mathbf{p}(\vartheta) \preccurlyeq \mathbf{p}(v_1\ldots v_{n+2}), \tag{5.11}$$

$$\left(\prod_{i=1}^{k+1}(r_i)^{N_i(1+\delta_i)}\right)(r_{k+2})^{(p-N_{k+1}+2)(1+\delta_{k+2})} \leqslant \rho_{v_1\ldots v_{n+2}}. \tag{5.12}$$

By (5.7), using the inequalities (5.8)–(5.9), (5.11)–(5.12) and (f), we obtain $\alpha(x)=\lim_{n\to\infty}\alpha_n$ by a direct calculation.

To prove (5.4), recall that F is the Moran set associated with $\mathcal{G}$ (see (5.2)). Again for a large n, write $n=\sum_{i=1}^{k}N_i+p$, where $1\leqslant p\leqslant N_{k+1}$. By (c), we have:

$$\inf_{v_n\in\mathcal{B}_n^*}\rho_{v_n} \geqslant (r_{k+1})^{1+\delta_{k+1}}, \qquad \sup_{v_1\in\mathcal{B}_1^*,\ldots,v_n\in\mathcal{B}_n^*}\rho_{v_1\ldots v_n} \leqslant \left(\prod_{i=1}^{k}(r_i)^{N_i(1-\delta_i)}\right)(r_{k+1})^{p(1-\delta_{k+1})}. \tag{5.13}$$

Using (5.13) and (f), we have:

$$\lim_{n\to\infty}\frac{\log(\inf_{v_n\in\mathcal{B}_n^*}\rho_{v_n})}{\log(\sup_{v_1\in\mathcal{B}_1^*,\ldots,v_n\in\mathcal{B}_n^*}\rho_{v_1\ldots v_n})}=0.$$

This implies that the condition (5.1) in Proposition 5.1 is fulfilled. Hence by Proposition 5.1, we have $\dim_H F = \liminf_{n\to\infty}s_n$, where s_n satisfies,

$$\sum_{v_1\in\mathcal{B}_1^*,\ldots,v_n\in\mathcal{B}_n^*}(\rho_{v_1\ldots v_n})^{s_n}=1. \tag{5.14}$$

It follows that

$$\dim_H F \geqslant \liminf_{n\to\infty}\frac{\log(\prod_{s=1}^{n}\#\mathcal{B}_s^*)}{-\log(\inf_{v_1\in\mathcal{B}_1^*,\ldots,v_n\in\mathcal{B}_n^*}\rho_{v_1\ldots v_n})}.$$

This, together with (5.10) and the following inequality,

$$\inf_{v_1\in\mathcal{B}_1^*,\ldots,v_n\in\mathcal{B}_n^*}\rho_{v_1\ldots v_n} \geqslant \left(\prod_{i=1}^{k}(r_i)^{N_i(1+\delta_i)}\right)(r_{k+1})^{p(1+\delta_{k+1})} \quad \text{(by (c))},$$

yields:

$$\dim_H F \geqslant \liminf_{k\to\infty}\frac{\tau_{U_0}^*(\alpha_k)+\delta_k(q_k+1)}{1+\delta_k} = \liminf_{k\to\infty}\tau_{U_0}^*(\alpha_k). \qquad \square$$

Proposition 5.3. *Let $q_1,q_2\in\mathbb{R}$. Suppose that $\alpha_1=\tau_{U_0}'(q_1)$, $\alpha_2=\tau_{U_0}'(q_2)$ exist. Then for any $0<\lambda<1$, $E_{U_0}(\lambda\alpha_1+(1-\lambda)\alpha_2)\neq\emptyset$, and*

$$\dim_H E_{U_0}\big(\lambda\alpha_1+(1-\lambda)\alpha_2\big) \geqslant \lambda\tau_{U_0}^*(\alpha_1)+(1-\lambda)\big(\tau_{U_0}^*(\alpha_2)\big).$$

In particular if $q_1=q_2$ and let $\alpha=\alpha_1=\alpha_2$, then $\dim_H E_{U_0}(\alpha)\geqslant\tau_{U_0}^*(\alpha)$.

Proof. The proposition is proved in a way similar to that of Proposition 5.2, through a more subtle Moran construction. Let $\{\delta_i\}_{i=1}^{\infty}$ be a positive sequence decreasing to 0. For each $i \in \mathbb{N}$ and $j \in \{1,2\}$, using Proposition 4.1 we construct $r_{i,j} > 0$, $k_{i,j} \in \mathbb{N}$ and $\mathcal{B}_{i,j} = \{u_{i,j,s}\colon 1 \leqslant s \leqslant k_{i,j}\} \subset \Omega$ such that

(a) $1 > r_{1,1} > r_{1,2} > r_{2,1} > r_{2,2} > \cdots$.
(b) $k_{i,j} \geqslant (r_{i,j})^{-\tau^*_{U_0}(\alpha_j)+\delta_i(|q_j|+1)}$.
(c) $(r_{i,j})^{1+\delta_i} \leqslant \rho_{u_{i,j,s}} \leqslant (r_{i,j})^{1-\delta_i}$, $\forall 1 \leqslant s \leqslant k_{i,j}$.
(d) $S_{u_{i,j,s}}(4U_0)$ $(1 \leqslant s \leqslant k_{i,j})$ are disjoint subsets of U_0 for each i, j.
(e) $(r_{i,j})^{\alpha_j+3\delta_i}\mathbf{p}(\vartheta) \preccurlyeq \mathbf{p}(u_{i,j,s}) \preccurlyeq (r_{i,j})^{\alpha_j-3\delta_i}\mathbf{p}(\vartheta)$, $\forall 1 \leqslant s \leqslant k_{i,j}$.

Let $0 < \lambda < 1$ be given. We construct a sequence of positive integers $\{N_{i,j}\}_{i\in\mathbb{N},1\leqslant j\leqslant 2}$ such that

(f) For any $i \in \mathbb{N}$, $|\frac{N_{i,1}\log r_{i,1}}{N_{i,1}\log r_{i,1}+N_{i,2}\log r_{i,2}} - \lambda| < 2^{-i}$, and
(g) $\max\{r_{i,1}^{N_{i,1}}, r_{i,2}^{N_{i,2}}\} < (r_{i+1,j})^{2^i}$ for $j = 1, 2$.

Also we select a sequence of positive integers $\{L_i\}_{i=1}^{\infty}$ such that

(h) For any $i \in \mathbb{N}$, $L_i \min_{1\leqslant j\leqslant 2}\{N_{i,j}|\log r_{i,j}|\} \geqslant 2^i \max_{1\leqslant j\leqslant 2}\{N_{i+1,j}|\log r_{i+1,j}|\}$.

Now we define a sequence $\{\mathcal{B}^*_i\}_{i=1}^{\infty}$ of subsets of Ω in the following manner:

$$\underbrace{\overbrace{\mathcal{B}_{1,1},\ldots,\mathcal{B}_{1,1}}^{N_{1,1}},\overbrace{\mathcal{B}_{1,2},\ldots,\mathcal{B}_{1,2}}^{N_{1,2}},\ldots,\overbrace{\mathcal{B}_{1,1},\ldots,\mathcal{B}_{1,1}}^{N_{1,1}},\overbrace{\mathcal{B}_{1,2},\ldots,\mathcal{B}_{1,2}}^{N_{1,2}}}_{L_1(N_{11}+N_{12})}$$

$$\underbrace{\overbrace{\mathcal{B}_{2,1},\ldots,\mathcal{B}_{2,1}}^{N_{2,1}},\overbrace{\mathcal{B}_{2,2},\ldots,\mathcal{B}_{2,2}}^{N_{2,2}},\ldots,\overbrace{\mathcal{B}_{2,1},\ldots,\mathcal{B}_{2,1}}^{N_{2,1}},\overbrace{\mathcal{B}_{2,2},\ldots,\mathcal{B}_{2,2}}^{N_{2,2}}}_{L_2(N_{21}+N_{22})}$$

$$\cdots$$

That is, we first let $\mathcal{B}_{1,1}$ appear in the sequence for $N_{1,1}$ times, then let $\mathcal{B}_{1,2}$ appear for $N_{1,2}$ times. Repeat this pattern for L_1 times. After that, we let $\mathcal{B}_{2,1}$ appear for $N_{2,1}$ times, then $\mathcal{B}_{2,2}$ appear for $N_{2,2}$ times. Repeat this pattern for L_2 times. Continuing this process we get the desired sequence, which is relabeled as $\{\mathcal{B}^*_i\}_{i=1}^{\infty}$. Let

$$\mathcal{G} = \left\{S_{v_1\ldots v_k}(\overline{U_0})\colon\ k \in \mathbb{N},\ v_i \in \mathcal{B}^*_i \text{ for } 1 \leqslant i \leqslant k\right\},$$

and

$$F = \bigcap_{n=1}^{\infty} \bigcup_{v_1\in\mathcal{B}^*_1,\ldots,v_n\in\mathcal{B}^*_n} S_{v_1v_2\ldots v_n}(\overline{U_0}).$$

Then $\mathcal{G}$ fulfills the Moran structure and F is the Moran set associated with $\mathcal{G}$. The condition (g) on $\{N_{i,j}\}$ is made so that the assumption (5.1) is satisfied. Hence by Proposition 5.1, $\dim_H F = \liminf_{n\to\infty} s_n$, where s_n satisfies:

$$\sum_{v_1\in\mathcal{B}^*_1,\ldots,v_n\in\mathcal{B}^*_n} (\rho_{v_1\ldots v_n})^{s_n} = 1. \tag{5.15}$$

The further conditions (f) and (h), together with (5.15) will guarantee that

$$\alpha(x) = \lambda\alpha_1 + (1-\lambda)\alpha_2, \quad \forall x \in F, \tag{5.16}$$

and

$$\dim_H F \geqslant \lambda\tau^*_{U_0}(\alpha_1) + (1-\lambda)\tau^*_{U_0}(\alpha_2). \tag{5.17}$$

The proofs of (5.16) and (5.17) are similar to those of (5.3) and (5.4) in Proposition 5.2, we just omit the details to avoid repetition. □

Our main result is the following theorem about the multifractal formalism.

Theorem 5.4. *Let μ be a self-similar measure associated with an IFS $\{S_i\}_{i=1}^m$ that satisfies the WSC and let $U_0 = U(x, r_0)$ be the open ball in* (2.5) *of the definition of WSC. Then*

(i) $E_{U_0}(\alpha) \neq \emptyset$ *if and only if* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.
(ii) *For any* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$,

$$\dim_H E_{U_0}(\alpha) = \tau^*_{U_0}(\alpha) \big(:= \inf\{\alpha q - \tau_{U_0}(q) \colon q \in \mathbb{R}\}\big). \tag{5.18}$$

Moreover, $\tau(q) = \tau_{U_0}(q)$ *for* $q \geqslant 0$, *and*

$$\dim_H E(\alpha) = \tau^*(\alpha) := \inf\{\alpha q - \tau(q) \colon q \in \mathbb{R}\}, \quad \forall \alpha \in [\alpha_{\min}, \tau'(0-)]. \tag{5.19}$$

Proof. If $E_{U_0}(\alpha) \neq \emptyset$, we can choose $\alpha(x) = \alpha$. By Corollary 3.2, $\alpha = \alpha(x) \in [\alpha_{\min}, \alpha_{\max}]$. Conversely, suppose $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. Since $\tau_{U_0}(\cdot)$ is concave, for any $\alpha \in (\alpha_{\min}, \alpha_{\max})$, there exist $q_1, q_2 \in \mathbb{R}$ such that $\tau'_{U_0}(\cdot)$ exists at q_1 and q_2 and $\tau'_{U_0}(q_1) \leqslant \alpha \leqslant \tau'_{U_0}(q_2)$. Hence α is a convex combination of $\alpha_j = \tau'_{U_0}(q_j)$ $(j = 1, 2)$ and Proposition 5.3 implies $E_{U_0}(\alpha) \neq \emptyset$ for $\alpha \in (\alpha_{\min}, \alpha_{\max})$. Moreover Proposition 5.2 implies that $E_{U_0}(\alpha) \neq \emptyset$ for $\alpha = \alpha_{\min}$ or $\alpha_{\max}$. Hence $E_{U_0}(\alpha) \neq \emptyset$ for any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. This completes the proof of (i).

To prove (ii), note that for any open set $V \subset \mathbb{R}^d$ with $\mu(V) > 0$, and $\alpha \in \mathbb{R}$ so that $E_V(\alpha) \neq \emptyset$, we always have:

$$\dim_H E_V(\alpha) \leqslant \tau^*_V(\alpha).$$

In fact, the above inequality holds for any compactly supported Borel probability measures on $\mathbb{R}^d$ (see, e.g., [2,21] for a proof). Combining this with Propositions 5.2 and 5.3, we have,

$$\dim_H E_{U_0}(\alpha) = \tau^*_{U_0}(\alpha),$$

if $\alpha = \alpha_{\min}$, $\alpha = \alpha_{\max}$, or $\alpha = \tau'_{U_0}(q)$.

Now, in general, we assume $\alpha \in (\alpha_{\min}, \alpha_{\max})$. Then there exists $t \in \mathbb{R}$ such that $\alpha \in [\tau'_{U_0}(t+), \tau'_{U_0}(t-)]$, and accordingly

$$\tau^*(\alpha) = \alpha t - \tau_{U_0}(t).$$

Denote $a = \tau'_{U_0}(t+)$, $b = \tau'_{U_0}(t-)$ and write $\alpha = \lambda a + (1-\lambda)b$ for some $\lambda \in [0, 1]$. Select $t_n \uparrow t$ and $s_n \downarrow t$ such that $a_n = \tau'_{U_0}(t_n)$ and $b_n = \tau'_{U_0}(s_n)$ exist. Then there exist $\lambda_n \in [0, 1]$ such that $\alpha = \lambda_n \tau'_{U_0}(t_n) + (1-\lambda_n)\tau'_{U_0}(s_n)$ and $\lim_{n\to\infty} \lambda_n = \lambda$. By Proposition 5.3, $\dim_H E_{U_0}(\alpha) \geqslant \lambda_n \tau^*(a_n) + (1-\lambda_n)\tau^*(b_n)$. Letting n tends to infinity, we have:

$$\begin{aligned}\dim_H E_{U_0}(\alpha) &\geqslant \lambda \tau^*_{U_0}(a) + (1-\lambda)\tau^*_{U_0}(b) = \lambda\big(at - \tau_{U_0}(t)\big) + (1-\lambda)\big(bt - \tau_{U_0}(t)\big)\\ &= \alpha t - \tau_{U_0}(t) = \tau^*_{U_0}(\alpha).\end{aligned}$$

Combining this with (5.18) we obtain $\dim_H E_{U_0}(\alpha) = \tau^*_{U_0}(\alpha)$. Hence we have proved that $\dim_H E_{U_0}(\alpha) = \tau^*(\alpha)$ for any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

To see the last part, by Proposition 3.1 we have $\tau(q) = \tau_{U_0}(q)$ for $q \geqslant 0$. To show (5.19), it suffices to show the lower bound. Let $\alpha \in [\alpha_{\min}, \tau'(0-)]$. Then $\alpha \in [\tau'_{U_0}(t+), \tau'_{U_0}(t-)]$ for some $t \geqslant 0$. Hence, we have:

$$\dim_H E(\alpha) \geqslant \dim_H E_{U_0}(\alpha) = \tau^*_{U_0}(\alpha) = \alpha t - \tau_{U_0}(t) = \alpha t - \tau(t) \geqslant \tau^*(\alpha).$$

This completes the proof of the theorem. □

Remark 5.5. Regarding of Theorem 5.4(ii), we have a slightly stronger result that for each $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, there is a Borel probability measure ν on $E_{U_0}(\alpha)$ such that

$$\liminf_{\delta\to 0} \frac{\log \nu(B(x, \delta))}{\log \delta} = \tau^*_{U_0}(\alpha) \quad \text{for } \nu\text{-a.e. } x. \tag{5.20}$$

Indeed, as proved in Propositions 5.2–5.3 and Theorem 5.4, there is a Moran set $F \subset E_{U_0}(\alpha)$ such that $\dim_H F = \tau^*_{U_0}(\alpha)$. The existence of ν then follows from Proposition 5.1.

Corollary 5.6. *The above theorem remain valid if U_0 is replaced by $U_u = S_u(U_0)$ for any word $u \in \mathcal{A}^*$. Furthermore, $\tau_{U_u}(q) = \tau_{U_0}(q)$ for all $q \in \mathbb{R}$.*

Proof. To see (iii), let $u \in \mathcal{A}^*$. Then by Lemma 2.5 and the definition of the L^q spectrum, we have $\tau_{S_u(U_0)} = \tau_{U_0}$. Since $S_u(U_0)$ also satisfies the maximality (2.5), the statements (i) and (ii) also hold if U_0 is replaced by $S_u(U_0)$. □

By Theorem 5.4, we have $\dim_H E(\alpha) = \tau^*(\alpha)$ for $\alpha \in [\alpha_{\min}, \tau'(0-)]$. It is interesting to consider the dimension spectrum $\dim_H E(\alpha)$ for those $\alpha \in (\tau'(0-), \alpha_{\max}]$. The following two corollaries determine the dimension spectrum under certain additional assumptions, which are satisfied for some concrete examples of self-similar measures (see Section 6).

Corollary 5.7. *Under the condition of Theorem 5.4, assume that $\tau(q) = \tau_{U_0}(q)$ for all $q < 0$. Then $\dim_H E(\alpha) = \tau^*(\alpha)$ for any $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.*

Proof. On one hand, the inequality $\dim_H E(\alpha) \leqslant \tau^*(\alpha)$ always holds for any compactly supported probability measure. On the other hand, by Theorem 5.4 and the assumption $\tau(q) = \tau_{U_0}(q)$ for all $q < 0$, we have $\dim_H E(\alpha) \geqslant \dim_H E_{U_0}(\alpha) = \tau^*_{U_0}(\alpha) = \tau^*(\alpha)$. □

Set $\mathcal{U} = \{U\colon U = U(x,r)$ attains the maximality in (2.5)$\}$ and $W = \bigcup_{U \in \mathcal{U}} U$.

Corollary 5.8. *Under the condition of Theorem 5.4, assume that $\dim_H K \backslash W = 0$. Then for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, $\dim_H E(\alpha) = \tau^*_{U_0}(\alpha)$.*

Proof. We first show that $\tau_{U_1} = \tau_{U_0}$ for any $U_1 \in \mathcal{U}$. Since $U_1 \cap K \neq \emptyset$, there exists a $u \in \mathcal{A}^*$ such that $S_u(U_0) \subset U_1$. It implies that $\tau_{U_1}(q) \leqslant \tau_{S_u(U_0)}(q) = \tau_{U_0}(q)$, where the last equality follows from Lemma 2.5. Symmetrically we have the opposite inequality.

Now observe that if $U(x,r) \in \mathcal{U}$, then $U(y,r') \in \mathcal{U}$ if y and r' ($r' < r$) are close enough to x and r. Hence each $U(x,r)$ is the union of some $U(y,r') \in \mathcal{U}$ with $y \in \mathbb{Q}^d$ and $r' \in \mathbb{Q}^+$. As a consequence, there exists a sequence $U_i \in \mathcal{U}$ such that $W = \bigcup_{i=1}^\infty U_i$. Since $\dim_H K \backslash W = 0$, $E(\alpha)$ differs only from $E_W(\alpha) = \bigcup_{i=1}^\infty E_{U_i}(\alpha)$ by a set of Hausdorff dimension 0. Hence by Theorem 5.4, we have:

$$\dim_H E(\alpha) = \dim_H E_W(\alpha) = \dim_H \left(\bigcup_{i=1}^\infty E_{U_i}(\alpha) \right) = \sup_i \dim_H E_{U_i}(\alpha) = \sup_i \tau^*_{U_i}(\alpha) = \tau^*_{U_0}(\alpha).$$

This completes the proof of the corollary. □

6. Examples and remarks

In the following we use some simple examples to illustrate the theorems.

Example 6.1. Consider the IFS $S_j(x) = \frac{1}{3}(x + 2j - 2)$ for $1 \leqslant j \leqslant 4$, let μ be the self-similar measure satisfying,

$$\mu = \sum_{j=1}^4 p_j \mu \circ S_j^{-1},$$

where $\mathbf{p} = (1/8, 3/8, 3/8, 1/8)$. The IFS has attractor $K = [0,3]$ and satisfies the WSC. The measure μ is just the 3-fold convolution of the standard Cantor measure. This example has some special interest as it is known that the dimension spectrum contains an isolated point and $\tau(q)$ contains one non-differentiable point at $q < 0$, and the multifractal formalism for μ breaks on an interval corresponding to the non-differentiable point $q_0 < 0$ [19,23]. The failure of the formalism is due to the fact that μ is too small on the intervals $[0, 3^{-n}]$ or $[3 - 3^{-n}, 3]$ ($n \in \mathbb{N}$) and a modified multifractal formalism is given in [13].

In the present situation, a simple calculation shows that we can take $\ell = 5$ and $U_0 = (5/18, 17/18)$ to attain the maximality in (2.5). To see this, we assume that $U_0 = U(x_0, r_0)$ attains the maximum in (2.5) with $0 < r_0 \leqslant 1$. Let $k \geqslant 1$ be the integer so that $3^{-k} < r_0 \leqslant 3^{-k+1}$. Then by the definition of Γ_r (see (2.1)), we have:

$$\Gamma_{r_0} = \{S_u\colon u \in \mathcal{A}^k\} = \{\phi_i(x) = 3^{-k}(x + 2i)\colon i = 0, 1, \ldots, (3^{k+1} - 3)/2\}.$$

Since $\Gamma_{r_0} = \Gamma_{3^{-k+1}}$, $U(x_0, 3^{-k+1})$ also attains the maximum in (2.5). Hence we may take $r_0 = 3^{-k+1}$. Then

$$\begin{aligned}\#\mathcal{S}_{r_0}(U_0) &= \#\{0 \leqslant i \leqslant (3^{k+1} - 3)/2\colon [2i/3^k, (2i + 3)/3^k] \cap U(x_0, 3^{-k+1}) \neq \emptyset\} \\ &= \#\{0 \leqslant i \leqslant (3^{k+1} - 3)/2\colon i \in \Delta = ((3^k x_0 - 6)/2, (3^k x_0 + 3)/2)\}.\end{aligned}$$

Since Δ is an open interval of length $9/2$ which contains at most 5 integral points, we have $\#\mathcal{S}_r(U_0) \leqslant 5$. A direct check shows that the maximality 5 can be attained if we let $k \geqslant 2$ and choose $x_0 = 3^{-k}(2i + 5.5)$ for any integer $i \in [0, (3^{k+1} - 11)/2]$. For example, we can take $k = 2$ and $x_0 = 11/18$ (corresponding to $i = 0$). In this case, $U_0 = (x_0 - 1/3, x_0 + 1/3) = (5/18, 17/18))$. It follows from Theorem 5.4 that $\mu|_{U_0}$ satisfies the multifractal formalism. As a direct check, we have:

$$W \supseteq \bigcup_{k=2}^{\infty} \bigcup_{i=0}^{\frac{3^{k+1}-11}{2}} (3^{-k}(2i + 5.5) - 3^{k-1}, 3^{-k}(2i + 5.5) + 3^{k-1}) = (0, 3).$$

Hence the condition of Corollary 5.8 is fulfilled, and we have $\dim_H E(\alpha) = \tau^*_{U_0}(\alpha)$ for $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. This gives an alternative proof of the result obtained in [13].

Example 6.2. More generally, consider the IFS $S_j(x) = \frac{1}{N}(x + j - 1)$ for $1 \leqslant j \leqslant m$, where $N, m \in \mathbb{N}$, $m > N \geqslant 2$. The attractor of the IFS is $K = [0, \frac{m-1}{N-1}]$. Let μ be the corresponding self-similar measure associated with a probability vector $(p_1, \ldots, p_m)$. The multifractal structure of μ was considered by Shmerkin in [36]. Using an argument analogous to Example 6.1, we can show that the integer ℓ in (2.4) is equal to the least integer not less than $2N + \frac{m-1}{N-1}$. Moreover, the maximality of (2.5) is attained for those intervals $U = U(x_0, N^{-k+1})$ such that $k \geqslant 2$ and $N^k x_0 - N - \frac{m-1}{N-1} \in (i - \{\frac{m-1}{N-1}\}, i)$ for some integer $i \in [0, \frac{(m-1)(N^k-1)}{N-1} - \ell - 1]$, where $\{\frac{m-1}{N-1}\}$ denotes the fractional part of $\frac{m-1}{N-1}$. A direct check also shows that $W \supseteq (0, \frac{m-1}{N-1})$. Hence $K \backslash W$ has Hausdorff dimension 0 and thus the result of Corollary 5.8 holds, i.e., $\dim_H E(\alpha) = \tau^*_{U_0}(\alpha)$ for $\alpha \in [\alpha_{\min}, \alpha_{\max}]$. This completes a result obtained by Shmerkin [36, Theorem 1.6], who proved a modified multifractal formalism on the range $(\alpha_{\min}, \alpha_{\max})$ under an additional assumption $m < 2N - 3$ (which forces μ to be an attractor of an infinite IFS without overlaps). Shmerkin also gave some further conditions on the probability vector $(p_1, \ldots, p_m)$ to guarantee $\tau = \tau_V$ for each open interval such that $\overline{V} \subset \mathrm{int}(K)$ and to verify the validity of the multifractal formalism (which is also a consequence of Corollary 5.7).

Example 6.3. Another simple example is the Bernoulli convolution associated with the IFS $\{\rho x, \rho x + 1\}$ with $\rho = \frac{\sqrt{5}-1}{2}$, which is also called *Erdös measure* [37]. There is a large literature concerning the fractal dimensions and multifractal structure of this measure (see [10] and references therein). It is the first example of self-similar measure found to have a phase transition [10] (see also [15]). Through a rather tedious calculation we can show that $\ell = 5$ and the maximum in (2.5) can be attained at $U_0 = U(x, \rho^2)$ for any $x \in (2\rho, 1 + \rho^2)$ (we omit the details). It is known that the measure satisfies the multifractal formalism [15].

There are more extensive class of examples of WSC studied in [22,10,39]. More generally, let $\{S_i\}_{i=1}^m$ be an IFS given by:

$$S_i(x) = (-1)^{m_i} \rho^{n_i} x + k_i, \quad i = 1, \ldots, m, \tag{6.1}$$

where $1/\rho$ is a Pisot number, $m_i \in \{0, 1\}$, $n_i \in \mathbb{N}$ and $k_i \in \mathbb{Z}$. Then the IFS satisfies the WSC according to an arithmetic property of Pisot numbers (see [21, p. 70] for a similar argument). Testud constructed some simple self-similar measures generated by IFS of the form $S_j(x) = \frac{\epsilon_j}{N} x + k_j$ ($\epsilon_j = \pm 1$) such that the dimension spectra are very wild and not concave at all [39].

For such IFS, one may design a finite algorithm to determine the corresponding integer ℓ and choose a suitable interval U_0 to attain the maximum in (2.5).

Theorems 1.1, 1.2 remain valid when the open ball U_0 is replaced by certain open sets. Indeed, let V be an arbitrary bounded open set so that $V \cap K \neq \emptyset$, and let $r > 0$. Then from the assumption of the WSC, one deduces that

$$\sup_{w \in \mathcal{A}^* \cup \{\vartheta\}} \#\mathcal{S}_{\rho_w r}\big(S_w(V)\big) < \infty.$$

Hence the supremum is attained at some $w \in \mathcal{A}^* \cup \{\vartheta\}$. Denote $\ell' = \#\mathcal{S}_{\rho_w r_0}(S_w(V))$. Let $V_0 = S_w(V)$ and $r_0 = \rho_w r$. Then as an analogue of Proposition 2.4,

$$\mathcal{S}_{\rho_u r_0}\big(S_u(V_0)\big) = \{S_{u\omega_i'}\colon 1 \leqslant i \leqslant \ell'\}, \quad \forall u \in \mathcal{A}^* \cup \{\vartheta\},$$

where $S_{\omega_i'}$, $1 \leqslant i \leqslant \ell'$, are distinct elements in $\#\mathcal{S}_{r_0}(V_0)$. All the results after Proposition 2.4 then remain valid when U_0 is replaced by V_0.

Example 6.4. Consider an IFS $\{S_i\}_{i=1}^5$ in $\mathbb{R}^2$ given by:

$$S_i(x, y) = (x/2, y/2) + a_i, \quad i = 1, \ldots, 5,$$

where $a_1 = (0, 0)$, $a_2 = (0, 1)$, $a_3 = (1, 0)$, $a_4 = (1, 1)$ and $a_5 = (1/2, 1/2)$. The attractor K of the IFS is $[0, 2]^2$. We can take $V_0 = (1/2, 3/2)^2$ (accordingly $r_0 = 1$ and $\ell' = 5$).

We remark that all the IFS in the above examples satisfy the *finite type condition*, a notion introduced by Ngai and Wang [26] which is stronger than the WSC [27]. Roughly speaking, in the definition of the finite type condition, we require not only (2.4) to hold, but also all the maps $S_u \circ S_v^{-1}$ with $S_u, S_v \in \mathcal{S}_r(U(x, r))$, $x \in \mathbb{R}$ and $r > 0$, form a finite set. It was shown in [9] that for a self-similar measure on $\mathbb{R}$, if its generating IFS is equi-contractive and satisfies the finite type condition, then the L^q-spectrum τ is always differentiable over $(0, +\infty)$, furthermore, an analogue of Theorem 1.2 holds [11]. The results are based on a dynamical representation for these measures through a non-trivial sub-shift coding, the thermodynamic formalism for matrix-valued functions as well as the multifractal structure of Lyapunov exponents for products of non-negative matrices. However for self-similar measures with the WSC, it seems difficult to set up such a sub-shift representation. We remark that the differentiability property of the L^q spectrum has been further studied for some specific non-equi-contractive IFS with the finite type condition [38]. It still remains open whether τ is differentiable over $(0, +\infty)$ for all self-similar measures satisfying the WSC.

For a general self-similar measure with the WSC, we conjectured that the multifractal formalism always holds whenever $\alpha = \tau'(q)$ for $q < 0$. So far, this is true for all known examples such that τ can be calculated explicitly. In particular, Testud [39] showed that this is true for a specific class of self-similar measures generated by IFS of form (6.1) under certain assumptions.

Finally, we remark that our main results can be extended to the following class of self-affine measures.

Example 6.5. Consider an IFS $\Phi = \{S_i\}_{i=1}^m$ in $\mathbb{R}^d$ given by:

$$S_i(x) = A^{-1}x + c_i, \quad i = 1, \ldots, m,$$

where A is a $d \times d$ integral expanding matrix such that all the eigenvalues of A have the same modulus and, $c_i \in \mathbb{Z}^d$ for all i. In this case, Φ is not necessary to be a self-similar IFS, but it has some similar properties as a self-similar IFS with the WSC. Indeed, let K denote the attractor of Φ. It is easy to check that there is an integer ℓ such that

$$\sup_{x \in \mathbb{R}^d,\, n \in \mathbb{N}} \#\big\{S_u\colon u \in \mathcal{A}^n,\ S_u(K) \cap A^n\big(U(x, 1)\big) \neq \emptyset\big\} = \ell. \tag{6.2}$$

Assume that the supremum in (6.2) is attained at $x = x_0$ and $n = n_0$. Take $U_0 = A^{n_0}(U(x_0, 1))$ and let ω_i ($i = 1, \ldots, \ell$) be words in $\mathcal{A}^{n_0}$ such that S_{ω_i}'s are different and $S_{\omega_i}(K) \cap U_0 \neq \emptyset$. As an analogue of Proposition 2.4, we have for any $u \in \mathcal{A}^k$ ($k \geqslant 0$),

$$\big\{S_v\colon v \in \mathcal{A}^{k+n_0} \text{ with } S_v(K) \cap S_u(U_0) \neq \emptyset\big\} = \{S_{u\omega_i}\colon i = 1, \ldots, \ell\}.$$

An essentially identical argument then shows that our main results (Theorems 1.1, 1.2) remain valid for any self-affine measure μ generated by Φ. This completes a recent result of Deng and Ngai [5], who showed that in this special self-affine case, τ is differentiable over $(0, \infty)$ and $\dim_H E(\alpha) = \tau^*(\alpha)$ for $\alpha = \tau'(q)$, $q > 0$.

Acknowledgements

The authors would like to thank Julien Barral for providing an abstract in French. The first author was partially supported by the RGC grant and direct grant in CUHK, Fok Ying Tong Education Foundation and NSFC (Grant 10571100). The second author was partially supported by the RGC grant in CUHK.

Appendix A

In this part we give a full proof of Proposition 3.3. We need the following lemma, which can be proved in a similar way to the proof of the Besicovitch covering lemma (see [24, pp. 32–33]).

Lemma A.1. *Let $\{B(x_i, r)\colon i \in \mathcal{I}\}$ be a family of disjoint closed balls in $\mathbb{R}^d$ of radius r. Then for $N > 1$, there exists an integer $n \leqslant (8N)^d$ such that the index set $\mathcal{I}$ can be partitioned into $\mathcal{I}_1, \dots, \mathcal{I}_n$, and for each $1 \leqslant j \leqslant n$, the balls in the family $\{B(x_i, Nr)\colon i \in \mathcal{I}_j\}$ are disjoint.*

Proof of Proposition 3.3. For the given $\delta, r_0 > 0$, we choose a small $0 < \epsilon < 1$ such that

$$(\alpha - \delta/2)\epsilon \leqslant \tau_V(q+\epsilon) - \tau_V(q) \leqslant (\alpha + \delta/2)\epsilon,$$

and

$$(\alpha - \delta/2)\epsilon \leqslant \tau_V(q) - \tau_V(q-\epsilon) \leqslant (\alpha + \delta/2)\epsilon,$$

continue by picking $0 < \gamma < \min\{\epsilon\delta/8, 1\}$ and $0 < r_1 < \min\{r_0, 1\}$ such that for all $0 < t < r_1$,

$$\Theta_V(q; t) \leqslant t^{\tau_V(q)-\gamma}, \qquad \Theta_V(q \pm \epsilon; t) \leqslant t^{\tau_V(q\pm\epsilon)-\gamma}. \tag{A.1}$$

We fix the targeted $0 < r < \min\{r_1, 3^{-1/\gamma}\}$ such that

$$r^{\tau_V(q)+\gamma} \leqslant \Theta_V(q; r). \tag{A.2}$$

(We will require further restrictions on r in the sequel.) By the definition of $\Theta_V(q; r)$, we can find a family $\mathcal{B} = \{B(x_i, r)\}$ of disjoint balls contained in V with centers in $\mathrm{supp}(\mu)$ such that

$$\Theta_V(q; r)/2 \leqslant \sum_{B \in \mathcal{B}} \mu(B)^q \leqslant \Theta_V(q; r).$$

Combining this with (A.1) and (A.2) yields,

$$r^{\tau_V(q)+2\gamma} \leqslant \sum_{B \in \mathcal{B}} \mu(B)^q \leqslant r^{\tau_V(q)-\gamma}. \tag{A.3}$$

Set

$$\mathcal{B}_1 = \{B \in \mathcal{B}\colon \mu(B) \geqslant r^{\alpha-\delta}\}, \qquad \mathcal{B}_2 = \{B \in \mathcal{B}\colon \mu(B) \leqslant r^{\alpha+\delta}\},$$

and $\mathcal{B}_3 = \mathcal{B} \backslash (\mathcal{B}_1 \cup \mathcal{B}_2)$. Then

$$\sum_{B \in \mathcal{B}_1} \mu(B)^q = \sum_{B \in \mathcal{B}_1} \mu(B)^{q+\epsilon} \cdot \mu(B)^{-\epsilon} \leqslant \Theta_V(q+\epsilon; r) r^{-\epsilon(\alpha-\delta)} \leqslant r^{\tau_V(q+\epsilon)-\gamma-\epsilon(\alpha-\delta)} \leqslant r^{\tau_V(q)+\epsilon\delta/2-\gamma}.$$

Similarly, we have:

$$\sum_{B \in \mathcal{B}_2} \mu(B)^q = \sum_{B \in \mathcal{B}_2} \mu(B)^{q-\epsilon} \cdot \mu(B)^{\epsilon} \leqslant \Theta_V(q-\epsilon; r) r^{\epsilon(\alpha+\delta)} \leqslant r^{\tau_V(q-\epsilon)-\gamma+\epsilon(\alpha+\delta)} \leqslant r^{\tau_V(q)+\epsilon\delta/2-\gamma}.$$

These two inequalities together with (A.3) and (A.2) imply:

$$\begin{aligned}
\sum_{B \in \mathcal{B}_3} \mu(B)^q &= \Bigg(\sum_{B \in \mathcal{B}} - \sum_{B \in \mathcal{B}_1} - \sum_{B \in \mathcal{B}_2}\Bigg) \mu(B)^q \\
&\geqslant r^{\tau_V(q)+2\gamma} - 2r^{\tau_V(q)+\epsilon\delta/2-\gamma} = r^{\tau_V(q)+3\gamma}\big(r^{-\gamma} - 2r^{\epsilon\delta/2-4\gamma}\big) \\
&\geqslant r^{\tau_V(q)+3\gamma}\big(r^{-\gamma} - 2\big) \geqslant r^{\tau_V(q)+3\gamma}.
\end{aligned} \tag{A.4}$$

Note that for each $B \in \mathcal{B}_3$, $\mu(B)^q \leqslant \max\{r^{(\alpha\pm\delta)q}\} = r^{\alpha q-\delta|q|}$. Hence

$$\sum_{B\in\mathcal{B}_3} \mu(B)^q \leqslant (\#\mathcal{B}_3) r^{\alpha q-\delta|q|}$$

which combining with (A.4) yields,

$$\#\mathcal{B}_3 \geqslant r^{-\tau_V^*(\alpha)+\delta|q|+3\gamma}. \tag{A.5}$$

Next we will choose the family of balls in the proposition. We conduct the construction by considering the two cases $q \leqslant 0$ and $q > 0$ separately.

Case (i): $q \leqslant 0$. Let $\mathcal{B}_3 = \mathcal{B}_3' \cup \mathcal{B}_3''$, where

$$\mathcal{B}_3' = \big\{B(x,r) \in \mathcal{B}_3\colon\ \mu\big(B(x,r/N)\big) < r^{\alpha+2\delta}\big\},$$

and

$$\mathcal{B}_3'' = \big\{B(x,r) \in \mathcal{B}_3\colon\ \mu\big(B(x,r/N)\big) \geqslant r^{\alpha+2\delta}\big\}.$$

Then by (A.1),

$$\sum_{B(x,r)\in\mathcal{B}_3'} \mu\big(B(x,r/N)\big)^{q-\epsilon} \leqslant \Theta_V(q-\epsilon; r/N) \leqslant (r/N)^{\tau_V(q-\epsilon)-\gamma} \leqslant (r/N)^{\tau_V(q)-\epsilon(\alpha+\delta/2)-\gamma}.$$

On the other hand, we have $\sum_{B(x,r)\in\mathcal{B}_3'} \mu(B(x,r/N))^{q-\epsilon} \geqslant (\#\mathcal{B}_3') r^{(\alpha+2\delta)(q-\epsilon)}$. If we choose r, in addition to $0 < r < \min\{r_1, 3^{-1/\gamma}\}$, such that $r^\gamma < N^{\tau_V(q)-\epsilon(\alpha+\delta/2)-\gamma}$, then the above two inequalities imply that

$$\#\mathcal{B}_3' \leqslant r^{\tau_V(q)-\alpha q+3\epsilon\delta/2-2\delta q-2\gamma} = r^{-\tau_V^*(\alpha)+3\epsilon\delta/2-2\delta q-2\gamma}. \tag{A.6}$$

By (A.5), (A.6) and (A.1), we have:

$$\begin{aligned}\#\mathcal{B}_3'' = \#\mathcal{B}_3 - \#\mathcal{B}_3' &\geqslant r^{-\tau_V^*(\alpha)+\delta|q|+3\gamma} - r^{-\tau_V^*(\alpha)+3\epsilon\delta/2-2\delta q-2\gamma}\\ &\geqslant r^{-\tau_V^*(\alpha)+\delta(|q|+1)} r^{-\delta}\big(r^{3\gamma} - r^{3\epsilon\delta/2-2\gamma}\big) \quad (\text{using } q \leqslant 0)\\ &> r^{-\tau_V^*(\alpha)+\delta(|q|+1)},\end{aligned}$$

where the last inequality follows from $r^{-\delta}(r^{3\gamma} - r^{3\epsilon\delta/2-2\gamma}) \geqslant r^{-4\gamma}(r^{3\gamma} - r^{4\gamma}) \geqslant r^{-\gamma} - 1 > 1$. Hence the family $\mathcal{B}_3''$ of balls satisfies properties (3.3) and (3.4) of the proposition when $q \leqslant 0$.

Case (ii): $q > 0$. Unlike the above case, we need to use $B(x, Nr)$ for the estimation. To ensure that $B(x, Nr) \subset V$, we need to consider an auxiliary open set U with $\overline{U} \subset V$ and $\tau_U(q) = \tau_V(q)$. All the definitions of $\epsilon, \delta, \gamma, r_1, r$ and $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ in case (i) should now be made using U instead of V. In addition, r_1 is asked to be small enough such that $B(x, Nr_1) \subset V$ for any $x \in U$. Because $\tau_U(q) = \tau_V(q)$ for $q > 0$, the formulas (A.3), (A.4) and (A.5) remain valid.

Let $\widetilde{\mathcal{B}}_3 = \{B(x,Nr)\colon\ B(x,r) \in \mathcal{B}_3\}$. By Lemma A.1, there exists $l \leqslant (8N)^d$ such that $\widetilde{\mathcal{B}}_3$ can be partitioned into subfamilies $\widetilde{\mathcal{C}}_1, \ldots, \widetilde{\mathcal{C}}_l$ such that the balls in each subfamily are disjoint. Without loss of generality we assume that $\#\widetilde{\mathcal{C}}_1 \geqslant \#\widetilde{\mathcal{B}}_3/l = \#\mathcal{B}_3/l$. Then for $0 < r < r_1/N$, (A.1) implies:

$$\sum_{B\in\widetilde{\mathcal{C}}_1} \mu(B)^q \leqslant \Theta_V(q; Nr) \leqslant (Nr)^{\tau_V(q)-\gamma}.$$

(Strictly speaking, because we are using U instead of V, we cannot apply (A.1) directly. However the above formula still holds by making r smaller if necessary.) Let $\widetilde{\mathcal{C}}_1 = \widetilde{\mathcal{C}}_1' \cup \widetilde{\mathcal{C}}_1''$, where

$$\widetilde{\mathcal{C}}_1' = \big\{B \in \widetilde{\mathcal{C}}_1\colon\ \mu(B) > (Nr)^{\alpha-2\delta}\big\}, \qquad \widetilde{\mathcal{C}}_1'' = \big\{B \in \widetilde{\mathcal{C}}_1\colon\ \mu(B) \leqslant (Nr)^{\alpha-2\delta}\big\}.$$

We have:

$$\sum_{B\in\widetilde{\mathcal{C}}_1} \mu(B)^q \geqslant \sum_{B\in\widetilde{\mathcal{C}}_1'} \mu(B)^q \geqslant \big(\#\widetilde{\mathcal{C}}_1'\big)(Nr)^{(\alpha-2\delta)q}.$$

It follows that

$$\#\widetilde{\mathcal{C}}_1' \leqslant (Nr)^{-\tau_V^*(\alpha)+2\delta q-\gamma}.$$

If we choose γ and r to satisfy the following additional conditions:

$$\gamma < \delta q/5, \qquad r^{-\gamma} > 2(8N)^d N^{-\tau_V^*(\alpha)+\delta q+4\gamma},$$

then

$$\begin{aligned}\#\widetilde{\mathcal{C}}_1'' &\geqslant \#\widetilde{\mathcal{C}}_1 - \#\widetilde{\mathcal{C}}_1' \geqslant \frac{1}{l}\#\mathcal{B}_3 - \#\widetilde{\mathcal{C}}_1' \\ &\geqslant (8N)^{-d}\cdot r^{-\tau_V^*(\alpha)+\delta|q|+3\gamma} - (Nr)^{-\tau_V^*(\alpha)+2\delta q-\gamma} \quad \text{(by (A.5))} \\ &\geqslant (Nr)^{-\tau_V^*(\alpha)+\delta|q|+4\gamma}\big(r^{-\gamma}(8N)^{-d}N^{-(-\tau_V^*(\alpha)+\delta q+4\gamma)} - (Nr)^{\delta q-5\gamma}\big) \\ &\geqslant (Nr)^{\tau_V^*(q)+\delta|q|+4\gamma} \geqslant (Nr)^{-\tau_V^*(\alpha)+\delta(|q|+1)}.\end{aligned}$$

Therefore the family $\widetilde{\mathcal{C}}_1''$ of balls satisfies an analogue of (3.3) and (3.4), in which the number r is replaced by Nr. □

References

[1] F. Ben Nasr, I. Bhouri, Y. Heurteaux, The validity of the multifractal formalism: results and examples, Adv. Math. 165 (2002) 264–284.
[2] G. Brown, G. Michon, J. Peyrière, On the multifractal analysis of measures, J. Statist. Phys. 66 (1992) 775–790.
[3] R. Cawley, R.D. Mauldin, Multifractal decompositions of Moran fractals, Adv. Math. 92 (1992) 196–236.
[4] I. Daubechies, J. Lagarias, Thermodynamic formalism for multifractal functions, Rev. Math. Phys. 6 (1994) 1033–1070.
[5] Q.R. Deng, S.M. Ngai, Multifractal formalism for self-affine measures with overlaps, preprint.
[6] G.A. Edgar, R.D. Mauldin, Multifractal decompositions of digraph recursive fractals, Proc. London Math. Soc. 65 (1992) 604–628.
[7] K.J. Falconer, Fractal Geometry. Mathematical Foundations and Applications, second ed., John Wiley & Sons, New York, 2003.
[8] A.H. Fan, K.S. Lau, Iterated function system and Ruelle operator, J. Math. Anal. Appl. 231 (1999) 319–344.
[9] D.J. Feng, Smoothness of the L^q-spectrum of self-similar measures with overlaps, J. London Math. Soc. 68 (2003) 102–118.
[10] D.J. Feng, The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers, Adv. Math. 195 (2005) 24–101.
[11] D.J. Feng, Lyapunov exponent for products of matrices and multifractal analysis. Part II: General matrices, Israel J. Math. 170 (2009) 355–394.
[12] D.J. Feng, Gibbs properties of self-conformal measures and the multifractal formalism, Ergodic Theory Dynam. Systems 27 (2007) 787–812.
[13] D.J. Feng, K.S. Lau, X.Y. Wang, Some exceptional phenomena in multifractal formalism. Part II, Asian J. Math. 9 (2005) 473–488.
[14] D.J. Feng, K.S. Lau, J. Wu, Ergodic limits on the conformal repellers, Adv. Math. 169 (2002) 58–91.
[15] D.J. Feng, E. Olivier, Multifractal analysis of the weak Gibbs measures and phase transition–applications to some Bernoulli convolutions, Ergodic Theory Dynam. Systems 23 (2003) 1751–1784.
[16] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, B.I. Shraiman, Fractal measures and their singularities: the characterization of strange sets, Phys. Rev. A (3) 33 (1986) 1141–1151.
[17] P. Hanus, R.D. Mauldin, M. Urbański, Thermodynamic formalism and multifractal analysis of conformal infinite iterated function systems, Acta Math. Hungar. 96 (2002) 27–98.
[18] Y. Heurteaux, Estimations de la dimension inférieure et de la dimension supérieure des mesures, Ann. Inst. H. Poincaré Probab. Statist. 34 (1998) 309–338.
[19] T.Y. Hu, K.S. Lau, Multifractal structure of convolution of the Cantor measure, Adv. in Appl. Math. 27 (1) (2001) 1–16.
[20] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.
[21] K.S. Lau, S.M. Ngai, Multifractal measures and a weak separation condition, Adv. Math. 141 (1999) 45–96.
[22] K.S. Lau, S.M. Ngai, H. Rao, Iterated function systems with overlaps and self-similar measures, J. London Math. Soc. 63 (2001) 99–116.
[23] K.S. Lau, X.Y. Wang, Some exceptional phenomena in multifractal formalism. Part I, Asian J. Math. 9 (2005) 275–294.
[24] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability, Cambridge University Press, 1995.
[25] R.D. Mauldin, M. Urbański, Dimensions and measures in infinite iterated function systems, Proc. London Math. Soc. 73 (1996) 105–154.
[26] S.M. Ngai, Y. Wang, Hausdorff dimension of self-similar sets with overlaps, J. London Math. Soc. 63 (2001) 655–672.
[27] N.T. Nguyen, Iterated function systems of finite type and the weak separation property, Proc. Amer. Math. Soc. 130 (2002) 483–487.
[28] L. Olsen, A multifractal formalism, Adv. Math. 116 (1995) 82–196.
[29] N. Patzschke, Self-conformal multifractal measures, Adv. in Appl. Math. 19 (1997) 486–513.
[30] Y. Peres, B. Solomyak, Existence of L^q dimensions and entropy dimension for self-conformal measures, Indiana Univ. Math. J. 49 (2000) 1603–1621.
[31] Ya. Pesin, Dimension Theory in Dynamical Systems. Contemporary Views and Applications, University of Chicago Press, 1997.
[32] Ya. Pesin, H. Weiss, The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples, Chaos 7 (1997) 89–106.
[33] D. Rand, The singularity spectrum $f(\alpha)$ for cookie-cutters, Ergodic Theory Dynam. Systems 9 (1989) 527–541.
[34] R.H. Riedi, B.B. Mandelbrot, Multifractal formalism for infinite multinomial measures, Adv. in Appl. Math. 16 (1995) 132–150.
[35] R.T. Rockafellar, Convex Analysis, Princeton University Press, 1970.
[36] P. Shmerkin, A modified multifractal formalism for a class of self-similar measures with overlap, Asian J. Math. 9 (2005) 323–348.
[37] N. Sidorov, A. Vershik, Ergodic properties of Erdös measure, the entropy of the goldenshift and related problems, Monatsh. Math. 126 (1998) 215–261.
[38] B. Testud, Mesures quasi-Bernoulli au sens faible: résultats et exemples, Ann. Inst. H. Poincaré Probab. Statist. 42 (2006) 1–35.
[39] B. Testud, Phase transitions for the multifractal analysis of self-similar measures, Nonlinearity 19 (2006) 1201–1217.
[40] M. Yuri, Multifractal analysis of weak Gibbs measures for intermittent systems, Comm. Math. Phys. 230 (2002) 365–388.
[41] M.P.W. Zerner, Weak separation properties for self-similar sets, Proc. Amer. Math. Soc. 124 (1996) 3529–3539.