Nonlinearity **21** (2008) 1227–1232 doi:10.1088/0951-7715/21/6/004

Open set condition and post-critically finite self-similar sets

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Received 10 November 2007, in final form 1 April 2008
Published 24 April 2008
Online at stacks.iop.org/Non/21/1227

Recommended by M Tsujii

Abstract
We prove that for a contractive self-similar iterated function systems, if the matrices are commensurable, then the post-critically finite property implies the open set condition.

Mathematics Subject Classification: 28A80, 37F20

1. Introduction

Throughout this note we assume that $\{S_j\}_{j=1}^m$ is an *iterated function system* (IFS) consisting of m contractive similitudes on $\mathbb{R}^d$:

$$S_j(x) = A_j(x + d_j), \qquad j = 1, \ldots, m, \tag{1.1}$$

where $A_j = \rho_j R_j$, $0 < \rho_j < 1$, $\{R_j\}_{j=1}^m$ are orthonormal $d \times d$ matrices and $d_j \in \mathbb{R}^d$. It is well known that there exists a unique compact subset $K \subset \mathbb{R}^d$ such that [H]

$$K = \bigcup_{j=1}^m S_j(K).$$

The compact set K is called a *self-similar* set. The IFS $\{S_j\}_{j=1}^m$ is said to have the *open set condition* (OSC) if there exists a bounded nonempty open set V such that

$$\bigcup_{j=1}^m S_j(V) \subseteq V \qquad \text{and} \qquad S_i(V) \cap S_j(V) = \emptyset \qquad \text{if } i \neq j.$$

The OSC asserts a separation property in the iteration, it is one of the most fundamental conditions in the study of the IFS and the attractors. However, other than the obvious cases, it is usually difficult to verify such a condition [SSW, BR]. Indeed, the following conjecture has not been answered.

Conjecture. Suppose $\#(S_i(K) \cap S_j(K)) < \infty,\ i \neq j$, then $\{S_i\}_{i=1}^m$ has the OSC.

We refer to the above condition on K as *finitely ramifiable*, that is, K becomes disconnected if we remove the finite set of points in the intersection. A simple example of this is the Sierpinski gasket. This property has been used by Lindstrøm [L] in the consideration of Brownian motion on the *nested fractals*. The more general class is the celebrated *post-critically finite* (p.c.f) self-similar sets introduced by Kigami [K1, K2], which has been used extensively in the study of the Laplacian on fractals. The question whether such IFS satisfies the OSC has also been raised.

The above conjecture for the finite ramifiable self-similar sets has been proved by Bandt and Rao in $\mathbb{R}^2$ [BR] for the special case that K is connected. Our goal in this note is to consider the problem on $\mathbb{R}^d$, but on the more restrictive p.c.f. self-similar sets. We will define the notion of p.c.f. in the following section after setting up some of the notation. To say that $\{A_i\}_{i=1}^m$ is commensurable, we mean the existence of a matrix A such that $A_i = A^{n_i}$ for some positive integers $n_i,\ 1 \leqslant i \leqslant m$. We prove the following theorem.

Theorem 1.1. *Suppose the IFS $\{S_j\}_{j=1}^m$ in (1.1) is p.c.f. and the associated $\{A_i\}_{i=1}^m$ is commensurable, then the IFS satisfies the OSC.*

2. Preliminaries and lemmas

Let $\Sigma = \{1, \ldots, m\}$, $\Sigma^* = \bigcup_{n\geqslant 0} \Sigma^n$ and $\Sigma^{\mathbb{N}} = \{\boldsymbol{i} = i_1 i_2 \ldots i_n \ldots : i_j \in \Sigma\}$. For any $\boldsymbol{i} = i_1 i_2 \ldots i_n \in \Sigma^n, \boldsymbol{j} = j_1 j_2 \ldots j_k \in \Sigma^k$, we let $\boldsymbol{ij} = i_1 i_2 \ldots i_n j_1 j_2 \ldots j_k$ be the concatenation; we also denote $\boldsymbol{j}_1 \boldsymbol{j}_2 \cdots \boldsymbol{j}_k$ by $\boldsymbol{j}^k$ if $\boldsymbol{j}_i = \boldsymbol{j}$ for all i. For any $n \geqslant 1$ and $\boldsymbol{i} = i_1 i_2 \ldots i_n \ldots \in \Sigma^{\mathbb{N}}$ (or $\boldsymbol{i} = i_1 i_2 \ldots i_k \in \Sigma^k, k \geqslant n$), we let $\boldsymbol{i}|_n = i_1 \ldots i_n$. Also, we let σ denote the (*left*) *shift* on $\Sigma^{\mathbb{N}}$, i.e. $\sigma(i_1 i_2 \ldots) = (i_2 i_3 \ldots)$.

For the family $\{S_j\}_{j=1}^m$ of similitudes, we let $S_{\boldsymbol{i}} = S_{i_1} \circ \cdots \circ S_{i_n},\ A_{\boldsymbol{i}} = A_{i_1} \cdots A_{i_n}$. Also we let $\pi(\boldsymbol{i})$ be the unique point in $\bigcap_{n=1}^{+\infty} S_{i_1 \ldots i_n}(K)$. It is clear that the following lemma holds:

Lemma 2.1. *Let $\boldsymbol{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$ and let $\boldsymbol{\tau}_1,\ \boldsymbol{\tau}_2, \ldots \in \Sigma^*$ be any sequence of finite words. Then for any $x \in \mathbb{R}^n$, the sequence $\{S_{i_1 i_2 \ldots i_n \boldsymbol{\tau}_n}(x)\}_{n=1}^{\infty}$ converges to $\pi(\boldsymbol{i})$.*

The following characterization of OSC is due to Bandt and Graf [BG] together with a result of Schief [Sch], where $S_{\boldsymbol{j}}^{-1} \circ S_{\boldsymbol{i}}$ describe the differences between the two maps $S_{\boldsymbol{j}}$ and $S_{\boldsymbol{i}}$.

Theorem 2.2. *Let $\{S_j\}_{j=1}^m$ be contractive similitudes and let*

$$\mathfrak{S} = \{S_{\boldsymbol{j}}^{-1} \circ S_{\boldsymbol{i}} : \boldsymbol{i}, \boldsymbol{j} \in \Sigma^*,\ \boldsymbol{i} \neq \boldsymbol{j}\}.$$

Then $\{S_j\}_{j=1}^m$ satisfies the OSC if and only if the identity map I is not in the closure of $\mathfrak{S}$.

We will use the contraposition form of this theorem to prove theorem 1.1. First we prove the following lemma.

Lemma 2.3. *For any $\{S_j\}_{j=1}^m$ of contractive similitudes, if I is in the closure of $\mathfrak{S}$, then there exist $\boldsymbol{i} = i_1 i_2 \ldots \in \Sigma^{\mathbb{N}}$, $\boldsymbol{j} = j_1 j_2 \ldots \in \Sigma^{\mathbb{N}}$ and $\boldsymbol{u}_n, \boldsymbol{v}_n \in \Sigma^*$ such that $|\boldsymbol{u}_n|, |\boldsymbol{v}_n| \geqslant n$, $\boldsymbol{u}_n|_n = i_1 \ldots i_n$, $\boldsymbol{v}_n|_n = j_1 \ldots j_n$, $i_1 \neq j_1$ and*

$$\lim_{n\to\infty} S_{\boldsymbol{v}_n}^{-1} \circ S_{\boldsymbol{u}_n} = I.$$

Remark. In the above case, the finite words $\boldsymbol{u}_n$ and $\boldsymbol{v}_n$ can be written in the forms $\boldsymbol{u}_n = i_1 \ldots i_n \tau_{n+1}^{(n)} \ldots \tau_{k_n}^{(n)}$ and $\boldsymbol{v}_n = j_1 \ldots j_n \sigma_{n+1}^{(n)} \ldots \sigma_{l_n}^{(n)}$, respectively.

Proof. By assumption, there exist two sequences $\{\boldsymbol{u}_n\}, \{\boldsymbol{v}_n\} \subset \Sigma^*$ such that $|\boldsymbol{u}_n|, |\boldsymbol{v}_n| \to \infty$ and $S_{\boldsymbol{v}_n}^{-1} \circ S_{\boldsymbol{u}_n} \to I$ as $n \to \infty$. Also by cancellation, we can assume that $\boldsymbol{u}_n|_1 \neq \boldsymbol{v}_n|_1$ for each n.

We use the diagonal method to select two subsequences from $\boldsymbol{u}_n$ and $\boldsymbol{v}_n$ to satisfy the requirement in the lemma: there exist $i_1, j_1 \in \Sigma$, $i_1 \neq j_1$ such that the set $E_1 = \{n > 0 : \boldsymbol{u}_n|_1 = i_1, \boldsymbol{v}_n|_1 = j_1\}$ is an infinite set. Inductively, we can find $i_1, \ldots, i_k, \ldots \in \Sigma$, $j_1, \ldots, j_k, \ldots \in \Sigma$ such that for each k, the set $E_k = \{n \in E_{k-1} : \boldsymbol{u}_n|_k = i_1 i_2 \ldots i_k, \boldsymbol{v}_n|_k = j_1 j_2 \ldots j_k\}$ is an infinite set. Hence we can choose an increasing sequence $n_k \in E_k$ such that the sequences $\{\boldsymbol{u}_{n_k}\}$ and $\{\boldsymbol{v}_{n_k}\}$ satisfy the lemma. □

For an IFS $\{S_j\}_{j=1}^m$ of contractive similitudes, we define

$$\mathcal{C}_K = \bigcup_{i,j\in\Sigma, i\neq j} (S_i(K) \cap S_j(K)), \qquad \mathcal{C} = \pi^{-1}(\mathcal{C}_K).$$

Following [K2], we say that $\{S_j\}_{j=1}^m$ has the p.c.f. property if the set $\mathcal{P} = \bigcup_{n=1}^\infty \sigma^n(\mathcal{C})$ is a finite set, where σ is the shift operator such that $\sigma(i_1 i_2 \ldots i_k \ldots) = i_2 i_3 \ldots i_k \ldots$. This condition implies that each $\boldsymbol{i} \in \mathcal{C}$ is eventually periodic in the following sense:

Proposition 2.4. *If $\{S_j\}_{j=1}^m$ has the p.c.f. property, then there exist integers $N, \ell > 0$ such that for any $\boldsymbol{i} = i_1 i_2 \ldots \in \mathcal{C}$, the sequence $\sigma^N(\boldsymbol{i})$ has period ℓ.*

Proof. Let N denote the number of elements in $\mathcal{P}$. For any $\boldsymbol{i} = i_1 i_2 \ldots \in \mathcal{C}$, the sequence $\{\sigma^n(\boldsymbol{i})\}_{n=1}^\infty$ has at most N elements; this means that for $n \geqslant N$, $\sigma^n(\boldsymbol{i}) = i_{n+1} i_{n+2} \ldots$ must repeat its predecessors. Let k and p be the smallest integers such that $\sigma^k(\boldsymbol{i}) = \sigma^{k+p}(\boldsymbol{i})$. It is easy to see that $k \leqslant N$ and $\sigma^k(\boldsymbol{i})$ is a periodic sequence with period p. Hence $\boldsymbol{i}$ is periodic starting from the index N. The lemma follows by letting ℓ be the least common divisor of the periods for all $\boldsymbol{i} \in \mathcal{P}$. □

Lemma 2.5. *Let $\boldsymbol{i}, \boldsymbol{j} \in \Sigma^{\mathbb{N}}$ be defined as in lemma 2.3, then we have*

$$\boldsymbol{i}, \boldsymbol{j} \in \mathcal{C} \qquad \text{and} \qquad \pi(\boldsymbol{i}) = \pi(\boldsymbol{j}).$$

If in addition $\{S_j\}_{j=1}^m$ has the p.c.f. property and the matrices $\{A_i\}_{i=1}^m$ are commensurable (i.e. $A_i = A^{n_i}$), then (using the notation in proposition 2.4), there exist $\boldsymbol{\alpha} \in \Sigma^p$ and $\boldsymbol{\beta} \in \Sigma^q$ where p, q are multiples of ℓ, such that

$$i_{N+1} i_{N+2} \ldots = \boldsymbol{\alpha} \cdots \boldsymbol{\alpha} \cdots, \qquad j_{N+1}\, j_{N+2} \ldots = \boldsymbol{\beta} \cdots \boldsymbol{\beta} \cdots$$

and $A_{\boldsymbol{\alpha}} = A_{\boldsymbol{\beta}}$.

Proof. It is easy to see that $\lim_{n\to\infty} S_{\boldsymbol{v}_n}^{-1} \circ S_{\boldsymbol{u}_n} = I$ implies $\lim_{n\to\infty}(S_{\boldsymbol{v}_n}(x) - S_{\boldsymbol{u}_n}(x)) = 0$, $x \in K$. Using the expressions in the remark of lemma 2.3, we have

$$\lim_{n\to\infty} (S_{j_1 \ldots j_n} \circ S_{\sigma_{n+1}^{(n)} \ldots \sigma_{h_n}^{(n)}}(x) - S_{i_1 \ldots i_n} \circ S_{\tau_{n+1}^{(n)} \ldots \tau_{k_n}^{(n)}}(x)) = 0 \qquad \forall\, x \in K.$$

On the other hand, lemma 2.1 ensures that

$$\lim_{n\to\infty} S_{i_1 \ldots i_n \tau_{n+1}^{(n)} \ldots \tau_{k_n}^{(n)}}(x) = \pi(\boldsymbol{i}) \qquad \text{and} \qquad \lim_{n\to\infty} S_{j_1 \ldots j_n \sigma_{n+1}^{(n)} \ldots \sigma_{h_n}^{(n)}}(x) = \pi(\boldsymbol{j}).$$

Hence $\pi(\boldsymbol{i}) = \pi(\boldsymbol{j})$, and this common point is in $S_{i_1}(K) \cap S_{j_1}(K)$. By the definition of $\mathcal{C}$, the first part of the lemma follows.

To prove the second statement, we observe that $\boldsymbol{i}, \boldsymbol{j} \in \mathcal{C}$ imply that $i_{N+1} \cdots i_{N+n} \cdots$ and $j_{N+1} \cdots j_{N+n} \cdots$ have period ℓ (proposition 2.4). By the commensurability of the A_is, we can write

$$A_{i_{N+1} \cdots i_{N+\ell}} = A^s \qquad \text{and} \qquad A_{j_{N+1} \cdots j_{N+\ell}} = A^t$$

for some integers $s, t > 0$. Hence $A_{i_{N+1} \cdots i_{N+t\ell}} = A^{st} = A_{j_{N+1} \cdots j_{N+s\ell}}$. The assertion follows by letting $p = t\ell$, $q = s\ell$ and

$$\boldsymbol{\alpha} = i_{N+1} \ldots i_{N+p}, \qquad \boldsymbol{\beta} = j_{N+1} \ldots j_{N+q}.$$

□

3. Proof of the theorem

Proof of theorem 1.1. Suppose on the contrary, $\{S_j\}_{j=1}^m$ does not satisfy the OSC. Then by theorem 2.2 and lemma 2.3–2.5, there exist $\boldsymbol{i} = i_1 i_2 \ldots, \boldsymbol{j} = j_1 j_2 \ldots \in \mathcal{C}$ with $i_1 \neq j_1$, and finite words $\boldsymbol{u}_n, \boldsymbol{v}_n \in \Sigma^*$ such that $\pi(\boldsymbol{i}) = \pi(\boldsymbol{j})$,

$$\boldsymbol{u}_n|_n = i_1 \ldots i_n, \qquad \boldsymbol{v}_n|_n = j_1 \ldots j_n, \qquad |\boldsymbol{u}_n| \geqslant n, \quad |\boldsymbol{v}_n| \geqslant n,$$

and

$$\lim_{n\to\infty} S_{\boldsymbol{v}_n}^{-1} \circ S_{\boldsymbol{u}_n} = I.$$

We let $\boldsymbol{\alpha} = i_{N+1} \ldots i_{N+p}$ and $\boldsymbol{\beta} = j_{N+1} \ldots j_{N+q}$ be the periodic segments of $\boldsymbol{i}$ and $\boldsymbol{j}$ as in lemma 2.5. Also we let

$$\boldsymbol{a} = i_1 \ldots i_N \boldsymbol{\alpha}, \qquad \boldsymbol{b} = j_1 \ldots j_N \boldsymbol{\beta}$$

and

$$h_n = \max\{k > 0 : \boldsymbol{u}_n|_{(N+p)+kp} = \boldsymbol{a}\boldsymbol{\alpha}^k\}, \qquad k_n = \max\{k > 0 : \boldsymbol{v}_n|_{(N+q)+kq} = \boldsymbol{b}\boldsymbol{\beta}^k\}.$$

Since $\boldsymbol{u}_n|_n = i_1 \ldots i_n$, $\boldsymbol{v}_n|_n = j_1 \ldots j_n$, we know that h_n and k_n are well defined and finite for n large (as $\boldsymbol{u}_n$ is a finite word). Furthermore, $k_n, \; h_n \to \infty$ as $n \to \infty$. Note that either $h_n \leqslant k_n$ for infinitely many n or the other way around. By passing to subsequence, we assume that $1 \leqslant h_n \leqslant k_n$ for all $n \geqslant 1$. Hence we can write

$$\boldsymbol{u}_n = i_1 \ldots i_n \boldsymbol{\tau}_n = \boldsymbol{a}\boldsymbol{\alpha}^{h_n} \boldsymbol{\alpha}_n, \qquad \boldsymbol{v}_n = j_1 \ldots j_n \boldsymbol{\tau}'_n = \boldsymbol{b}\boldsymbol{\beta}^{h_n} \boldsymbol{\beta}_n, \tag{3.1}$$

where $\boldsymbol{\alpha}_n, \; \boldsymbol{\beta}_n \in \Sigma^*$ are finite words. This means that for $|\boldsymbol{\alpha}_n| > p$, then $\boldsymbol{\alpha}_n|_p \neq \boldsymbol{\alpha}$.

Next we let $x_1, \; x_2 \in K$ be the fixed point of $S_{\boldsymbol{\alpha}}$ and $S_{\boldsymbol{\beta}}$, respectively, i.e.

$$x_1 = S_{\boldsymbol{\alpha}}(x_1), \qquad x_2 = S_{\boldsymbol{\beta}}(x_2).$$

We claim that

$$\lim_{n\to\infty} S_{\boldsymbol{\alpha}_n}(x) = x_1, \qquad \lim_{n\to\infty} S_{\boldsymbol{\beta}_n}(x) = x_2, \qquad \forall\, x \in K. \tag{3.2}$$

Indeed by lemma 2.5 and the proof there, we have

$$S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}}(x_1) = S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}(x_2) = \pi(\boldsymbol{i}) = \pi(\boldsymbol{j}), \qquad \forall\, n > 0, \tag{3.3}$$

which also equals $S_{\boldsymbol{a}}(x_1) = S_{\boldsymbol{b}}(x_2)$. Since $\lim_{n\to\infty} S^{-1}_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}\boldsymbol{\beta}_n} \circ S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}\boldsymbol{\alpha}_n} = I$ and $S_{\boldsymbol{\beta}_n}$ is contractive, using the notations defined in (3.1), we have, for any $x \in K$,

$$\lim_{n\to\infty} (S^{-1}_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}} \circ S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}\boldsymbol{\alpha}_n}(x) - S_{\boldsymbol{\beta}_n}(x)) = 0. \tag{3.4}$$

Note that $A_{\boldsymbol{\alpha}} = A_{\boldsymbol{\beta}}$ (lemma 2.5). Since $\{A_i\}_{i=1}^m$ is commensurable, we let $A_{\boldsymbol{a}} = A^{r_1}$, $A_{\boldsymbol{b}} = A^{r_2}$ and $A_{\boldsymbol{\alpha}} = A^r$. Then by using

$$S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}}(x) - S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}}(x_1) = A^{r_1 + rh_n}(x - x_1)$$

and

$$S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}^{-1}(y) - S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}^{-1}(y') = A^{-r_2-rh_n}(y-y'),$$

we have

$$\begin{aligned} S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}^{-1} \circ S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}\boldsymbol{\alpha}_n}(x) - x_2 & \\ &= S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}^{-1} \circ S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}\boldsymbol{\alpha}_n}(x) - S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}^{-1} \circ S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}(x_2) \\ &= A^{-r_2-rh_n}(S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}\boldsymbol{\alpha}_n}(x) - S_{\boldsymbol{b}\boldsymbol{\beta}^{h_n}}(x_2)) \\ &= A^{-r_2-rh_n}(S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}\boldsymbol{\alpha}_n}(x) - S_{\boldsymbol{a}\boldsymbol{\alpha}^{h_n}}(x_1)) \qquad \text{(by (3.3))} \\ &= A^{r_1-r_2}(S_{\boldsymbol{\alpha}_n}(x) - x_1). \end{aligned}$$

This together with (3.4) yields

$$\lim_{n\to\infty}(A^{r_1-r_2}(S_{\boldsymbol{\alpha}_n}(x)-x_1) - (S_{\boldsymbol{\beta}_n}(x)-x_2)) = 0, \qquad x \in K.$$

For each $x \in K$, the sequences $\{S_{\boldsymbol{\alpha}_n}(x)\}$ and $\{S_{\boldsymbol{\beta}_n}(x)\}$ have converging subsequences. Without loss of generality, assume that $\lim_{n\to\infty} S_{\boldsymbol{\alpha}_n}(x) = y_1$ and $\lim_{n\to\infty} S_{\boldsymbol{\beta}_n}(x) = y_2$. Then the above relation implies that

$$A^{r_1-r_2}(y_1 - x_1) = y_2 - x_2. \tag{3.5}$$

Using (3.3), (3.5), and noting that $A_{\boldsymbol{a}} = A^{r_1}$ and $A_{\boldsymbol{b}} = A^{r_2}$, we see that

$$S_{\boldsymbol{a}}(y_1) - S_{\boldsymbol{b}}(y_2) = S_{\boldsymbol{a}}(x_1) + A^{r_1}(y_1-x_1) - S_{\boldsymbol{b}}(x_2) - A^{r_2}(y_2-x_2) = 0.$$

This implies $S_{\boldsymbol{a}}(y_1) = S_{\boldsymbol{b}}(y_2) \in S_{i_1}(K) \cap S_{j_1}(K)$. By the p.c.f. assumption, there exists $\boldsymbol{\omega} = \omega_1\omega_2\cdots \in \Sigma^{\mathbb{N}}$ such that $y_1 = \pi(\boldsymbol{\omega})$ and $\boldsymbol{a}\boldsymbol{\omega} \in \mathcal{C}$. Note that $\boldsymbol{a}\boldsymbol{\omega} = i_1\cdots i_N\boldsymbol{\alpha}\boldsymbol{\omega}$, proposition 2.4 and lemma 2.5 imply that $\boldsymbol{\alpha}\boldsymbol{\omega}$ has period ℓ, hence, $\boldsymbol{\omega} = \boldsymbol{\alpha}\boldsymbol{\alpha}\ldots$ by the definition of $\boldsymbol{\alpha}$. Therefore $y_1 = x_1$, and also $y_2 = x_2$ by (3.5). Since x_1 and x_2 are independent of $x \in K$. Hence the claim follows.

The claim also shows that $\boldsymbol{\alpha}_n$ and $\boldsymbol{\beta}_n$ are nonempty words for n large and that $|\boldsymbol{\alpha}_n| \to \infty$. To arrive at a contradiction, we can apply a similar proof of lemma 2.3 to the sequence $\{\boldsymbol{\alpha}_n\}$, to obtain a $\boldsymbol{u} = u_1u_2\ldots u_n\ldots \in \Sigma^{\mathbb{N}}$ and a subsequence $\{\boldsymbol{\alpha}_{q_n}\}$ such that $\boldsymbol{\alpha}_{q_n}|_n = u_1\ldots u_n$ for all n. Hence by the above claim and lemma 2.1 we have

$$x_1 = \lim_{n\to\infty} S_{\boldsymbol{\alpha}_n}(x) = \lim_{n\to\infty} S_{\boldsymbol{\alpha}_{q_n}}(x) = \pi(\boldsymbol{u}).$$

Therefore by (3.3),

$$\pi(\boldsymbol{a}\boldsymbol{u}) = S_{\boldsymbol{a}}(\pi(\boldsymbol{u})) = S_{\boldsymbol{a}}(x_1) = \pi(\boldsymbol{i}) = \pi(\boldsymbol{j}).$$

We conclude that $\boldsymbol{a}\boldsymbol{u} \in \mathcal{C}$. Note that $\boldsymbol{a} = i_1\ldots i_N\boldsymbol{\alpha}$, so proposition 2.4 implies that $\boldsymbol{\alpha}u_1u_2\ldots u_n\cdots$ has period ℓ. Hence $\boldsymbol{\alpha}_n|_p = \boldsymbol{\alpha}$ for infinite many $n > 0$, which contradicts the construction in (3.1) that $\boldsymbol{\alpha}_n|_p \neq \boldsymbol{\alpha}$ for all n. Hence $\{S_j\}_{j=1}^m$ has the OSC.

Acknowledgment

The research is partially supported by an HKRGC grant.

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