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On a generalized dimension of self-affine fractals

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For a $d\times d$ expanding matrix A, we define a pseudo-norm $w(x)$ in terms of A and use this pseudo-norm (instead of the Euclidean norm) to define the Hausdorff measure and the Hausdorff dimension $\dim_H^w E$ for subsets E in $\mathbf{R}^d$. We show that this new approach gives convenient estimations to the classical Hausdorff dimension $\dim_H E$, and in the case that the eigenvalues of A have the same modulus, then $\dim_H^w E$ and $\dim_H E$ coincide. This setup is particularly useful to study self-affine sets T generated by $\phi_j(x) = A^{-1}(x+d_j)$, $d_j \in \mathbf{R}^d$, $j = 1,\dots,N$. We use it to investigate the fractality of T for the case that $\{\phi_j\}_{j=1}^N$ satisfying the open set condition as well as the cases without the open set condition. We extend some well-known results in the self-similar sets to the self-affine sets.

1 Introduction

Let $M_d(\mathbf{R}^d)$ denote the set of $d\times d$ matrices and let $\mathcal{A}$ be a finite family of $A_j \in M_d(\mathbf{R}^d)$, $j=1,\dots,N$, of expanding matrices (i.e., all the eigenvalues of A_j have moduli > 1). Let $\mathcal{D} = \{d_1,\dots,d_N\} \subset \mathbf{R}^d$ be a *digit set*. We define the affine maps $\phi_j(x) = A_j^{-1}(x+d_j)$ and call $\{\phi_j(x)\}_{j=1}^N$ a self-affine *iterated function system* (IFS). If there is an attractor $T = T(\mathcal{A},\mathcal{D})$ satisfying $T = \bigcup_{j=1}^N \phi_j(T)$, T is called a *self-affine set*, and a *self-similar set* if all the A_j's are similar matrices (i.e., $A_j = \rho_j R_j$, where $\rho_j > 0$ and R_j is an orthonormal matrix). It is well-known that if the A_j's are expanding similarity matrices, the self-similar set always exists; it is not the case for the self-affine sets, one needs to use the joint spectral radius (see e.g., [10]). Nevertheless for the special case that all A_j equal to A, the self-affine set T always exists under the expanding condition and sometimes we use $T(A,\mathcal{D})$ to denote T to specify the A and $\mathcal{D}$. In general T or its boundary ∂T (if T has non-void interior) are fractal sets. One of the basic question in fractal geometry is to study the dimension of T. For the self-similar sets with certain separating conditions (e.g., open set condition [6], weak separation condition ([13], [14]), finite type condition [20]), there are methods to calculate their dimensions ([6], [9], [20], [24]). However the situation becomes much more complicated for self-affine sets. The problem stems from the non-uniform contraction of the A_j^{-1}'s in different directions, whereas the contraction of a self-similar set T is uniform in each direction.

In [4] Falconer obtained a formula of the Hausdorff dimension and box dimension of the self-affine set T for almost all vectors $(d_1,\dots,d_N)$ in $\mathbf{R}^{Nd}$. The formula depends on the product of singular values of the affine maps and is quite difficult to handle. This approach has been studied in more detail and the bounds of dimensions of T were estimated ([5], [4], [8], [25], [1]). However there are very few cases of which the Hausdorff dimensions can be calculated exactly. The only well-known case that this can be done is the example of McMullen [19] and Bedford [2] in $\mathbf{R}^2$ that all A_j are equal to $A = \left(\begin{smallmatrix} n & 0\\ 0 & m\end{smallmatrix}\right)$, $0<n<m$, $\mathcal{D} \subseteq \{0,1,\dots,n-1\}\times\{0,1,\dots,m-1\} \subseteq \mathbf{Z}^2$. There were extensions of McMullen's example with the same A and with (i) a graph-directed system or subshift of finite type plus the open set condition ([11], [12], [21], [22]); (ii) a more general $\mathcal{D} \subset \mathbf{Z}^2$ [9] (the

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number of $\mathcal{D}$ is unrestricted, hence unlike the previous cases, the corresponding IFS might not satisfy the *open set condition* (OSC)). However, another natural extension, namely the case that all A_j are equal to $A = \left(\begin{smallmatrix} n & 1 \\ 0 & n \end{smallmatrix}\right)$, has not been studied before. This case is a special case in the paper.

In this paper we consider the self-affine sets where all the dilation matrices A_j are equal, say, to A. We set up a new approach of using a pseudo-norm $w(x)$ (instead of the Euclidean norm) to define the Hausdorff measure $\mathcal{H}_w^s$, the Hausdorff dimension $\dim_H^w$ and the box dimension $\dim_B^w$. The pseudo-norm is defined in terms of A so as to absorb the non-uniform contractility from A (Section 2). This idea was introduced by Lemarié–Rieusset [15] to study the multi-resolution analysis in $\mathbf{R}^d$ with respect to an expanding matrix A. It was also used by Cohen, Gröchenig and Villemoes [3] to study the regularity of multivariate refinable functions. A simple relationship of the new and the old definitions of dimension is as follows.

Theorem 1.1 *Let A be an expanding matrix with $|\det(A)| = q$. Then for any subset E of $\mathbf{R}^d$*

$$\frac{\ln q}{d \ln \lambda_{\max}} \dim_H^w E \le \dim_H E \le \frac{\ln q}{d \ln \lambda_{\min}} \dim_H^w E$$

and

$$\frac{\ln q}{d \ln \lambda_{\max}} \underline{\dim}_B^w E \le \underline{\dim}_B E \le \overline{\dim}_B E \le \frac{\ln q}{d \ln \lambda_{\min}} \overline{\dim}_B^w E,$$

where $\lambda_{\max}$, $\lambda_{\min}$ are the maximum and minimum moduli of the eigenvalues of A.

The theorem provides a crude estimation of the Hausdorff dimension of a set E. It also follows immediately that if the eigenvalues of A have the same modulus λ, then $\dim_H E = (\ln q / d \ln \lambda) \dim_H^w E$. This class of matrices includes the similar matrices, the details will be discussed in Section 3 and an illustrating example in Section 5.

For self-affine sets, unlike the classical case, the relation of the new Hausdorff dimension and the box dimension is simple.

Theorem 1.2 *Let $T := T(A, \mathcal{D})$ be a self-affine set. Then $\dim_H^w E = \dim_B^w E$.*

One of the most important conditions in the study of self-similar sets is the open set condition (OSC) of the IFS. We have an extension of Schief's well-known result [23] to self-affine sets.

Theorem 1.3 *Let $A \in M_d(\mathbf{R})$ with $|\det A| = q$, let $\mathcal{D} = \{d_1, \dots, d_N\} \subset \mathbf{R}^d$ and let*

$$\phi_j(x) = A^{-1}(x + d_j), \quad j = 1, \dots, N.$$

If $\{\phi_j\}_{j=1}^N$ satisfies the OSC, then we can choose the open set O in the definition of the OSC such that $O \cap T \neq \emptyset$. Furthermore we have $\dim_H^w(T(A, \mathcal{D})) = d \ln N / \ln q := s$ and $0 < \mathcal{H}_w^s(T(A, \mathcal{D})) < \infty$.

For the case that $\{\phi_j\}_{j=1}^N$ does not satisfy the open set condition, we restrict our attention to the case that A is an integral matrix and $\mathcal{D} \subset \mathbf{Z}^d$. In this case we say that $T(A, \mathcal{D})$ is an *integral self-affine set.* The corresponding $\{\phi_j\}_{j=1}^N$ actually satisfies the weak separation condition in [14]. Following from the approach in [9], we reduce the recurrence of the IFS $\{\phi_j\}_{j=1}^N$ to a graph directed IFS with the OSC, which yields an adjacency matrix B. We prove (Theorem 5.4)

Theorem 1.4 *Let $T = T(A, \mathcal{D})$ be an integral self-affine set and let B be the adjacency matrix of T, then*

$$\dim_H^w T = d \ln \lambda_B / \ln q,$$

where λ_B is the spectral radius of the adjacency matrix B.

If the interior of T is nonempty, then we can express the boundary ∂T by another graph directed system. By using this we construct a corresponding adjacency matrix B' for ∂T and we have an analogous theorem for ∂T (Theorem 5.5).

We remark that it is more difficult to construct the pseudo-norm for two or more expanding matrices, as it may happen that the product of two expanding matrices may not be expanding, e.g., for $A_1 = \left(\begin{smallmatrix} 2 & n \\ 0 & 2 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 2 & 0 \\ n & 2 \end{smallmatrix}\right)$, the product of them is not an expanding matrix when $n \ge 3$. In comparison with Falconer's approach, we see that the singular value function $\phi^s(A)$ in [4], [2] involves the product of the $[s]+1$ (or s if it is an integer) smallest singular values of A, whereas in our approach, the pseudo-norm depends on $q = |\det A|$, which is the absolute value of the product of the d eigenvalues of A (see (2.1)). The former one gives more accurate estimation, but the treatment of the present one is more or less like the self-similar case which is easier to use.

The paper is organized as follows. In Section 2, we introduce the pseudo-norm $w(x)$ and lay down some basic properties. In Section 3, we define the various dimensions with respect to $w(x)$ and prove Theorem 1.1 (Theorems 3.1 and 3.5). The open set condition (OSC) and Theorem 1.3 (Theorems 4.2 and 4.4) are considered in Section 4. In Section 5, we prove Theorem 1.2 (Theorem 5.1), we also study the integral self-affine sets without assuming the OSC, and following the approach in [9] we sketch the proof of Theorem 1.4. Some illustrative examples are also provided.

2 The pseudo-norm

Let $M_d(\mathbf{R})$ denote the class of $d \times d$ matrices and let $A \in M_d(\mathbf{R})$ be expanding (i.e., all its eigenvalues have modulus > 1) with $|\det A| = q \in \mathbf{R}$. Since we can renorm $\mathbf{R}^d$ with an equivalent norm $|||\cdot|||$ so that $|||x||| < |||Ax|||$ for all $x \neq 0$ [16], we can assume without loss of generality that A has the property $||x|| \leq ||Ax||$ where the norm $||\cdot||$ is the Euclidean norm and the equality holds only for $x = 0$. Following the approach of [15], we introduce a "pseudo-norm" associated with A as follows: For $0 < \delta < 1/2$, we choose a positive C^∞ function $\phi_\delta(x)$ with support in B_δ such that $\varphi_\delta(-x) = \varphi_\delta(x)$ and $\int \varphi_\delta(x)\,dx = 1$. (Here $B_\delta = B(0,\delta)$ and $B(x,r)$ is the closed ball of center at x with radius r.) Consider the region $V = AB_1 \setminus B_1$, then by our convention that $||x|| < ||Ax||$ for $x \neq 0$, V is an annular region. It is clear that $\mathbf{R}^d \setminus \{0\}$ is the disjoint union of $\{A^kV\}_{k=-\infty}^{\infty}$. We let $h(x) = \chi_V * \varphi_\delta(x)$ be the convolution of the indicator function χ_V and $\varphi_\delta(x)$. Following [15], we define

$$w(x) = \sum_{n=-\infty}^{\infty} q^{-n/d} h(A^n x), \qquad x \in \mathbf{R}^d. \tag{2.1}$$

In the following we provide some basic properties of $w(x)$, some of them can be found in [15], [3].

Proposition 2.1 *The $w(x)$ defined in* (2.1) *is a C^∞ function on $\mathbf{R}^d$ and satisfies*

(i) $w(x) = w(-x)$, $w(x) = 0$ *if and only if* $x = 0$;

(ii) $w(Ax) = q^{1/d} w(x) \geq w(x)$ *for all* $x \in \mathbf{R}^d$;

(iii) *there exists an integer* $p > 0$ *such that for each* $x \in \mathbf{R}^d$, *the sum in* (2.1)) *representing* $w(x)$ *has at most* p *terms* $\neq 0$ *and*

$$\alpha \leq w(x) \leq pq^{p/d}, \qquad x \in V, \tag{2.2}$$

where $\alpha = \inf_{x \in V} h(x) > 0$.

Proof. Properties (i) and (ii) are straight forward from the definition. To prove (iii) we observe that h is supported by the region $V + B_\delta$, hence $\operatorname{supp} h \cap B_{1-2\delta} = \emptyset$. We can find a positive integer k such that for all $n \geq k$.

$$A^{-n}(V + B_\delta) \subset B^\circ_{1-2\delta}.$$

We claim that V intersects only the sets in $\{\operatorname{supp}(h \circ A^n)\}_{n=-k}^{k}$. In fact,

$$V \cap (\operatorname{supp} h \circ A^n) = V \cap (A^{-n} \operatorname{supp} h) = A^{-n}(A^n V \cap \operatorname{supp} h).$$

If $n > k$, we use the middle expression to conclude that

$$V \cap (A^{-n} \operatorname{supp} h) \subseteq V \cap B^\circ_{1-2\delta} = \emptyset;$$

if $n < -k$, let $m = -n$, we use the third expression, hence

$$A^{-m} V \cap \operatorname{supp} h \subseteq A^{-m} V \cap (V + B_\delta) \subseteq B^\circ_{1-2\delta} \cap (V + B_\delta) = \emptyset.$$

The claim follows from this. It follows that for $x \in V, w(x)$ is a sum of at most $p = 2k+1$ non-zero terms, that is, $w(x) = \sum_{n=-k}^{k} q^{-n/d} h(A^n x)$. Since $h(x) \leq 1$ for $x \in \mathbf{R}^d$ and $h(x) \geq \alpha$ for $x \in V$, it follows from $w \geq h$ on V that $w(x)$ has bounds as in (2.2). By (ii) the sum of the p terms can be extended to all $x \in \mathbf{R}^d$. □

We remark that (i) and (ii) in the above proposition defines $w(x) \in C^\infty$ (because $h(x) \in C^\infty$ and the expression of $w(x)$ in (2.1) is a finite sum in any compact set) uniquely up to constant bounds, i.e., if $w_i(x) \in C^\infty$, $i = 1, 2$, satisfy (i) and (ii), then there exists $C > 0$ such that $C^{-1} w_2(x) \leq w_1(x) \leq C w_2(x)$ for all $x \in \mathbf{R}^d$. Indeed from (i) and the continuity of $w_i(x)$, there exist C_1 such that

$$\sup_{x \in V} w_1(x) \leq C_1 \inf_{x \in V} w_2(x).$$

Note that $\mathbf{R}^d \setminus \{0\} = \bigcup_{n=-\infty}^{\infty} A^n V$, a disjoint union, hence for any $0 \neq x \in \mathbf{R}^d$, we can write $x = A^n x_0$ for some $x_0 \in V$ and a unique n. It follows that

$$w_1(x) = w_1(A^n x_0) = q^{n/d} w_1(x_0) \leq C_1 q^{n/d} w_2(x_0) = C_1 w_2(x).$$

We can interchange the role of $w_1(x)$ and $w_2(x)$ and the equivalence follows.

Lemma 2.2 *There exists $\beta > 0$ such that for any $x_1, x_2 \in \mathbf{R}^d$ with $w(x_1) \leq w(x_2)$, then $w(x_1 + x_2) \leq \beta w(x_2)$.*

Proof. For the given x_i, $i = 1, 2$, there exist integers l_1 and l_2 such that $x_i = A^{l_i} y_i$ for some $y_i \in V$. According to Proposition 2.1 (ii), we have $w(x_i) = q^{l_i/d} w(y_i)$ and by (iii), we have

$$\alpha q^{l_1/d} \leq w(x_1) \leq w(x_2) \leq p q^{p/d} q^{l_2/d}.$$

Consequently,

$$l_1 - l_2 \leq d \ln(p q^{p/d}/\alpha)/\ln q := \gamma.$$

Write $y_2 + A^{l_1 - l_2} y_1 = A^k x_0$ where $x_0 \in V$ and denote $\theta = \max_{x \in V} \|x\|$, we have

$$\|A^k x_0\| \leq \|y_2\| + \|A\|^\gamma \|y_1\| \leq (1 + \|A\|^\gamma)\theta.$$

Let k_0 be the largest integer k which satisfies

$$\|A^k x\| \leq (1 + \|A\|^\gamma)\theta$$

for some $x \in V$. Hence

$$w(x_1 + x_2) = w(A^{l_2 + k} x_0) = q^{(l_2 + k)/d} w(x_0) \leq q^{(l_2 + k_0)/d} w(x_0).$$

By Proposition 2.1 (iii), $w(x_0) \leq p q^{p/d}$ and $q^{l_2/d} \leq \alpha^{-1} w(x_2)$. Substituting these two inequalities to the above expression, we obtain

$$w(x_1 + x_2) \leq (p q^{p/d}/\alpha) q^{k_0/d} w(x_2).$$

The conclusion follows by taking β to be the constant in the above expression (note that k_0 is independent of x_1 and x_2). □

Let $B_w(x, r) = \{y : w(x - y) \leq r\}$ be the closed w-ball with center at x and radius r and let $\operatorname{diam}_w E = \sup\{w(x - y) : x, y \in E\}$. The following is an immediate consequence of Lemma 2.2.

Corollary 2.3 *Let β be as in Lemma 2.2, then*

(i) *for any $x, y \in \mathbf{R}^d$, $w(x + y) \leq \beta \max\{w(x), w(y)\}$;*

(ii) *for any $x \in \mathbf{R}^d$, $\operatorname{diam} B_w(x, r) \leq \beta r$.*

The pseudo-norm $w(x)$ is comparable with the Euclidean norm $\|x\|$ through $\lambda_{\max}$ and $\lambda_{\min}$, the maximal and minimal moduli of the eigenvalues of A.

Proposition 2.4 *Let A be an expanding matrix with $|\det A| = q$. Then for any $0 < \epsilon < \lambda_{\min} - 1$, there exist $C > 0$ (depending on ϵ) such that*

$$C^{-1}\|x\|^{\ln q/d\ln(\lambda_{\max}+\epsilon)} \leq w(x) \leq C\,\|x\|^{\ln q/d\ln(\lambda_{\min}-\epsilon)}, \qquad \|x\| > 1,$$

and

$$C^{-1}\|x\|^{\ln q/d\ln(\lambda_{\min}-\epsilon)} \leq w(x) \leq C\,\|x\|^{\ln q/d\ln(\lambda_{\max}+\epsilon)}, \qquad \|x\| \leq 1.$$

Proof. According to the properties of the spectral radius of the matrix A, it is not difficult to show that there exists C_1 such that

$$C_1^{-1}(\lambda_{\min}-\epsilon)^k\|x\| \leq \|A^k x\| \leq C_1(\lambda_{\max}+\epsilon)^k\|x\|$$

holds for all $k \geq 0$ and $x \in \mathbf{R}^d$. Note that $\mathbf{R}^d \setminus \{0\} = \bigcup_{k\in\mathbf{Z}} A^k V$. For $x \in \mathbf{R}^d$ with $\|x\| > 1$, then $x = A^k x_1$ for some $x_1 \in V$ and $k \geq 0$ (as $V = AB_1 \setminus B_1$). By (2.1) and (2.2) we have

$$\alpha q^{k/d} \leq w(x) \leq p q^{p/d} q^{k/d}.$$

If we write $\lambda_{\max} + \epsilon = q^{\ln(\lambda_{\max}+\epsilon)/\ln q}$, then the above implies that

$$(\lambda_{\max}+\epsilon)^k \leq \left(\alpha^{-1}w(x)\right)^{d\ln(\lambda_{\max}+\epsilon)/\ln q}.$$

Hence

$$\|x\| = \|A^k x_1\| = \|A^{k+1}A^{-1}x_1\| \leq C_1(\lambda_{\max}+\epsilon)^{k+1}\|A^{-1}x_1\| \leq C_2 w(x)^{d\ln(\lambda_{\max}+\epsilon)/\ln q}$$

with C_2 defined in the obvious way. By using the same argument, we have

$$\|x\| \geq C_3 w(x)^{d\ln(\lambda_{\min}-\epsilon)/\ln q}$$

for some $C_3 > 0$. This yields the estimation for $\|x\| > 1$. The proof of the case for $\|x\| \leq 1$ is similar by using $x = A^{-k}x_2$ for some $k > 0$ and $x_2 \in V$. □

3 The dimensions

In the following we use $w(x)$ to replace the Euclidean norm to define a measure analogous to the Hausdorff measure: For any subset $E \subset \mathbf{R}^d$, we define

$$\widetilde{\mathcal{H}}^\alpha_{w,\delta}(E) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}_w E_i)^\alpha : E \subseteq \bigcup E_i,\ \mathrm{diam}_w E_i \leq \delta\right\} \tag{3.1}$$

Since $\widetilde{\mathcal{H}}^\alpha_{w,\delta}(E)$ is increasing when δ tends to 0, we can define

$$\mathcal{H}^\alpha_w(E) = \lim_{\delta\to 0}\widetilde{\mathcal{H}}^\alpha_{w,\delta}(E).$$

By using Proposition 2.4 and the routine procedure, it is direct to show that $\mathcal{H}^\alpha_w$ is an outer measure and is a regular measure on the family of Borel subsets in $\mathbf{R}^d$. We call $\mathcal{H}^\alpha_w$ the *α-Hausdorff measure* with respect to w. By Proposition 2.1 (ii), we have

$$\mathcal{H}^\alpha_w(AE) = q^{\alpha/d}\mathcal{H}^\alpha_w(E).$$

As usual we define $\dim^w_H E = \inf\{\alpha : \mathcal{H}^\alpha_w E = 0\} = \sup\{\alpha : \mathcal{H}^\alpha_w E = \infty\}$ and refer to this as the *Hausdorff dimension* of E with respect to w.

Theorem 3.1 *Let $A \in M_d(\mathbf{R})$ be an expanding matrix with $|\det A| = q$. Then for any subset E of $\mathbf{R}^d$,*

$$\frac{\ln q}{d \ln \lambda_{\max}} \dim_H^w E \le \dim_H E \le \frac{\ln q}{d \ln \lambda_{\min}} \dim_H^w E,$$

where $\dim_H E$ is the standard Hausdorff dimension.

Proof. Let $0 < \epsilon < \lambda_{\min} - 1$ and let $\alpha_\epsilon = \ln(\lambda_{\min} - \epsilon)/\ln q$. For any $E \subset \mathbf{R}^d$ such that $\operatorname{diam} E$ (with respect to the Euclidean norm) is small enough, Proposition 2.4 implies that there exists $C > 0$ (depends only on ϵ) such that

$$\operatorname{diam} E \le C \left(\sup_{x,y \in E} w(x-y) \right)^{d\alpha_\epsilon} = C(\operatorname{diam}_w E)^{d\alpha_\epsilon}.$$

Let δ be small enough and $s = \dim_H^w E$. By the definition of Hausdorff measure, we have

$$\begin{aligned}
\widetilde{\mathcal{H}}^s_{w,\delta}(E) &= \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam}_w E_i)^s : E \subseteq \bigcup E_i,\ \operatorname{diam}_w E_i \le \delta \right\} \\
&\ge C_1 \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} E_i)^{s/d\alpha_\epsilon} : E \subseteq \bigcup E_i,\ \operatorname{diam} E_i \le \delta_1 \right\} \\
&= C_1 \mathcal{H}^{s/d\alpha_\epsilon}_{\delta_1}(E).
\end{aligned}$$

where $\delta_1 = C\delta^{d\alpha_\epsilon}$ and $C_1 = C^{-s/d\alpha_\epsilon}$. Consequently $\widetilde{\mathcal{H}}^s_w(E) \ge C_1 \mathcal{H}^{s/d\alpha_\epsilon}(E)$. This yields

$$\dim_H E \le s/d\alpha_\epsilon = \big(\ln q/(d \ln(\lambda_{\min} - \epsilon))\big) \dim_H^w E.$$

By taking $\epsilon \to 0$, the second inequality of the theorem follows. The proof of the first inequality is similar. □

Corollary 3.2 *Let A be an expanding matrix with $\lambda_{\min} = \lambda_{\max}$. Then for any subset E of $\mathbf{R}^d$, $\dim_H E = \dim_H^w E$.*

We remark that Theorem 3.1 gives a convenient way to estimate the bounds of the standard Hausdorff dimension in terms of $\dim_H^w$. The latter can be calculated for many important classes of self-affine sets and will be discussed in the following sections. The simplest family of matrices with $\lambda_{\min} = \lambda_{\max}$ are the 2×2 matrices such that the characteristic polynomials have complex roots or double roots. The class of matrices with this property which are not similitudes, are the strict upper (or lower) triangular matrices with the same entries in their diagonal line. For higher dimensions, recall that a square matrix A can be reduced to a Jordan normal form through a unitary P: $A = P^{-1}DP$; also A is a similar matrix if and only if D is diagonal. In general, D is not a diagonal matrix, we see that there is a much larger class of matrices whose eigenvalues have equal moduli and Corollary 3.2 indeed provides a useful alternative to study the Hausdorff dimension of self-affine sets with such matrices.

We can use a similar idea to define the box dimension with respect to w. Let E be any bounded subset of $\mathbf{R}^d$. We say that a cover $\{U_i\}_{i=1}^{\infty}$ of E is a r-cover with respect to w (or simply a r-cover of E) if $\operatorname{diam}_w(U_i) = r$ for each i. Let $N_r^w(E)$ be the smallest number of sets that are in the r-covers of E. The upper and lower *box dimensions* of E with respect to w are defined by

$$\overline{\dim}_B^w E = \limsup_{r \to 0} \frac{\ln N_r^w(E)}{-\ln r} \quad \text{and} \quad \underline{\dim}_B^w E = \liminf_{r \to 0} \frac{\ln N_r^w(E)}{-\ln r}$$

respectively. If the two expressions are equal, we denote the common value by $\dim_B^w E$ and call it the box dimension of E with respect to w.

The box dimension can also be defined by packing of w-balls. Let $M_r^w(E)$ be the maximum number of disjoint w-balls with centers at E and radii r.

Proposition 3.3 *For any bounded set E in $\mathbf{R}^d$, we have*

$$\overline{\dim}_B^w E = \limsup_{r\to 0} \frac{\ln M_r^w(E)}{-\ln r}, \quad \underline{\dim}_B^w E = \liminf_{r\to 0} \frac{\ln M_r^w(E)}{-\ln r}.$$

Proof. Let $m = M_r^w(E)$ and let $\{B_w(x_i,r)\}_{i=1}^m$ be disjoint w-balls with center at E, then for any r-cover $\{U_i\}$ of E, each U_i contains at most one point among the set $\{x_1,\dots,x_m\}$. Hence, $M_r^w(E) \le N_r^w(E)$. On the other hand, the maximality of $M_r^w(E)$ implies that: for any $x \in E$, there exists i such that $B_w(x,r)\cap B_w(x_i,r) \neq \emptyset$. Let $z \in B_w(x,r)\cap B_w(x_i,r)$ and $y \in B_w(x,r)$, then by Corollary 2.3

$$\begin{aligned} w(y-x_i) &\le \beta \max\{w(y-z), w(z-x_i)\} \\ &\le \beta \max\{\beta \max\{w(y-x), w(x-z)\}, r\} \\ &\le \beta^2 r. \end{aligned}$$

Hence, $\{B_w(x_i,\beta^2 r)\}_{i=1}^m$ is a cover of E. Again by Corollary 2.3, we can choose $U_i \supseteq B_w(x_i,\beta^2 r)$ with $\operatorname{diam}_w U_i = \beta^3 r$, then $\{U_i\}_{i=1}^m$ is a $\beta^3 r$-cover of E. This implies that $N_{\beta^3 r}^w(E) \le M_r^w(E)$. The conclusion follows by the definition. □

Proposition 3.4 *For any bounded subset E in $\mathbf{R}^d$, we have $\dim_H^w E \le \underline{\dim}_B^w E$.*

Proof. Let $s = \underline{\dim}_B^w E$ $(< \infty)$, then for any $\epsilon > 0$, there exists a decreasing sequence $\{r_n\}_{n=1}^\infty$ which converges to zero and for n large,

$$\frac{\ln N_{r_n}^w(E)}{-\ln r_n} \le s+\epsilon.$$

Let $m_n = N_{r_n}^w(E)$, then $m_n \le r_n^{-(s+\epsilon)}$ and there is a r_n-cover $\{U_i\}_{i=1}^{m_n}$ of E. This implies that

$$\widetilde{\mathcal{H}}_{w,r_n}^{s+\epsilon}(E) \le \sum_{i=1}^{m_n} (\operatorname{diam}_w(U_i))^{s+\epsilon} = m_n r_n^{s+\epsilon} \le 1.$$

Hence $\mathcal{H}_w^{s+\epsilon}(E) \le 1$ and $\dim_H^w E \le s$. □

Theorem 3.5 *Let $A \in M_d(\mathbf{R})$ be an expanding matrix with $|\det A| = q$. Then for any bounded set E of $\mathbf{R}^d$,*

$$\frac{\ln q}{d\ln\lambda_{\max}} \underline{\dim}_B^w E \le \underline{\dim}_B E \le \overline{\dim}_B E \le \frac{\ln q}{d\ln \lambda_{\min}} \overline{\dim}_B^w E,$$

where $\underline{\dim}_B E$ and $\overline{\dim}_B E$ are the standard lower and upper box dimension.

Proof. It follows from the definition of upper box dimension with respect to w that for $n = N_r^w(E)$, there exist a r-cover $\{U_i\}_{i=1}^n$ of E. Let $\eta = d\ln(\lambda_{\min}-\epsilon)/\ln q$. For r small enough, we can use Proposition 2.4 to obtain

$$\operatorname{diam} U_i \le C_1 \Big(\sup_{x,\, y\in U_i} w(x-y) \Big)^\eta \le C_1 r^\eta.$$

Hence $N_{C_1 r^\eta}(E) \le N_r^w(E)$. This implies that $\eta\, \overline{\dim}_B E \le \overline{\dim}_B^w E$. By letting $\epsilon \to 0$, we have

$$\overline{\dim}_B E \le \frac{\ln q}{d\ln\lambda_{\min}} \overline{\dim}_B^w E.$$

The lower bound estimation is similar and is omitted. □

4 The self-affine sets and the open set condition

Let $A \in M_d(\mathbf{R})$ be an expanding matrix, and let $\mathcal{D} = \{d_1, d_2, \ldots, d_N\} \subset \mathbf{R}^d$ be a digit set. Without loss of generalization we can assume that $d_1 = 0$. We define the *iterated function system* (IFS) by $\phi_j(x) = A^{-1}(x + d_j)$, $j = 1, \ldots, N$, then there is a unique $T := T(A, \mathcal{D})$ satisfying

$$T = \bigcup_{i=1}^{N} \phi_i(T).$$

We call this T the *self-affine set* generated by the IFS $\{\phi_j\}_{j=1}^N$. It is easy to see that each $x \in T$ has a radix expression $x = \sum_{n=1}^{\infty} A^{-n} d_{j_n}$, $d_{j_n} \in \mathcal{D}$. Let $\Sigma = \{1, 2, \ldots, N\}$, $\Sigma^n = \{(j_1 \ldots j_n) : 1 \le j_i \le N\}$ and $\Sigma^* = \bigcup_{n=1}^{\infty} \Sigma^n$. For $J = (j_1 \ldots j_n) \in \Sigma^*$, we use $|J| = n$ to denote the length of the multi-index J. Let

$$\mathcal{D}_1 = \mathcal{D}, \quad \mathcal{D}_n = \mathcal{D} + A\mathcal{D}_{n-1}, \ \ n \ge 2, \quad \text{and} \quad \mathcal{D}_\infty = \bigcup_{n=1}^{\infty} \mathcal{D}_n.$$

The map ϕ_J is defined as $\phi_J(x) = \phi_{j_1} \circ \cdots \circ \phi_{j_n}(x)$. By a simply calculation, we have $\phi_J(x) = A^{-n}(x + d_J)$ with

$$d_J = d_{j_n} + A d_{j_{n-1}} + \cdots + A^{n-1} d_{j_1} \in \mathcal{D}_n.$$

For any subset $X \subset \mathbf{R}^d$, we write $X_J = \phi_J(X)$ $(= A^{-n}(X + d_J))$. We say that the IFS $\{\phi_j(x)\}_{j=1}^N$ (or the pair $(A, \mathcal{D})$) satisfies the *open set condition* (OSC) if there exists an open O such that

$$\bigcup_i \phi_i(O) \subseteq O \quad \text{and} \quad \phi_i(O) \cap \phi_j(O) = \emptyset \ \text{ if } i \ne j.$$

Since $d_1 = 0$, it is easy to see that $0 \in \overline{O}$, the closure of O.

Proposition 4.1 *Suppose $(A, \mathcal{D})$ satisfies the OSC. Then*

(i) $\#\mathcal{D}_k = N^k$ *for all* $k \ge 1$ *and* $\mathcal{D}_\infty = \{d_J : J \in \Sigma^*\}$ *is uniformly discrete, i.e., there exists* $\eta > 0$ *such that* $\|x - y\| \ge \eta$ *for all* $x \ne y \in \mathcal{D}_\infty$*;*

(ii) *For any* $\eta > 0$, $C(\eta) = \sup_{k \ge 1} \sup_{x \in \mathbf{R}^d} \#\{J \in \Sigma^k : B(x, \eta) \cap (T + d_J) \ne \emptyset\} < \infty$.

Proof. Let $d_I \ne d_J \in \mathcal{D}_\infty$. We can assume without loss of generality that $|I| = |J| = k$ (because we can always adjust the length J in d_J by letting $J' = (0 \ldots 0J)$ and $d_J = d_{J'}$). By the OSC, we have

$$\phi_I(O) \cap \phi_J(O) = \emptyset \quad \text{for} \quad I \ne J, \quad I, J \in \Sigma^k.$$

This is equivalent to $(O + (d_I - d_J)) \cap O = \emptyset$. By the simple fact that $\eta = \inf_{x \in \mathbf{R}^d}\{\|x\| : (O + x) \cap O = \emptyset\} > 0$, we see that $|d_I - d_J| \ge \eta$ and (i) follows. Part (ii) is a direct consequence of (i). □

The following extends the well-known result of self-similar sets:

Theorem 4.2 *Let A be an expanding matrix with* $|\det A| = q$ *and let* $\mathcal{D} = \{0 = d_1, \ldots, d_N\} \subset \mathbf{R}^d$. *Suppose* $(A, \mathcal{D})$ *satisfies the OSC. Then* $\dim_H^w(T(A, \mathcal{D})) = d \ln N / \ln q := s$ *and* $0 < \mathcal{H}_w^s(T(A, \mathcal{D})) < \infty$.

Note that $N = q^{s/d}$. In the proof we will make use of the probability measure that satisfies the following invariant identity [6]:

$$\mu = N^{-1} \sum_{i=1}^{N} \mu \circ \phi_i^{-1}. \tag{4.1}$$

We need a lemma,

Lemma 4.3 *With the above assumptions and notations, we have*

(i) *there exists* $C_1 > 0$ *such that for any* $J \in \Sigma^k$, $\mu(T_J) \le C_1 N^{-k}$*;*

(ii) *there exists* $C_2 > 0$ *such that for any set* E *with* $\operatorname{diam}_w(E) < \delta < 1$, $\mu(E) \le C_2 \delta^s$.

Proof. Note that $J \in \Sigma^k$, $T_J = \phi_J(T) = A^{-k}(T + d_J)$. Iterating (4.1) k times yields

$$\mu = N^{-k} \sum_{|J|=k} \mu \circ \phi_J^{-1}.$$

Hence

$$\begin{aligned}\mu(T_J) &= N^{-k} \sum_{|J|=k} \mu(\phi_I^{-1}\phi_J(T)) \\ &\le N^{-k}\#\{I \in \Sigma^k : \phi_I^{-1}\phi_J(T) \cap T \ne \emptyset\} \\ &= N^{-k}\#\{I \in \Sigma^k : (T + d_J) \cap (T + d_I) \ne \emptyset\}.\end{aligned}$$

Let $\eta = \operatorname{diam}(T)$. Then $T + d_J \subseteq B(d_J, \eta)$. By Proposition 4.1 (ii) we have $\mu(T_J) \le C_1 N^{-k}$ with $C_1 = C(\eta)$. To prove (ii) we assume that $\operatorname{diam}_w(E) < \delta$. Let k be the positive integer such that $q^{-k/d} \le \delta < q^{-(k-1)/d}$. For $|J| = k$, we observe that $E \cap T_J \ne \emptyset$ is equivalent to $E \cap A^{-k}(T + d_J) \ne \emptyset$, which in tern is equivalent to $A^k E \cap (T + d_J) \ne \emptyset$. By Proposition 2.1 (ii),

$$\operatorname{diam}_w\left(A^k E\right) = \sup\left\{w(x-y) :\ x, y \in A^k E\right\} \le q^{k/d}\delta < q^{1/d}.$$

This together with Proposition 2.4 allows us to estimate the diameter of E with respect to the Euclidean norm: take any fixed small $\epsilon > 0$, if $\operatorname{diam}\left(A^k E\right) > 1$, then

$$\operatorname{diam}\left(A^k E\right) \le C \operatorname{diam}_w\left(A^k E\right)^{d\ln(\lambda_{\max}+\epsilon)/\ln q} \le C\, q^{\ln(\lambda_{\max}+\epsilon)/\ln q};$$

in general we have

$$\operatorname{diam}\left(A^k E\right) \le \max\left\{1, C q^{\ln(\lambda_{\max}+\epsilon)/\ln q}\right\} := 2\eta.$$

Let $C(\eta)$ be the corresponding constant in Proposition 4.1 (ii) with respect to the ball $B(x, \eta) \supseteq A^k E$. Then

$$\#\left\{J \in \Sigma^k :\ E \cap T_J \ne \emptyset\right\} = \#\left\{J \in \Sigma^k :\ A^k E \cap (T + d_J) \ne \emptyset\right\} \le C(\eta).$$

Let $C_2 = C_1 C(\eta)$, it follows that

$$\mu(E) \le \sum\{\mu(T_J) : T_J \cap E \ne \emptyset,\ |J| = k\} \le C(\eta)\left(C_1 N^{-k}\right) = C_2 q^{-sk/d} \le C_2 \delta^s.$$

□

Proof of Theorem 4.2. We first show that $\mathcal{H}_w^s(T) < \infty$. Let O be the open set as in the definition of OSC, for any $J \in \Sigma^k$, $\phi_J(O) = A^{-k}(O + d_J)$ implies that $\operatorname{diam}_w \phi_J(O) = q^{-k/d}\operatorname{diam}_w(O) =: \delta_k$, then $\left\{\phi_J\left(\overline{O}\right) : J \in \Sigma^k\right\}$ is a δ_k-cover of T. Consequently

$$\begin{aligned}\widetilde{\mathcal{H}}^s_{w,\delta_k}(T) &\le \sum_{|J|=k} \left(\operatorname{diam}_w \phi_J\left(\overline{O}\right)\right)^s = \sum_{|J|=k} \left(q^{-k/d} \operatorname{diam}_w \overline{O}\right)^s \\ &= N^{-k} \sum_{|J|=k} \left(\operatorname{diam}_w \overline{O}\right)^s = \left(\operatorname{diam}_w \overline{O}\right)^s.\end{aligned}$$

Letting $k \to \infty$, we have $\mathcal{H}_w^s(T) \le \left(\operatorname{diam}_w \overline{O}\right)^s < \infty$.

To prove $\mathcal{H}_w^s(T) > 0$, we make use of the self-affine measure μ in Lemma 4.3. For any cover $\{E_i\}_{i=1}^{\infty}$ of T with $\operatorname{diam}_w E_i \le \delta$, we denote $\operatorname{diam}_w E_i = \delta_i$, then by Lemma 4.3,

$$1 = \mu(T) \le \sum_i \mu(E_i) \le C_2 \sum_i \delta_i^s = C_2 \sum_i (\operatorname{diam}_w E_i)^s.$$

From the definition, we have $\widetilde{\mathcal{H}}^s_{w,\delta}(T) \ge C_2^{-1}\mu(T) > 0$. The conclusion follows by letting $\delta \to 0$. □

For the open set O in the OSC, it is technically important to choose the open set such that $O \cap T \neq \emptyset$ (the O such that $O \cap T = \emptyset$ is not so useful in the iteration of the maps, e.g., let T be the standard Cantor set generated by $\phi_1(x) = x/3$, $\phi_2(x) = (x+2)/3$ and let $O = (0,1) \setminus T$). Schief [23] proved that for the self-similar maps, such O exists (but it is not easy to find except for the obvious cases). In the following we extend his theorem to the self-affine case in term of the pseudo-norm setup. For convenience we say that $(A, \mathcal{D})$ satisfies the *strong open set condition* (SOSC) if this additional condition $O \cap T \neq \emptyset$ is satisfied.

Theorem 4.4 *The following statements are equivalent:*

(i) $(A, \mathcal{D})$ *satisfies the OSC ;*

(ii) $(A, \mathcal{D})$ *satisfies the SOSC ;*

(iii) $\mathcal{D}_\infty$ *is uniformly discrete and* $\#\mathcal{D}_k = N^k$ *for all* $k \geq 1$.

Proof. It is trivial that (ii) implies (i). That (i) implies (iii) is proved in Proposition 4.1. We need only prove that (iii) implies (ii).

Let $E^\delta = \{x : w(x-y) < \delta \text{ for some } y \in E\}$ denote the δ-neighborhood of a set E with respect to w. For $J \in \Sigma^k$, we denote $T_J^\delta = \left(T^\delta\right)_J$, i.e.,

$$T_J^\delta = \{\phi_J(x) : w(x-y) < \delta \text{ for some } y \in T\}.$$

Let

$$\Gamma_\delta(J) = \left\{I \in \Sigma^k : T_I \cap T_J^\delta \neq \emptyset\right\} \quad \text{and} \quad \gamma_\delta = \sup_{J \in \Sigma^*} \#\Gamma_\delta(J).$$

Note that $T_I \cap T_J^\delta \neq \emptyset$ is equivalent to $(T + d_I) \cap \left(T^\delta + d_J\right) \neq \emptyset$. Since $\#\mathcal{D}_k = N^k$ implies that $d_I \neq d_J$ if $I \neq J \in \Sigma^k$ for all $k \geq 1$ and $\mathcal{D}_\infty$ is uniformly discrete, by Proposition 4.1 (ii) (with $B(x, \eta)$ where $x \in T^\delta + d_J$, $\eta = \operatorname{diam} T^\delta$), we have $\gamma_\delta < \infty$. We can choose $J_0 \in \Sigma^*$ to attain the supremum, i.e., $\#\Gamma_\delta(J_0) = \gamma_\delta$. Note that $\Gamma_\delta(IJ_0) \supseteq I\,\Gamma_\delta(J_0)$, by the maximality of $\#\Gamma_\delta(J_0)$, we have

$$\Gamma_\delta(IJ_0) = I\,\Gamma_\delta(J_0) \quad \text{for each} \quad I \in \Sigma^*. \tag{4.2}$$

Let β be as in Lemma 2.2 and let $\delta' = \delta/\beta$, define

$$O = \bigcup_{J \in \Sigma^*} T_{JJ_0}^{\delta'}.$$

We claim that this O is the open set for the SOSC. It is clear that O is a bounded open set, $O \cap T \neq \emptyset$ and

$$\phi_i(O) = \bigcup_{J \in \Sigma^*} T_{iJJ_0}^{\delta'} \subseteq O, \quad i = 1, 2, \ldots, N.$$

We have to show that $\{\phi_j(O)\}_{j=1}^N$ are disjoint. If $\phi_i(O) \cap \phi_j(O) \neq \emptyset$ for some $i \neq j$, then there exist two multi-indices J_1 and J_2 such that

$$T_{iJ_1J_0}^{\delta'} \cap T_{jJ_2J_0}^{\delta'} \neq \emptyset.$$

Without loss of generality we assume that $|J_1| \leq |J_2|$. Let $y \in T_{iJ_1J_0}^{\delta'} \cap T_{jJ_2J_0}^{\delta'}$ be arbitrary. Observing that $T_{iJ_1J_0}^{\delta'} = A^{-|ij_1J_0|}\left(T^{\delta'} + d_{iJ_1J_0}\right)$, there exist $y_i, y_j \in T^{\delta'}$ such that

$$y = A^{-|iJ_1J_0|}\left(y_i + d_{iJ_1J_0}\right) = A^{-|jJ_2J_0|}\left(y_j + d_{jJ_2J_0}\right).$$

We can choose y such that there exist $z_i, z_j \in T$ satisfying

$$w(z_j - y_j) \leq w(z_i - y_i) < \delta'. \tag{4.3}$$

It follows that $w(A^{-l}(z_j - y_j)) \le w(z_j - y_j)$ for all $l \ge 0$ (Proposition 2.1 (ii)). By Lemma 2.2,

$$\begin{aligned} w(\phi_{iJ_1J_0}(z_i) - \phi_{jJ_2J_0}(z_j)) &= w(\phi_{iJ_1J_0}(z_i) - y + y - \phi_{jJ_2J_0}(z_j)) \\ &= w(A^{-|iJ_1J_0|}(z_i - y_i) + A^{-|jJ_2J_0|}(y_j - z_j)) \\ &< q^{-|iJ_1J_0|/d}\delta'\beta \\ &= q^{-|iJ_1J_0|/d}\delta. \end{aligned} \tag{4.4}$$

On the other hand, since $i \ne j$, (4.2) implies that $jJ \notin \Gamma_\delta(iJ_1J_0)$ for all $|J| = |J_1J_0|$, that is,

$$T_{jJ} \cap T^\delta_{iJ_1J_0} = \emptyset \quad \text{for all} \quad |J| = |J_1J_0|.$$

Hence $T_j \cap T^\delta_{iJ_1J_0} = \emptyset$. Observing that $\phi_{jJ_2J_0}(z_j) \in T_j$, we have

$$w(\phi_{iJ_1J_0}(z_i) - \phi_{jJ_2J_0}(z_j)) \ge \min_{y \notin T^\delta_{iJ_1J_0}} w(\phi_{iJ_1J_0}(z_i) - y) \ge q^{-|iJ_1J_0|/d}\delta,$$

which contradicts (4.4). Hence T satisfies the SOSC. □

The following is a convenient sufficient condition for the OSC, a corollary of Theorem 4.4 (iii).

Corollary 4.5 *Let A be an integral matrix and let $\mathcal{D} \subset \mathbf{Z}^d$ be a set of coset representatives of $\mathbf{Z}^d/A\mathbf{Z}^d$ (i.e., $d_i + A\mathbf{Z} \ne d_j + A\mathbf{Z}$ for distinct $d_i, d_j \in \mathcal{D}$). Then $(A, \mathcal{D})$ satisfies the OSC.*

5 The integral self-affine sets with overlaps

In this section we do not assume $(A, \mathcal{D})$ to satisfy the OSC, in other words, we allows the corresponding IFS $\{\phi_j\}_{j=1}^N$ to have overlap. We first show that for self-affine sets, the box dimension and the Hausdorff dimension with respect to w are the same.

Theorem 5.1 *Let $T =: T(A, \mathcal{D})$ be a self-affine set. Then $\dim_B^w T = \dim_H^w T$.*

Proof. Let $s = \dim_H^w T$. In view of Proposition 3.4, we only need to show that $\overline{\dim}_B^w T \le s$. We make use of the w-ball packing expression of the box dimension in Proposition 3.3. Let $a = \mathrm{diam}_w T$. For any $t > s$, we consider $0 < r < a$ such that

$$M_r^w(T) > (ar^{-1})^t \tag{5.1}$$

i.e., $\ln M_r^w(T)/(-\ln a^{-1}r) > t\ (> s)$. Let $m = M_r^w(T)$, by definition there exist disjoint w-balls $\{B_w(x_i, r)\}_{i=1}^m$ of radii r with centers $x_i \in T$. Let $k \ge 1$ be the integer such that

$$aq^{-k/d} \le r < aq^{-(k-1)/d}.$$

Since $T = \bigcup_{J \in \Sigma^k} \phi_J(T)$ and $x_i \in T$ for $1 \le i \le m$, there exist $y_i \in T$ and $J_i \in \Sigma^k$ such that $x_i = \phi_{J_i}(y_i)$. Observe that for any $x \in T$,

$$w(\phi_{J_i}(x) - x_i) = w(A^{-k}(x - y_i)) \le q^{-k/d}\mathrm{diam}_w T \le r.$$

This implies that $\phi_{J_i}(T) \subseteq T \cap B_w(x_i, r)$ so that $\{\phi_{J_i}(T)\}_{i=1}^m$ are disjoint closed sets. Now consider the IFS $\{\phi_{J_i}\}_{i=1}^m$, let E be the attractor. It is clear that $E \subset T$ and $\{\phi_{J_i}(E)\}_{i=1}^m$ are disjoint closed subsets. This implies that $\{\phi_{J_i}\}_{i=1}^m$ satisfies the OSC. By Theorem 4.2

$$\dim_H^w T \ge \dim_H^w E = \frac{d \ln m}{\ln q^k} \ge \frac{(k-1)\ln m}{-k \ln a^{-1} r} > \frac{(k-1)t}{k}. \tag{5.2}$$

By the hypothesis $\dim_H^w T = s$, (5.2) cannot hold for k sufficiently large, i.e., for r sufficiently small. This implies that for r sufficiently small, (5.1) cannot hold. Hence we have $M_r^w(T) \le (ar^{-1})^t$ for r sufficiently small so that

$$\limsup_{r \to 0} \frac{\ln M_r^w(T)}{-\ln r} \le t.$$

Since $t > s$ is arbitrary, we conclude that $\overline{\dim}_B^w T \le s$. □

Let $M_d(\mathbf{Z})$ denote the set of $d \times d$ matrices with integral entries. We call $T(A, \mathcal{D})$ an *integral self-affine set* if $A \in M_d(\mathbf{Z})$ and the digit set $\mathcal{D} \subset \mathbf{Z}^d$. For the special case that the digit set satisfies $\#\mathcal{D} = |\det A| = q$ and $T =: T(A, \mathcal{D})$ has positive Lebesgue measure, T tiles $\mathbf{R}^d$, that is, there exists a countable set $\mathcal{T}$ (tiling set) such that

$$\bigcup_{t \in \mathcal{T}} (T + t) = \mathbf{R}^d \quad \text{and} \quad (T + t) \cap (T + s) = \emptyset, \quad \text{for all} \quad t \neq s \in \mathcal{T}.$$

T is called a *self-affine tile* and $\mathcal{D}$ a *tile digit set*. In this case $(A, \mathcal{D})$ satisfies the OSC. This special class of self-affine sets has been studied in great detail in the literatures (cf., e.g., [16], [2]). Our consideration is the general case that $\#\mathcal{D}$ may not equal to $|\det A|$, but we need the self-affine tiles to define an auxiliary system in our consideration.

Lemma 5.2 *Let $A \in M_d(\mathbf{Z})$ be an expanding matrix, then for any bounded set E, there exists a tile digit set $\mathcal{C} \subset \mathbf{Z}^d$ such that $E \subseteq T(A, \mathcal{C})$.*

Proof. Let $\mathcal{E} = \{e_1, \dots, e_q\}$ be a complete set of coset representatives of the additive group $\mathbf{Z}^d / A\mathbf{Z}^d$, it is well-known that $T(A, \mathcal{E})$ is a tile (see e.g., [16]). There exists a ball $B(x_0, r)$ contained in $T(A, \mathcal{E})$, therefore by checking the radix expansion of $x \in T(A, \mathcal{E})$, it is easy to show that for any $l \in \mathbf{N}$, $B(lx_0, lr) \subseteq T(A, l\mathcal{E})$. Note also that $T(A, l\mathcal{E} - k) = T(A, l\mathcal{E}) - (A - I)^{-1}k$, where I is the identity matrix. Let $\mathcal{C} = l\mathcal{E} - k$, we can choose l large enough and $k \in \mathbf{Z}^d$ such that $E \subseteq T(A, \mathcal{C})$. □

For a bounded set E, we can use the tile $T(A, \mathcal{C}) \supseteq E$ to construct a mesh of partitions of E. We use this to reformulate the counting in the definition of the box dimension in Section 3. Let $\psi_j(x) = A^{-1}(x + c_j)$, $j = 1, 2, \dots, q$ and let Σ_q^n denote the corresponding index sets for $\psi_J = \psi_{j_1} \circ \cdots \circ \psi_{j_n}$, $J = (j_1, \dots, j_n) \in \Sigma_q^n$. Let

$$P_n(E) = \#\{J \in \Sigma_q^n : \ E \cap \psi_J(T(A, \mathcal{C})) \neq \emptyset\}.$$

Proposition 5.3 *For any bounded set E and tile $T(A, \mathcal{C})$ with $E \subseteq T(A, \mathcal{C})$, we have*

$$\overline{\dim}_B^w E = \frac{d}{\ln q} \limsup_{n \to \infty} \frac{\ln P_n(E)}{n}, \quad \underline{\dim}_B^w E = \frac{d}{\ln q} \liminf_{n \to \infty} \frac{\ln P_n(E)}{n}.$$

Proof. Let $\Gamma = T(A, \mathcal{C})$. Since $\Gamma = \bigcup_{J \in \Sigma_q^n} \psi_J(\Gamma)$ and $E \subseteq \Gamma$,

$$\{\psi_J(\Gamma) : J \in \Sigma_q^n \text{ and } E \cap \psi_J(\Gamma) \neq \emptyset\}$$

is a r_n-cover of E with $r_n = q^{-n/d} \operatorname{diam}_w \Gamma$. Hence $N_{r_n}^w(E) \leq P_n(E)$. On the other hand, let $m = N_{r_n}^w(E)$ and let $\{U_i\}_{i=1}^m$ be a r_n-cover of E. For any $J \in \Sigma_q^n$ with $E \cap \psi_J(\Gamma) \neq \emptyset$, $U_i \cap \psi_J(\Gamma) \neq \emptyset$ for some i, which is equivalent to $A^n U_i \cap (\Gamma + c_J) \neq \emptyset$. Since $\operatorname{diam}_w A^n U_i = q^{n/d} r_n = \operatorname{diam}_w \Gamma$ (Proposition 2.1), by Proposition 4.1 (ii) with $\eta = \operatorname{diam} \Gamma$,

$$\#\{J \in \Sigma_q^n : U_i \cap \psi_J(\Gamma) \neq \emptyset\} \leq C(\eta).$$

Hence $P_n(E) \leq C(\eta) N_{r_n}^w(E)$, this implies that $P_n(E)$ and $N_{r_n}^w(E)$ are equivalent and the lemma follows. □

Next we give a result for the dimension of the integral self-affine set $T =: T(A, \mathcal{D})$ (Theorem 5.4), which also yields an algorithm for the calculation. Since the approach is the same as in [9], we only outline the main idea here. For the self-affine set $T =: T(A, \mathcal{D})$, we construct an auxiliary self-affine tile $\Gamma =: T(A, \mathcal{C})$ (as in Lemma 5.2) satisfying

$$T \subset \Gamma^o \quad \text{and} \quad \Gamma + \mathcal{D} \subseteq A\Gamma.$$

Write $\psi_j(x) = A^{-1}(x + c_j)$, $j = 1, 2, \dots, q$. We use $\{\psi_J(\Gamma) : J \in \Sigma_q^k\}_{k \geq 1}$ to form a nested family of partitions and to select a graph-directed set using these Γ_J: for $J \in \Sigma_q^k$, let

$$\Delta(J) = \{d_I - c_J : \ I \in \Sigma_N^k, \ \psi_J(\Gamma) \cap \phi_I(\Gamma) \neq \emptyset\}. \tag{5.3}$$

(Here Σ_N^k is the Σ^k used before, the N is to emphasis the ϕ_I with index coming from $\mathcal{D} = \{d_1, \ldots, d_N\}$.) Note that $\psi_J(\Gamma) \cap \phi_I(\Gamma) \neq \emptyset$ if and only if $\Gamma \cap (\Gamma + (d_I - c_J)) \neq \emptyset$. Hence $\Delta(J)$ is used to record those partitions $\psi_J(\Gamma)$ that intersects $\phi_I(\Gamma)$. If we let $\mathcal{S}_k = \{J \in \Sigma_q^k : \Delta(J) \neq \emptyset\}$, then

$$T = \bigcap_{k=1}^{\infty} \left(\bigcup_{J \in \mathcal{S}_k} \psi_J(\Gamma) \right). \tag{5.4}$$

The crux of this construction is that $\{\Delta(J) : J \in \Sigma_q^*\}$ is actually a finite set. This allows us to construct a graph-directed system to reproduce T in view of (5.4). The vertices set is the set of all distinct $\Delta(J)$ together with the "root" $\Delta(0) = \{0\}$ (define $\Delta(0J) = \Delta(J)$) of the iteration, labelled as

$$V = \{\Delta(0), \Delta(J_1), \ldots, \Delta(J_m)\} = \{v_0, v_1, \ldots, v_m\}.$$

The iteration of ψ_J in the notion of the graph-directed system on V is

$$\Delta(J_i) \longrightarrow \Delta(J_i 1)\Delta(J_i 2)\ldots\Delta(J_i q), \quad i = 0, 1, \ldots, m. \tag{5.5}$$

We can write down the corresponding directed edges $E = \{E_{ij}\}_{i,j=0}^{m}$ on V are

$$E_{i,j} = \{c_s \in \mathcal{C} : \Delta(J_i s) = \Delta(J_j),\ 1 \leq s \leq q\}.$$

If we let

$$\phi_{i,j}^e(x) = A^{-1}(x + e), \quad e \in E_{i,j} \neq \emptyset, \quad i, j = 0, 1, \ldots, m,$$

then according to [9, Proposition 3.3], there are nonempty compact subsets $\{F_0 = T, F_1, \ldots, F_m\}$ satisfying the following graph-directed relation for T

$$F_i = \bigcup_{k=0}^{\infty} \bigcup_{e \in E_{i,j}} \phi_{ij}^e(F_j) = \bigcup_{j=0}^{m} A^{-1}(F_j + E_{ij}), \quad j = 0, 1, \ldots, m. \tag{5.6}$$

From the graph-directed relationship (5.6) we can define an $(m+1) \times (m+1)$ matrix B with the (i,j)-th entry given by

$$b_{ij} = \#E_{ij}, \quad i, j = 0, 1, \ldots, m$$

[6, p. 48], B is called the *adjacency* matrix of T. The adjacency matrix is used to count the number of paths of the graph-directed set in the iteration. Let e be the $(m+1)$-vector with all entries equal to 1 and let e_i be an $(m+1)$-vector with the i-th entry 1 and zero otherwise. It is not difficult to prove that $\#\mathcal{S}_n = e_0 B^n e$ where $\mathcal{S}_n$ is used in (5.4) [9, Proposition 4.1].

Theorem 5.4 *Let $T = T(A, \mathcal{D})$ be an integral self-affine set and let B be the adjacency matrix of T, then*

$$\dim_H^w T = \dim_B^w T = \frac{d \ln \lambda_B}{\ln q}$$

where λ_B is the spectral radius of the matrix B.

Proof. According to the Perron–Frobenius theorem, there exists a nonnegative vector $u \neq 0$ such that $\lambda_B u = Bu$, then $\lambda_B^n u = B^n u$ for all $n \geq 1$. We claim that $u_0 > 0$. In fact, since $u_j > 0$ for some j and there is a path from F_0 to F_j, i.e., there exist an integer k such that b_{0j}^k, the $(0,j)$ entry of B^k, is positive, this implies that $u_0 > 0$. Hence

$$\lambda_B = \left(\frac{e_0^t B^n u}{u_0} \right)^{1/n} \leq \lim_{n \to \infty} \left(e^t B^n e \right)^{1/n} = \lim_{n \to \infty} \|B^n\|^{1/n} = \lambda_B,$$

that is,

$$\lambda_B = \lim_{n\to\infty} (\#\mathcal{S}_n)^{1/n}.$$

On the other hand by the relation in (5.4), we have

$$\{J \in \Sigma_q^n : T \cap \psi_J(\Gamma) \neq \emptyset\} = \{J : J \in \mathcal{S}_n\}.$$

Hence, by Proposition 5.3, $\dim_B^w T = (d/\ln q)\lim_{n\to\infty} \ln(\#\mathcal{S}_n)/n = d\ln\lambda_B/\ln q$, and the conclusion follows by Theorem 5.1. □

From (5.4) and $\psi_J(\Gamma) = \bigcup_{i=1}^q \psi_{Ji}(\Gamma)$ for $J \in \Sigma_q^*$, it is easy to see that $\psi_J(\Gamma) \subseteq T$ if and only if $\Delta(JI) \neq \emptyset$ for all $I \in \Sigma_q^*$. Those vertices $\Delta(J)$ are termed *interior vertices*, they generate the interior of T and can be obtained in finite steps by using an algorithm developed by the above approach (see the following examples for an illustration). To obtain the boundary of T, we simply delete all the interior vertices, and there is a graph-directed subsystem for the boundary ∂T of T; similar to the above, there is an adjacency matrix B' of ∂T [9] to calculate the dimension of the boundary.

Theorem 5.5 *Let $T = T(A,\mathcal{D})$ be an integer self-affine set. If the Lebesgue measure of T is positive, let B' be the adjacency matrix of ∂T, then*

$$\dim_H^w \partial T = \dim_B^w \partial T = \frac{d\ln\lambda_{B'}}{\ln q}$$

where $\lambda_{B'}$ is the spectral radius of the matrix B'.

We omit the proof as it is similar to the proof of Theorem 5.4 (see [9] for detail). Instead, we use the following two examples to demonstrate this approach.

Example 5.6 Let $A = 3$ and $\mathcal{D} = \{0,1,3,4\}$, then $T^o \neq \emptyset$ and $\dim_H \partial T(A,\mathcal{D}) = 0$.

We remark that it is not difficult to show that $T = [0,2]$ and the statement is trivial. However it will serve as a good example to illustrate the algorithm in connection with the theorems.

Proof. Let $\{\phi_j\}_{j=1}^4$ be defined by $\phi_j(x) = 3^{-1}(x + d_j)$, $d_j = 0,1,3,4$, then it is an overlapping IFS. Let $\mathcal{C} = \{-1,2,5\}$, and let $\psi_j(x) = 3^{-1}(x + c_i)$ with $c_i = -1,2,5$, it is easy to check that $T(3,\mathcal{C}) = T(3, 3\{0,1,2\} - 1) = [-1/2, 3/2] =: \Gamma$. We use $\{\psi_i\}_{i=1}^3$ as the auxiliary system in the setup preceding Theorem 5.4. Let $\Delta(0) = \{0\} =: v_0$, according to (5.3) we have

$$\Delta(1) = \{1,2\} =: v_1, \quad \Delta(2) = \{-1,-2,1,2\} =: v_2, \quad \Delta(3) = \{-1,-2\} =: v_3$$

and by (5.5),

$$\begin{array}{lll} \Delta(11) = \emptyset, & \Delta(12) = v_1, & \Delta(13) = v_2, \\ \Delta(21) = v_2, & \Delta(22) = v_2, & \Delta(23) = v_2, \\ \Delta(31) = v_2, & \Delta(32) = v_3, & \Delta(33) = \emptyset. \end{array}$$

We see that the second generation already repeats all the vertices, hence $V = \{v_0, v_1, v_2, v_3\}$. The graph-directed relationship of $T(A,\mathcal{D})$ (as in (5.5)) is

$$\begin{array}{ll} v_0 \longrightarrow v_1v_2v_3, & v_1 \longrightarrow v_1v_2, \\ v_2 \longrightarrow v_2v_2v_2, & v_3 \longrightarrow v_2v_3. \end{array}$$

It is clear that there is only one interior vertex v_2, hence $T^o \neq \emptyset$. We omit v_2 and obtain the graph-directed relationship of $\partial T(A,\mathcal{D})$: $\partial V = \{v_0, v_1, v_3\}$ and

$$v_0 \longrightarrow v_1v_3, \quad v_1 \longrightarrow v_1, \quad v_3 \longrightarrow v_3.$$

The adjacency matrix of ∂T is

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and its eigenvalues are $\{0,1,1\}$. Hence, by Theorem 5.5, $\dim_H \partial T = 0$.

Example 5.7 Let $A = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right)$ and let $\mathcal{D}_1 = \{(0,0)^t, (0,1)^t\}$, $\mathcal{D}_2 = \{(0,0)^t, (0,1)^t, (3,0)^t\}$ and $\mathcal{D}_3 = \{(0,0)^t, (0,1)^t, (3,0)^t, (3,1)^t\}$. Then

$$\dim_H T(A,\mathcal{D}_i) = \dim_H^w T(A,\mathcal{D}_i) = \ln(i+1)/\ln 2, \quad i = 1,2,3.$$

Moreover $T(A,\mathcal{D}_3)$ is a tile and the dimension of its boundary is

$$\dim_H \partial T(A,\mathcal{D}_3) = \dim_H^w \partial T(A,\mathcal{D}_3) = 1.$$

We remark that $T(A,\mathcal{D}_3)$ actually plays a special role in the tiling theory. It was first introduced by Kenyon [11] to show that every self-replicate tiling for $T(A,\mathcal{D}_3)$ is non-periodic. Based on this example, Lagarias and Wang [18] characterized the class of tiles T generated by the *standard digit sets* $\mathcal{D}$ (e.g., $\mathcal{D}$ is a complete set of coset representative of $\mathbf{Z}^d/A\mathbf{Z}^d$) that has Lebesgue measure > 1.

Proof. Since the two eigenvalue of A equals 2, the two Hausdorff dimensions are equal (Corollary 3.2); by Corollary 4.5 , $(A,\mathcal{D}_i)$ satisfies the OSC, and by Theorem 4.2, the dimensions of $T(A,\mathcal{D}_i)$ follows.

That $T(A,\mathcal{D}_3)$ is a tile is known in [16]. To calculate the boundary of $T(A,\mathcal{D}_3)$, we follow the approach outlined above (see [9] for detail) and make use of Mathematica to find the directed-graph system and the adjacency matrix B'.

To simplify calculation we first let $\mathcal{D}' = \{(0,0)^t, (0,1)^t, (1,0)^t, (1,1)^t\}$, it is clear from (5.7) that the two tiles $T(A,\mathcal{D}_3)$ and $T(A,\mathcal{D}')$ have isomorphic boundaries on each side. Therefore $\dim_H \partial T(A,\mathcal{D}_3) = \dim_H \partial T(A,\mathcal{D}')$. We can replace $\mathcal{D}_3$ by $\mathcal{D}'$. Also we let $\mathcal{D} = \mathcal{D}' + A\mathcal{D}'$ and consider the same tiles $T(A^2,\mathcal{D})$ $(= T(A,\mathcal{D}))$ (using A^2 can reduce the number of iterations by a factor of 2).

Next we let $\mathcal{C}' = \{(0,0)^t, (0,1)^t, (2,0)^t, (2,1)^t\}$ and let $\Gamma =: T\big(A^2,\, \mathcal{C}' + A\mathcal{C}' - (1,0)^t\big)$. Then

$$\Gamma = T\big(A^2,\, \mathcal{C}' + A\mathcal{C}'\big) - \big(A^2 - I\big)^{-1}(1,0)^t = T(A,\mathcal{C}') - (1/3,0)^t \supseteq T(A,\mathcal{D}').$$

(We remark that $T(A,\mathcal{D}') \not\subseteq \Gamma^o$, which does not satisfy the condition in the above algorithm, however the missing part is only the horizontal top and bottom (see the geometry of this tile in the sequel) which does not affect our calculation.) With the help of Mathematica, we find the vertices set V of 24 elements in two iterations, and we can find the edges from the relation (5.5)

$$\Delta(J) \longrightarrow \Delta(J\cdot 1)\Delta(J\cdot 2)\ldots\Delta(J\cdot 16).$$

We check that there are 10 interior vertices, by eliminating those we use the remaining vertices to form the graph-directed subsystem and the adjacency matrix B' for the boundary of $T(A,\mathcal{D})$:

$$\begin{pmatrix}
1 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2
\end{pmatrix}.$$

Its eigenvalues are $\{0,0,0,0,0,1,1,1,1,1,4,4,4,4\}$. By Theorem 5.5 the Hausdorff dimension of the boundary of the self-affine set $T(A,\mathcal{D})$ is 1.

The geometry of $T(A,\mathcal{D}_3)$ can also be obtained by the radix expansions. We note that $(x,y)^t \in T(A,\mathcal{D}_3)$ if and only if

$$(x,y)^t = \sum_{k=1}^{\infty} A^{-k}(3d_{k,1}, d_{k,2})^t,$$

where all $d_{k,1}, d_{k,2} \in \{0,1\}$ can be chosen arbitrarily. By observing that $A^{-k} = 2^{-k}\begin{pmatrix} 2 & k2^{-1} \\ 0 & 2 \end{pmatrix}$, we have

$$y = \sum_{k=1}^{\infty} 2^{-k} d_{k,2}, \quad x = 3\left(\sum_{k=1}^{\infty} 2^{-k} d_{k,1}\right) - g(y), \tag{5.7}$$

where $g(y) = \sum_{k=1}^{\infty} k2^{-(k+1)} d_{k,2}$. From (5.7) we see that $0 \le y \le 1$ and also for each fixed value of y the allowed values of x form intervals of length 3. Therefore, the tile T has horizontal top and bottom at $y=0$ and $y=1$ respectively. The other two sides are $\{(-g(y),y)^t : 0 \le y \le 1\}$ and $\{(3-g(y),y)^t : 0 \le y \le 1\}$ (see Figure 1).

Fig. 1 The graph of $T(A,\ \mathcal{D}_3)$

The boundary of $T(A,\mathcal{D}_3)$ in the above example can also be calculated directly by using (5.7). The main estimation is on the graph

$$E =: \{(x, g(x)) : x \in [0,1]\}.$$

For each n and for $J = (j_1,\dots,j_n)$, we let $a_J = \sum_{k=1}^{n} 2^{-k} d_{j_k}$, $b_J = \sum_{k=1}^{n} k2^{-k-1} d_{j_k}$. We divide $[0,1]$ into dyadic intervals $[a_J,\ a_J + 1/2^{n+1})$. The image of these intervals under g is contained in

$$\left[b_J,\ b_J + \sum_{k=n+1}^{\infty} k2^{-k-1}\right).$$

Note that $\sum_{k=n+1}^{\infty} k2^{-(k+1)} = (n+1)2^{-(n-1)}$. It follows that the family of rectangles

$$[a_J,\ a_J + 1/2^{n+1}) \times [b_J,\ b_J + 2(1+n)2^{-n}], \quad |J| = n,$$

is a cover of E with diameter $2^{-n}(1+4(n+1))^{1/2}$. By the definition of Hausdorff measure, for any $\eta > 0$ and $\delta > 0$, there exists n large enough such that $2^{-n}(1+4(1+n))^{1/2} < \delta$. Hence

$$\begin{aligned} H_\delta^{1+\eta}(E) &= \inf\left\{\sum_{k=1}^{\infty} \operatorname{diam}(U_k)^{1+\eta} :\ E \subseteq \bigcup U_k \text{ and } \operatorname{diam}(U_k) < \delta\right\} \\ &\le \sum_{d_k\in\{0,1\},\ 1\le k\le n} 2^{-n(1+\eta)}(1+4(1+n))^{(1+\eta)/2} \\ &= 2^{-n\eta}(1+4(1+n))^{(1+\eta)/2} \end{aligned}$$

which tends to zero when $\delta \to 0$ (hence $n \to \infty$). This lead to $\mathcal{H}^{1+\eta}(E) = 0$ and therefore $\dim_H E \le 1$. On the other hand $\dim_H E \ge 1$ is clear as E is a graph on $[0,1]$.

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